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Congruence lattices of intransitive G-Sets and flat M-Sets

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Abstract: An M-Set is a unary algebra $\langle X, M\rangle$ whose set M of operations is a monoid of transformations of X; $\langle X, M\rangle$ is a G-Set if M is a group. A lattice L is said to be represented by an M-Set $\langle X, M\rangle$ if the congruence lattice of $\langle X, M\rangle$ is isomorphic to L. Given an algebraic lattice L, an invariant $\mathbf{\Pi}(L)$ is introduced here. $\mathbf{\Pi}(L)$ provides substantial information about properties common to all representations of L by intransitive G-Sets. $\mathbf{\Pi}(L)$ is a sublattice of L (possibly isomorphic to the trivial lattice), a Π-product lattice. A Π-product lattice $\Pi(\{L_i : i \in I\})$ is determined by a so-called multiset of factors $\{L_i : i \in I\}$. It is proven that if $\mathbf{\Pi}(L) \cong \Pi(\{L_i : i \in I\})$, then whenever L is represented by an intransitive G-Set $\mathbf{Y}$, the orbits of $\mathbf{Y}$ are in a one-to-one correspondence β with the factors of $\mathbf{\Pi}(L)$ in such a way that if $|I| > 2$, then for all $i \in I$, $L_{\beta(i)} \cong Con(\mathbf{X}_i)$; if $|I| = 2$, the direct product of the two factors of $\mathbf{\Pi}(L)$ is isomorphic to the direct product of the congruence lattices of the two orbits of $\mathbf{Y}$. Also, if $\mathbf{\Pi}(L)$ is the trivial lattice, then L has no representation by an intransitive G-Set. A second result states that algebraic lattices that have no cover-preserving embedded copy of the six-element lattice $A(1)$ are representable by an intransitive G-Set if and only if they are isomorphic to a Π-product lattice. All results here pertain to a class of M-Sets that properly contain the G-Sets — the so-called flat M-Sets, those M-Sets whose underlying sets are disjoint unions of transitive subalgebras.

Keywords: unary algebra; congruence lattice; intransitive G-Sets; M-Sets; representations of lattices
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