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Free associative algebras, noncommutative Gröbner bases, and universal associative envelopes for nonassociative structures

Abstract: First, we provide an introduction to the theory and algorithms for noncommutative Gröbner bases for ideals in free associative algebras. Second, we explain how to construct universal associative envelopes for nonassociative structures defined by multilinear operations. Third, we extend the work of Elgendy (2012) for nonassociative structures on the 2-dimensional simple associative triple system to the 4- and 6-dimensional systems.

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