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On certain non-constructive properties of infinite-dimensional vector spaces

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Abstract: In set theory without the axiom of choice (AC), we study certain non-constructive properties of infinite-dimensional vector spaces. Among several results, we establish the following: (i) None of the principles $\mathrm{AC}^{\mathrm{LO}}$ (AC for linearly ordered families of nonempty sets)—and hence $\mathrm{AC}^{\mathrm{WO}}$ (AC for well-ordered families of nonempty sets)— $\mathrm{DC}({<}\kappa)$ (where κ is an uncountable regular cardinal), and "for every infinite set X, there is a bijection $f\colon X \to \{0,1\} \times X$", implies the statement "there exists a field F such that every vector space over F has a basis" in ZFA set theory. The above results settle the corresponding open problems from Howard and Rubin "Consequences of the axiom of choice", and also shed light on the question of Bleicher in "Some theorems on vector spaces and the axiom of choice" about the set-theoretic strength of the above algebraic statement. (ii) "For every field F, for every family $\mathcal{V} = \{V_i\colon i \in I\}$ of nontrivial vector spaces over F, there is a family $\mathcal{F} = \{f_i : i \in I\}$ such that $f_i \in F^{V_i}$ for all $i \in I$, and f_i is a nonzero linear functional" is equivalent to the full AC in ZFA set theory. (iii) "Every infinite-dimensional vector space over $\mathbb{R}$ has a norm" is not provable in ZF set theory.

Keywords: choice principle; vector space; base for vector space; nonzero linear functional; norm on vector space; Fraenkel–Mostowski permutation models of ZFA + ¬AC; Jech–Sochor first embedding theorem

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