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On sets of discontinuities of functions continuous on all lines

Abstract: Answering a question asked by K. C. Ciesielski and T. Glatzer in 2013, we construct a C^1-smooth function f on $[0, 1]$ and a closed set $M \subset \operatorname{graph} f$ nowhere dense in $\operatorname{graph} f$ such that there does not exist any linearly continuous function on $\mathbb{R}^2$ (i.e., function continuous on all lines) which is discontinuous at each point of M. We substantially use a recent full characterization of sets of discontinuity points of linearly continuous functions on $\mathbb{R}^n$ proved by T. Banakh and O. Maslyuchenko in 2020. As an easy consequence of our result, we prove that the necessary condition for such sets of discontinuities proved by S. G. Slobodnik in 1976 is not sufficient. We also prove an analogue of this Slobodnik's result in separable Banach spaces.

Keywords: linear continuity; discontinuity sets; Banach space
AMS Subject Classification: 26B05, 46B99

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