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Linear extensions of relations between vector spaces

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Abstract: Let X and Y be vector spaces over the same field K. Following the terminology of Richard Arens [Pacific J. Math. 11 (1961), 9–23], a relation F of X into Y is called linear if $\lambda F(x) \subset F(\lambda x)$ and $F(x) + F(y) \subset F(x + y)$ for all $\lambda \in K \setminus \{0\}$ and $x, y \in X$.
After improving and supplementing some former results on linear relations, we show that a relation Φ of a linearly independent subset E of X into Y can be extended to a linear relation F of X into Y if and only if there exists a linear subspace Z of Y such that $\Phi(e) \in Y|Z$ for all $e \in E$. Moreover, if E generates X, then this extension is unique.
Furthermore, we also prove that if F is a linear relation of X into Y and Z is a linear subspace of X, then each linear selection relation Ψ of $F|Z$ can be extended to a linear selection relation Φ of F. A particular case of this Hahn-Banach type theorem yields an easy proof of the existence of a linear selection function f of F such that $f \circ F^{-1}$ is also a function.

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