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On the solvability of systems of linear equations over the ring $\mathbb{Z}$ of integers

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Abstract: We investigate the question whether *a system $(E_i)_{i\in I}$ of homogeneous linear equations over $\mathbb{Z}$ is non-trivially solvable in $\mathbb{Z}$ provided that each subsystem $(E_j)_{j\in J}$ with $|J| \leq c$ is non-trivially solvable in $\mathbb{Z}$* where c is a fixed cardinal number such that $c < |I|$. Among other results, we establish the following. (a) The answer is 'No' in the finite case (i.e., I being finite). (b) The answer is 'No' in the denumerable case (i.e., $|I| = \aleph_0$ and c a natural number). (c) The answer in case that I is uncountable and $c \leq \aleph_0$ is 'No relatively consistent with ZF', but is unknown in ZFC. For the above case, we show that "*every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$*" **implies** (1) the Axiom of Countable Choice (2) the Axiom of Choice for families of non-empty finite sets (3) the Kinna–Wagner selection principle for families of sets each order isomorphic to $\mathbb{Z}$ with the usual ordering, and is **not implied by** (4) the Boolean Prime Ideal Theorem (BPI) in ZF (5) the Axiom of Multiple Choice (MC) in ZFA (6) $\mathsf{DC}_{<\kappa}$ in ZF, for every regular well-ordered cardinal number κ. We also show that the related statement "*every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has an uncountable subsystem with a non-trivial solution in $\mathbb{Z}$*" (1) is provable in ZFC (2) is not provable in ZF (3) does not imply "every uncountable system of linear homogeneous equations over $\mathbb{Z}$, each of its countable subsystems having a non-trivial solution in $\mathbb{Z}$, has a non-trivial solution in $\mathbb{Z}$" in ZFA.

Keywords: Axiom of Choice; weak axioms of choice; linear equations with coefficients in $\mathbb{Z}$; infinite systems of linear equations over $\mathbb{Z}$; non-trivial solution of a system in $\mathbb{Z}$; permutation models of ZFA; symmetric models of ZF

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