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Construction of Mendelsohn designs by using quasigroups of (2,q)-varieties

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Abstract: Let q be a positive integer. An algebra is said to have the property $(2,q)$ if all of its subalgebras generated by two distinct elements have exactly q elements. A variety $\mathcal{V}$ of algebras is a variety with the property $(2,q)$ if every member of $\mathcal{V}$ has the property $(2,q)$. Such varieties exist only in the case of q prime power. By taking the universes of the subalgebras of any finite algebra of a variety with the property $(2,q)$, $2<q$, blocks of Steiner system of type $(2,q)$ are obtained. The stated correspondence between Steiner systems of type $(2,3)$ and the finite algebras of the varieties with the property $(2,3)$ is a folklore. There are also more general and significant results on $(2,q)$-varieties which can be considered as a part of an "algebraic theory of Steiner systems". Here we discuss another connection between the universal algebra and the theory of combinatorial designs, and that is the relationship between the finite algebras of such varieties and Mendelsohn designs. We prove that these algebras can be used not only as models of Steiner systems, but for construction of Mendelsohn designs, as well. For any two elements a and b of a groupoid, we define a sequence generated by the pair (a,b) in the following way: $w_0=a$, $w_1=b$, and $w_k=w_{k-2}\cdot w_{k-1}$ for $k\geq 2$. If there is an integer $p>0$ such that $w_p=a$ and $w_{p+1}=b$, then for the least number with this property we say that it is the period of the sequence generated by the pair (a,b). Then the sequence can be represented by the cycle $(w_0,w_1,\ldots,w_{p-1})$. The main purpose of this paper is to show that all of the sequences generated by pairs of distinct elements in arbitrary finite algebra of a variety with the property $(2,q)$ have the same periods (we say it is the period of the variety), and they contain unique appearance of each ordered pair of distinct elements. Thus, the cycles with period p obtained by a finite quasigroup of a variety with the property $(2,q)$ are the blocks (all of them of order p) of a Mendelsohn design.

Keywords: Mendelsohn design; quasigroup; $(2,q)$-variety; t-design
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