Totality of product completions

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Abstract. Categories whose Yoneda embedding has a left adjoint are known as total categories and are characterized by a strong cocompleteness property. We introduce the notion of multitotal category $\mathcal{A}$ by asking the Yoneda embedding $\mathcal{A} \to [\mathcal{A}^{op}, \mathcal{S}et]$ to be right multiadjoint and prove that this property is equivalent to totality of the formal product completion $\Pi\mathcal{A}$ of $\mathcal{A}$. We also characterize multitotal categories with various types of generators; in particular, the existence of dense generators is inherited by the formal product completion iff measurable cardinals cannot be arbitrarily large.

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1. Introduction

The concept of totality, introduced by Street and Walters [15], is a strong property of categories (implying completeness and cocompleteness — and more, see [14], [11]) which, nevertheless, most "current" categories enjoy. Recall that a category $\mathcal{A}$ is called *total* if its Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \to [\mathcal{A}^{op}, \mathcal{S}et]$ is right adjoint. Since solid functors (i.e., "good" faithful, right adjoint functors) $\mathcal{A} \to \mathcal{X}$ lift totality from $\mathcal{X}$ to $\mathcal{A}$ (see [17]), totality of $\mathcal{S}et$ and of its small-indexed powers is responsible for the totality of many important types of categories. For example, it allows us to conclude that all cocomplete, cowellpowered categories with a generator are total. These include locally presentable categories and monadic or topological categories over $\mathcal{S}et$.

In the present paper we investigate the totality of the free product completion $\Pi\mathcal{A}$ of a category $\mathcal{A}$ (dual to the free coproduct completion $\mathrm{Fam}\,\mathcal{A}^{op}$). The motivation is to describe the appropriate strong property of categories which are not cocomplete, but only *multicocomplete*, i.e., every small diagram has a multicolimit (as introduced by Y. Diers); examples are the category of linearly ordered sets, the category of fields, the category of local rings, all locally multipresentable categories in the sense of Diers [8], etc. Usually, "multi-concepts" for $\mathcal{A}$ are easily seen to be equivalent to the corresponding concepts for $\Pi\mathcal{A}$, e.g., a category $\mathcal{A}$ is

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multicocomplete iff $\Pi\mathcal{A}$ is cocomplete, and a functor $F : \mathcal{A} \to \mathcal{B}$ is *right multiadjoint* iff $\Pi F : \Pi\mathcal{A} \to \Pi\mathcal{B}$ is right adjoint, see [7]. We call a category *multitotal* if its Yoneda embedding $Y_{\mathcal{A}}$ is right multiadjoint; the question naturally arising is whether $\mathcal{A}$ is multitotal iff $\Pi\mathcal{A}$ is total. The affirmative answer in Theorem 3.6 of this paper unexpectedly turns out to have a somewhat involved proof. This theorem enables us to establish quite easily analogues of the results of [17], [3] in the "multi" context. Namely, we are able to characterize multitotal categories with various types of generators; in particular the above mentioned examples of multicocomplete categories are all multitotal. In fact, any multicocomplete, cowellpowered category with a generator is multitotal. While it is easy to see that the existence of a (strong or regular) generator in a category with a multi-initial object gives the same for its product completion, the corresponding property for dense generator turns out to be more involved. In fact, the question of whether a dense generator exists in $\Pi(\mathcal{A})$ whenever it exists in $\mathcal{A}$ depends on the set-theoretical assumption (M) that measurable cardinals are not arbitrarily large. One direction follows from Isbell's result in [10] that $\mathcal{S}et^{op}$ has a dense generator iff (M) holds; the converse direction is more difficult.
Analogously to the role of solid functors for totality, multisolid functors play an essential role in detecting multitotal categories. Multisolid functors $U : \mathcal{A} \to \mathcal{X}$ were already introduced (under a different name) and characterized in [18] by the property that the product-preserving extension $\Pi U : \Pi\mathcal{A} \to \Pi\mathcal{X}$ is solid. For $\mathcal{X}$ multicocomplete and $\mathcal{A}$ cowellpowered, they are precisely the faithful right multiadjoint functors for which $\mathcal{A}$ is multicocomplete, as shown more recently in [13].

2. Review of total categories and solid functors

2.1. A diagram $H : \mathcal{D} \to \mathcal{A}$ in a category $\mathcal{A}$ is said to be *small-partitioned* [12] if for all $A \in \mathcal{A}$ the comma category $(A \downarrow H)$ has only a small set of connected components. Thus, every small diagram is small-partitioned. An example of a (generally large) small-partitioned diagram is the diagram of elements of any functor from $\mathcal{A}$ to $\mathcal{S}et$. Recall from [11] that the following conditions are equivalent for $\mathcal{A}$:

(i) $\mathcal{A}$ is total, i.e., the Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \longrightarrow [\mathcal{A}^{op}, \mathcal{S}et]$ has a left adjoint;

(ii) $\operatorname{colim} H$ exists in $\mathcal{A}$ whenever the diagram $H : \mathcal{D} \to \mathcal{A}$ is small-partitioned;

(iii) $\operatorname{colim} H$ exists in $\mathcal{A}$ whenever, for all $A \in \mathcal{A}$, $\operatorname{colim} \mathcal{A}(A, H-)$ exists in $\mathcal{S}et$.

2.2. Total categories are trivially cocomplete (i.e., have colimits of all small diagrams), but they are also complete — indeed, they are "as complete as a category with small hom-sets can possibly be". In fact, recall that a category $\mathcal{A}$ is called *hypercomplete* [5] if every diagram $H : \mathcal{D} \to \mathcal{A}$ for which $\lim \mathcal{A}(A, H-)$ exists in $\mathcal{S}et$ for all $A \in \mathcal{A}$, has a limit in $\mathcal{A}$. The following conditions are equivalent ([4]):

(i) $\mathcal{A}$ is hypercomplete;

(ii) $\lim H$ exists in $\mathcal{A}$ for every diagram $H : \mathcal{D} \to \mathcal{A}$ with the property that for all $A \in \mathcal{A}$, there is only a small set of cones $\Delta A \to H$ in $\mathcal{A}$.

Every total category is hypercomplete, see [4].

2.3. Since solid functors detect totality, we briefly recall this notion. For a functor $U : \mathcal{A} \to \mathcal{X}$, a *$U$-sink* σ with codomain $X \in \mathcal{X}$ is a (possibly large) family of $\mathcal{A}$-objects A_i and of $\mathcal{X}$-morphisms $x_i : UA_i \to X$ $(i \in I)$. Let $(X \downarrow U)_\sigma$ be the full subcategory of $(X \downarrow U)$ of all objects $y : X \to UB$ such that for every $i \in I$ there exists a morphism $f_i : A_i \to B$ in $\mathcal{A}$ with $Uf_i = y \cdot x_i$. The functor U is *solid* (formerly semi-topological [16]) if U is faithful and if for every U-sink σ the category $(X \downarrow U)_\sigma$ has an initial object. The following conditions are equivalent for every functor $U : \mathcal{A} \to \mathcal{X}$ (see [16], [4], [11]):

(i) U is solid;

(ii) U has a left adjoint, and there is a class $\mathcal{E}$ of morphisms in $\mathcal{A}$ containing all isomorphisms and being closed under composition with them, such that

1. the counits of U belong to $\mathcal{E}$;
2. $\mathcal{A}$ is *$\mathcal{E}$-cocomplete*, that is: a. the pushout of a morphism in $\mathcal{E}$ along any morphism exists in $\mathcal{A}$ and belongs to $\mathcal{E}$, and b. the cointersection of a (possibly large) family of morphisms in $\mathcal{E}$ with common domain exists in $\mathcal{A}$ and belongs to $\mathcal{E}$.

In particular, every faithful right adjoint functor $U : \mathcal{A} \to \mathcal{X}$ defined on a cocomplete and cowellpowered category $\mathcal{A}$ is solid. The connection with totality is given by the following

Theorem. *Let $U : \mathcal{A} \to \mathcal{X}$ be a functor. Then:*

(1) *If $\mathcal{X}$ is total and U solid, then also $\mathcal{A}$ is total* (see [17]).

(2) *If $\mathcal{A}$ is total and U faithful and right adjoint, then U is solid* (see [3]).

Recall that a *generator* in a category $\mathcal{A}$ is a set $\mathcal{G}$ of objects such that the canonical functor $\mathcal{A} \to \mathcal{S}et^{\mathcal{G}}$, $A \mapsto (\mathcal{A}(G, A))_{G \in \mathcal{G}}$, is faithful. Since $\mathcal{S}et$ and all its small-indexed powers are total, the above theorem shows in particular:

Corollary ([4])**.** *Every cocomplete, cowellpowered category with a generator is total.*

2.4. Recall that the *free product completion* $\Pi\mathcal{A}$ of a category $\mathcal{A}$ has objects $A = (A_i)_{i \in I}$ given by small-indexed families of $\mathcal{A}$-objects A_i, and a morphism $f : A \to B = (B_j)_{j \in J}$ in $\Pi\mathcal{A}$ is given by a function $\varphi : J \to I$ and a family $f_j : A_{\varphi(j)} \to B_j$ $(j \in J)$ of $\mathcal{A}$-morphisms (with the obvious composition and identity maps), see [7]. Writing $SA = I$ and $Sf = \varphi$, one has a functor

$$S : (\Pi\mathcal{A})^{op} \longrightarrow \mathcal{S}et.$$

Whenever necessary we write $S_{\mathcal{A}}$ instead of S for distinction. In the terminology of [6], $\Pi\mathcal{A} = (\mathrm{Fam}(\mathcal{A}^{op}))^{op}$. We denote by

$$J_{\mathcal{A}} : \mathcal{A} \longrightarrow \Pi\mathcal{A}$$

the canonical embedding which identifies objects of $\mathcal{A}$ with singleton families. Every functor $F : \mathcal{A} \to \mathcal{B}$ into a category with products has a product-preserving extension $\overline{F} : \Pi\mathcal{A} \to \mathcal{B}$ which is unique up to natural isomorphism; hence, $[\mathcal{A}, \mathcal{B}]$ is equivalent to the full subcategory of product-preserving functors in $[\Pi\mathcal{A}, \mathcal{B}]$.

2.5. A *multicolimit* of a diagram $H : \mathcal{D} \to \mathcal{A}$ in $\mathcal{A}$ is given by a colimit of $J_{\mathcal{A}}H$ in $\Pi\mathcal{A}$; hence, it is a small-indexed family $(L_i)_{i\in I}$ of $\mathcal{A}$-objects together with cocones $\lambda_i : H \to \Delta L_i$, such that every cocone $H \to \Delta B$ in $\mathcal{A}$ factors uniquely through λ_i, for a unique $i \in I$. Already Diers [7] proved that every small diagram in $\mathcal{A}$ has a multicolimit if and only if $\Pi\mathcal{A}$ is cocomplete. As we shall deal with large diagrams, we need a precise analysis of this result. For $H : \mathcal{D} \to \Pi\mathcal{A}$ first assume that the limit $K = \lim SH^{op}$ exists in $\mathcal{S}et$; hence, every element $\alpha \in K$ is given by a compatible family of elements $\alpha_D \in SHD$, $D \in \mathcal{D}$. Every $\alpha \in K$ defines a diagram

$$H_\alpha : \mathcal{D} \to \mathcal{A} \quad \text{with} \quad H_\alpha D = (HD)_{\alpha_D},\ H_\alpha d = (Hd)_{\alpha_{D'}}\,,$$

for all $d : D \to D'$ in $\mathcal{D}$. (Note that $H_\alpha d$ is well-defined since $(SHd)(\alpha_{D'}) = \alpha_D$.)

Lemma ([7]). *A diagram $H : \mathcal{D} \to \Pi\mathcal{A}$ has a colimit in $\Pi\mathcal{A}$ if SH^{op} has a limit K in $\mathcal{S}et$ and H_α has a multicolimit in $\mathcal{A}$, for every $\alpha \in K$.*

Proof: For every $\alpha \in K$, a multicolimit of H_α is given by a small family of cocones $\lambda_{\alpha,i} : H_\alpha \to \Delta L_{\alpha,i}$, $i \in I_\alpha$. With $L = (L_{\alpha,i})_{\alpha\in K, i\in I_\alpha}$, this defines a cocone $\lambda : H \to \Delta L$ when we put

$$(\lambda_D)_{\alpha,i} = (\lambda_{\alpha,i})_D : (HD)_{\alpha_D} = H_\alpha D \to L_{\alpha,i}$$

for every $D \in \mathcal{D}$. In order to see that every cocone $\beta : H \to \Delta B$ factors uniquely through $\lambda_{\alpha,i}$, for a unique pair (α, i), we may without loss of generality assume $B \in \mathcal{A}$. The naturality of the family β_D $(D \in \mathcal{D})$ defines a uniquely determined element $\alpha \in K$, and the morphisms $\beta_D : (HD)_{\alpha_D} \to B$ define in fact a cocone $\overline{\beta} : H_\alpha \to \Delta B$. Hence, there are uniquely defined $i \in I_\alpha$ and $f : L_{\alpha,i} \to B$ in $\mathcal{A}$ with $\overline{\beta} = \Delta f \cdot \lambda_{\alpha,i}$, which gives the desired factorization $\beta = \Delta f \cdot \lambda$. □

2.6. A functor $U : \mathcal{A} \to \mathcal{X}$ is *right multiadjoint* if its extension

$$\Pi U : \Pi\mathcal{A} \longrightarrow \Pi\mathcal{X}$$

with $\Pi U \cdot J_{\mathcal{A}} = J_{\mathcal{X}} \cdot U$ is right adjoint ([7]). Analogously one defines U to be *multisolid* ("strongly localizing semitopological" in [18]) if ΠU is solid; this means that U is faithful, and that for every U-sink σ as in 2.3 the category $(X \downarrow U)_\sigma$ has a multi-initial object (i.e., a multicolimit of the empty diagram).

The characterization 2.3 of solid functors should lead to a characterization of multisolid functors when we exploit it for ΠU in lieu of U. In fact, let us agree that a *counit* of a right multiadjoint functor $U : \mathcal{A} \to \mathcal{X}$ at $A \in \mathcal{A}$ is simply the counit of ΠU at $A \in \Pi\mathcal{A}$. For a class $\mathcal{E}$ of morphisms in $\mathcal{A}$ we call $\mathcal{A}$ *multi-$\mathcal{E}$-cocomplete* if the multipushout of a morphism in $\mathcal{E}$ along any morphism exists in $\mathcal{A}$, with every component of it belonging to $\mathcal{E}$, and if the multicointersection of any family of morphisms in $\mathcal{E}$ with common domain exists in $\mathcal{A}$, with every component of it belonging to $\mathcal{E}$. Hence, if $\mathcal{A}$ is multi-$\mathcal{E}$-cocomplete, then $\Pi\mathcal{A}$ is $\mathcal{E}^{\Pi}$-cocomplete, with $\mathcal{E}^{\Pi}$ those morphisms of $\Pi\mathcal{A}$ whose components lie in $\mathcal{E}$; conversely if $\Pi\mathcal{A}$ is $\mathcal{F}$-cocomplete for a class of morphisms in $\Pi\mathcal{A}$, then $\mathcal{A}$ is multi-$\mathcal{F}^1$-cocomplete, with $\mathcal{F}^1$ the class of those morphisms of $\mathcal{A}$ which appear as a component of some morphism in $\mathcal{F}$. These observations prove the following results which essentially appeared in [13]:

Proposition. *Equivalent are for a functor $U : \mathcal{A} \to \mathcal{X}$:*

(i) *$U : \mathcal{A} \to \mathcal{X}$ is multisolid;*
(ii) *$U : \mathcal{A} \to \mathcal{X}$ is right multiadjoint, and there is a class $\mathcal{E}$ of morphisms in $\mathcal{A}$ containing all isomorphisms and being closed under composition with them, such that*
   1. *the counits of U lie in $\mathcal{E}$,*
   2. *$\mathcal{A}$ is multi-$\mathcal{E}$-cocomplete.*

Corollary ([13]). *If $\mathcal{X}$ is multicocomplete and $\mathcal{A}$ cowellpowered, a faithful functor $U : \mathcal{A} \to \mathcal{X}$ is multisolid if and only if $\mathcal{A}$ is multicocomplete.*

3. Multitotal categories

3.1 Definition. A category $\mathcal{A}$ is called *multitotal* if the Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \to [\mathcal{A}^{op}, \mathcal{S}et]$ is right multiadjoint.

It is easy to prove a "multiversion" of the characterization 2.1 of total categories:

3.2 Proposition. *The following conditions are equivalent for a category $\mathcal{A}$:*

(i) *$\mathcal{A}$ is multitotal;*
(ii) *every small-partitioned diagram $H : \mathcal{D} \to \mathcal{A}$ has a multicolimit in $\mathcal{A}$;*
(iii) *every diagram $H : \mathcal{D} \to \mathcal{A}$ for which* $\operatorname{colim} \mathcal{A}(A, H-)$ *exists in $\mathcal{S}et$ for all $A \in \mathcal{A}$, has a multicolimit in $\mathcal{A}$.*

Proof: (i)⇒(ii). A small-partitioned diagram $H : \mathcal{D} \to \mathcal{A}$ defines a functor $E : \mathcal{A}^{op} \to \mathcal{S}et$ which assigns to an object A the set of connected components of $(A \downarrow H)$. A multicolimit $\lambda_i : H \to \Delta L_i$ $(i \in I)$ is obtained from a $\Pi Y_{\mathcal{A}}$-universal arrow $(\eta_i : E \to Y_{\mathcal{A}} L_i)_{i \in I}$ for $E \in [\mathcal{A}^{op}, \mathcal{S}et]$: one just evaluates $(\eta_i)_{HD} : EHD \to \mathcal{A}(HD, L_i)$ at the component $(D, 1_{HD})$ in $(HD \downarrow H)$ to define $(\lambda_i)_D$, for every $i \in I$ and $D \in \mathcal{D}$.

(ii)⇒(i). For any $E \in [\mathcal{A}^{op}, \mathcal{S}et]$ one considers a multicolimit $(\lambda_i)_{i \in I}$ of the small-partitioned forgetful functor $H : \operatorname{el} E \to \mathcal{A}$, where $\operatorname{el} E$ is the "element category" with objects (A, x), $A \in \mathcal{A}$, $x \in EA$. Then the $\Pi Y_{\mathcal{A}}$-universal arrow

$(\eta_i)_{i\in I}$ for E is obtained as $(\eta_i)_A(x) = (\lambda_i)_{(A,x)}$, for every $i \in I$, $A \in \mathcal{A}$ and $x \in EA$.

(ii)$\Leftrightarrow$(iii) follows from the fact that, given a diagram $H : \mathcal{D} \to \mathcal{A}$ and $A \in \mathcal{A}$, $\operatorname{colim} \mathcal{A}(A, H-)$ exists in $\mathcal{S}et$ iff $(A \downarrow H)$ has just a set of connected components. □

3.3 Corollary. *If $\Pi\mathcal{A}$ is total, then $\mathcal{A}$ is multitotal.*

Proof: We check condition (ii) of 3.2. Given a small-partitioned diagram $H : \mathcal{D} \to \mathcal{A}$, it is easy to see that then also $J_{\mathcal{A}}H : \mathcal{D} \to \Pi\mathcal{A}$ is small-partitioned, so that $\operatorname{colim} J_{\mathcal{A}}H$ exists in $\Pi\mathcal{A}$, by hypothesis. But this is, by definition, a multicolimit of H in $\mathcal{A}$. □

3.4. The question which remains is whether multitotality of $\mathcal{A}$ is also a sufficient condition for $\Pi\mathcal{A}$ to be total. In other words: does right adjointness of $\Pi Y_{\mathcal{A}}$ imply right adjointness of $Y_{\Pi\mathcal{A}}$? For every category $\mathcal{A}$ denote by $[(\Pi\mathcal{A})^{op}, \mathcal{S}et]_{\amalg}$ the full subcategory of $[(\Pi\mathcal{A})^{op}, \mathcal{S}et]$ of all coproduct-preserving functors; then one has a functor Σ which makes the upper triangle of the diagram (1) below commutative (up to natural isomorphism):

$$
\begin{tikzcd}
 & \Pi\mathcal{A} \arrow[dl, "\Pi Y_{\mathcal{A}}"'] \arrow[dr, "Y_{\Pi\mathcal{A}}"] & \\
\Pi[\mathcal{A})^{op}, \mathcal{S}et] \arrow[rr, "\Sigma"] & & [(\Pi\mathcal{A})^{op}, \mathcal{S}et] \\
[\mathcal{A}^{op}, \mathcal{S}et] \arrow[u, hook] \arrow[rr, "\sim"] & & [(\Pi\mathcal{A})^{op}, \mathcal{S}et]_{\amalg} \arrow[u, hook]
\end{tikzcd}
\tag{1}
$$

Σ is the product-preserving extension of the functor that assigns to $E \in [\mathcal{A}^{op}, \mathcal{S}et]$ the functor ΣE with $(\Sigma E)(C_k)_K = \amalg_{k\in K} EC_k$; of course, $\Sigma E \in [(\Pi\mathcal{A})^{op}, \mathcal{S}et]_{\amalg}$. It is clear that the restriction of Σ creates an equivalence of categories, as indicated in (1). Now it is a straightforward exercise to show that the existence of a left adjoint to $\Pi Y_{\mathcal{A}}$ gives a $(Y_{\Pi\mathcal{A}})$-universal arrow for every $F \in [\mathcal{A}^{op}, \mathcal{S}et]_{\amalg}$, hence a "partial left adjoint" to $Y_{\Pi\mathcal{A}}$. We now establish the existence of "a totally defined" left adjoint.

3.5 Example. The category $\Pi\mathcal{S}et$ is total. To see this, we check the conditions of Corollary 2.3. First observe that $\Pi\mathcal{S}et$ is certainly cocomplete (see 2.5). Now, a morphism $f : A \to B$ in $\Pi\mathcal{S}et$ is an epimorphism if and only if $\varphi = Sf : SB \to SA$ is injective and every map $f_j : A_{\varphi(j)} \to B_j$ $(j \in J = SB)$ is epic in $\mathcal{S}et$ (see 6.3 in [18]). This characterization shows that wellpoweredness and cowellpoweredness of $\mathcal{S}et$ give cowellpoweredness of $\Pi\mathcal{S}et$. Finally, $\Pi\mathcal{S}et$ has a (single-object) generator, namely the triple $(1, \emptyset, \emptyset)$: the singleton set 1 and (one copy of) the empty set are needed to distinguish distinct morphisms $f, g : A \to B$ in $\Pi\mathcal{S}et$ with $Sf = Sg$, and two copies of $\emptyset$ are needed to distinguish f and g in case $Sf \neq Sg$.

3.6 Proposition. *The functor $S_{\mathcal{A}} : (\Pi\mathcal{A})^{op} \to \mathcal{S}et$*

- *has a left adjoint $L_{\mathcal{A}}$ if $\mathcal{A}$ has a terminal object;*
- *has a right adjoint $R_{\mathcal{A}}$ if $\mathcal{A}$ has a multi-initial object.*

Proof: For 1 terminal in $\mathcal{A}$ and every set I, let $L_{\mathcal{A}}I = (1)_I$ be the constant I-indexed family with value 1. Then $\mathrm{id}_I : I \to S_{\mathcal{A}}L_{\mathcal{A}}I$ serves as an $S_{\mathcal{A}}$-universal arrow. For an initial object $O = (O_t)_{t\in T}$ in $\Pi\mathcal{A}$ and every set I, let $R_{\mathcal{A}}I = (O_i)_{(t,i)\in T\times I}$. Then the projection $S_{\mathcal{A}}R_{\mathcal{A}}I \to I$ is an $S_{\mathcal{A}}$-couniversal arrow. $\Box$

3.7 Theorem. *A category $\mathcal{A}$ is multitotal if and only if $\Pi\mathcal{A}$ is total.*

Proof: We need to prove necessity (see 3.3). To this end we assume $\mathcal{A}$ to be multitotal and prove totality of $\Pi\mathcal{A}$ by checking condition 2.1 (ii). Hence, let $H : \mathcal{D} \to \Pi\mathcal{A}$ be a small-partitioned diagram in $\Pi\mathcal{A}$. From 3.6 one has adjoint situations

$$L_{\mathcal{S}et} \dashv S_{\mathcal{S}et} \quad \text{and} \quad S_{\mathcal{A}} \dashv R_{\mathcal{A}},$$

and we can form

$$\overline{H} = L^{op}_{\mathcal{S}et}S^{op}_{\mathcal{A}}H : \mathcal{D} \to \Pi\mathcal{S}et.$$

By adjointness, for all $X \in \Pi\mathcal{S}et$ there are canonical isomorphisms

$$(X \downarrow \overline{H}) \cong (S_{\mathcal{S}et}X \downarrow S^{op}_{\mathcal{A}}H) \cong (R_{\mathcal{A}}S_{\mathcal{S}et}X \downarrow H),$$

so that $\overline{H}$ must be small-partitioned since H is. Consequently, $\mathrm{colim}\,\overline{H}$ exists in $\Pi\mathcal{S}et$, by 3.5. This is a limit of $\overline{H}^{op}$ in $(\Pi\mathcal{S}et)^{op}$, which is preserved by the right adjoint functor $S_{\mathcal{S}et}$. Since

$$S_{\mathcal{S}et}\overline{H}^{op} = S_{\mathcal{A}}H^{op},$$

we see that a limit of $S_{\mathcal{A}}H^{op}$ exists in $\mathcal{S}et$.

In order to see that $\mathrm{colim}\,H$ exists in $\Pi\mathcal{A}$, according to Lemma 2.5 it is sufficient to show that for every $\alpha \in K = \lim S_{\mathcal{A}}H^{op}$ in $\mathcal{S}et$, the diagram $H_\alpha : \mathcal{D} \to \mathcal{A}$ has a multicolimit in $\mathcal{A}$. In fact, since $\mathcal{A}$ is multitotal, thanks to 3.2 it suffices to prove that H_α is small-partitioned. Hence, given $A \in \mathcal{A}$, we must show that $A \downarrow H_\alpha$ has only a small set of connected components.

Let $O = (O_t)_{t\in T}$ be initial in $\Pi\mathcal{A}$ and form $\overline{A} = A \times O$ in $\Pi\mathcal{A}$. Every object (D, f), with $f : A \to H_\alpha D$, of $(A \downarrow H_\alpha)$ defines an object $(D, \overline{f})$ of $(\overline{A} \downarrow H)$, as follows: for each $i \in S_{\mathcal{A}}HD - \{\alpha_D\}$, let $\overline{f}_i : O_t \to (HD)_i$ be the morphism determined by initiality of O, and for $i = \alpha_D$ let $\overline{f}_i = f$. Since H is small-partitioned, $(\overline{A} \downarrow H)$ has only a set of connected components. Hence, it now suffices to show that, for any pair of objects (D, f), (D', f') in $(A \downarrow H_\alpha)$, any zig-zag of $(\overline{A} \downarrow H)$ between $(D, \overline{f})$, $(D', \overline{f'})$ gives a zig-zag of $(A \downarrow H_\alpha)$ between (D, f), (D', f').

Consider the first step of the zig-zag between $\overline{f}$, $\overline{f'}$, which is given by one of the following commutative triangles:

(2)
$$\begin{tikzcd} & \overline{A} \arrow[dl, "\overline{f}"'] \arrow[dr, "g_1"] & \\ HD \arrow[rr, "Hd_1"] & & HD_1 \end{tikzcd} \qquad \begin{tikzcd} & \overline{A} \arrow[dl, "\overline{f}"'] \arrow[dr, "g_1"] & \\ HD & & HD_1 \arrow[ll, "Hd_1"'] \end{tikzcd}$$

(a) Case $g_1 = Hd_1 \cdot \overline{f}$. Since $(S_{\mathcal{A}}Hd_1)(\alpha_{D_1}) = \alpha_D$, we know that $S_{\mathcal{A}}g_1 = S_{\mathcal{A}}\overline{f} \cdot S_{\mathcal{A}}Hd_1$ takes α_{D_1} to the index of A. Hence, the morphism $(g_1)_{\alpha_{D_1}} : A \to (HD_1)_{\alpha_{D_1}} = H_\alpha D_1$ gives us the first step of the zig-zag between (D, f), (D', f').

(b) Case $\overline{f} = Hd_1 \cdot g_1$. Since $(S_{\mathcal{A}}Hd_1)(\alpha_D) = \alpha_{D_1}$, we know that, again, $S_{\mathcal{A}}g_1$ takes α_{D_1} to the index of A. Hence, also in this case $(g_1)_{\alpha_{D_1}}$ gives the first step of the zig-zag between (D, f), (D', f').

Inductively one finishes the proof of the last claim, so that the proof of the Theorem is complete. □

3.8 Corollary ([18])**.** *Every multitotal category $\mathcal{A}$ is* connectively hypercomplete, *that is, every connected diagram $H : \mathcal{D} \to \mathcal{A}$ for which* $\lim \mathcal{A}(A, H-)$ *exists in $\mathcal{S}et$ for all $A \in \mathcal{A}$, has a limit in $\mathcal{A}$.*

Proof: With 3.7 and 2.2, $\Pi\mathcal{A}$ is total and therefore hypercomplete. Consider a connected diagram $H : \mathcal{D} \to \mathcal{A}$ with the indicated property; equivalently, with the property that there is only a small set of cones $\Delta A \to H$ for every $A \in \mathcal{A}$. Since $\mathcal{D}$ is connected, it is easy to see that there is only a small set of cones $\Delta A \to J_{\mathcal{A}}H$ for every $A \in \Pi\mathcal{A}$, so that $(L_i)_{i\in I} = \lim J_{\mathcal{A}}H$ exists in $\Pi\mathcal{A}$. Again, connectedness of $\mathcal{D}$ determines a unique index i_0 such that $L_{i_0} = \lim H$. □

Corollary. *Every multitotal category $\mathcal{A}$ has equalizers, pullbacks and intersections of* (*arbitrarily large*) *families of monomorphisms.*

3.9. Theorem 3.7 makes it easy to establish the interrelationship between multitotal categories and multisolid functors:

Theorem. *Let $U : \mathcal{A} \to \mathcal{X}$ be a functor. Then:*

(1) *if $\mathcal{X}$ is multitotal and U multisolid, then also $\mathcal{A}$ is multitotal;*
(2) *if $\mathcal{A}$ is multitotal and U faithful and right multiadjoint, then U is multisolid.*

Proof: (1) The hypotheses together with Theorem 3.7 imply that $\Pi\mathcal{X}$ is total and $\Pi U : \Pi\mathcal{A} \to \Pi\mathcal{X}$ is solid, so that $\Pi\mathcal{A}$ is total, see Theorem 2.5. Hence, $\mathcal{A}$ is multitotal.

(2) Using again Theorem 3.7 we see that $\Pi\mathcal{A}$ is total and ΠU is faithful and right adjoint, by hypothesis, so that ΠU is solid by Theorem 2.5. Hence, U is multisolid. □

3.10. If $\mathcal{A}$ has a generator $\mathcal{G}$, one may apply Proposition 2.6 and then Theorem 3.9(1) to the canonical functor $\mathcal{A} \to \mathcal{S}et^{\mathcal{G}}$ to obtain:

Corollary. *Every cowellpowered category with a generator and multicolimits is multitotal.*

In the next section, we study multitotal categories with various types of generators.

4. Multitotal categories with generators

4.1. Recall that a generator $\mathcal{G}$ of $\mathcal{A}$ is *strong* if a morphism $f : A \to B$ in $\mathcal{A}$ is an isomorphism whenever all maps $\mathcal{A}(G, f) : \mathcal{A}(G, A) \to \mathcal{A}(G, B)$ are bijective, $G \in \mathcal{G}$; in other words, if the canonical functor $\mathcal{A} \to \mathcal{S}et^{\mathcal{G}}$ is conservative. In order to relate a (strong) generator of a category $\mathcal{A}$ to the category $\Pi\mathcal{A}$, denote by O an initial object of $\Pi\mathcal{A}$ (i.e., a multi-initial object of $\mathcal{A}$), provided that it exists. For every $A \in \mathcal{A}$, we put $\overline{A} = A \times O$ (product in $\Pi\mathcal{A}$). For $\mathcal{G}$ a set of objects in $\mathcal{A}$, let

$$\overline{\mathcal{G}} = \{\overline{G} \mid G \in \mathcal{G}\} \cup \{O \times O\};$$

and for $\mathcal{H}$ a set of objects in $\Pi\mathcal{A}$, let $\mathcal{H}^1$ be the set of objects of $\mathcal{A}$ which are components of objects in $\mathcal{H}$. Then we have the following:

Lemma. (1) *If $\mathcal{G}$ is a (strong) generator of $\mathcal{A}$ and if $\mathcal{A}$ has a multi-initial object, then $\overline{\mathcal{G}}$ is a (strong) generator of $\Pi\mathcal{A}$.*

(2) *If $\mathcal{H}$ is a (strong) generator of $\Pi\mathcal{A}$, then $\mathcal{H}^1$ is a (strong) generator of $\mathcal{A}$.*

Proof: (1) If $\mathcal{G}$ is a generator of $\mathcal{A}$ and if $p, q : A \to B$ are distinct morphisms of $\Pi\mathcal{A}$, then there is $k \in SB$ such that either $Sp(k) \neq Sq(k)$, or $Sp(k) = l = Sq(k)$ and $p_k \neq q_k : A_l \to B_k$. In the former case, consider $h : O \times O \to A$ with $Sh(Sp(k)) \neq Sh(Sq(k))$, then $ph \neq qh$; in the latter case, choose $h_0 : G \to A_l$ with $p_k h_0 \neq q_k h_0$ and $G \in \mathcal{G}$, and let $h : \overline{G} \to A$ be a morphism with l-component h_0, then $ph \neq qh$.

If $\mathcal{G}$ is a strong generator of $\mathcal{A}$ and if $f : A \to B$ is a morphism in $\Pi\mathcal{A}$ such that $\Pi\mathcal{A}(G, f)$ is bijective for all $G \in \overline{\mathcal{G}}$, then (a) Sf is bijective since, on the one hand, the bijectivity of $\Pi\mathcal{A}(O \times O, f)$ is clearly equivalent to the bijectivity of $\mathcal{S}et^{op}(2, Sf)$, and, on the other hand, $\mathcal{S}et^{op}(2, Sf)$ is surjective (injective) iff Sf is injective (surjective, respectively); and (b) each component f_k of f is an isomorphism since, thanks to the morphisms from $G \times O$, $G \in \mathcal{G}$, to B, $\mathcal{A}(G, f_k)$ is bijective. Thus f is an isomorphism, hence, $\overline{\mathcal{G}}$ is a strong generator.

(2) If $\mathcal{H}$ is a generator of $\Pi\mathcal{A}$ and $p, q : A \to B$ are distinct morphisms of $\mathcal{A}$, choose $h : H \to A$ with $ph \neq qh$ in $\Pi\mathcal{A}$. The unique component $h_1 : H_1 \to A$ of h then fulfills $ph_1 \neq qh_1$ and $H_1 \in \mathcal{H}^1$. If $\mathcal{H}$ is a strong generator of $\Pi\mathcal{A}$, it follows easily that also $\mathcal{H}^1$ is a strong generator of $\mathcal{A}$, from the observation that, given $H \in \mathcal{H}$ and a morphism $f : A \to B$ in $\mathcal{A}$, $\Pi\mathcal{A}(H, f)$ is essentially the map $\Pi_{i \in SH}\mathcal{A}(H_i, f)$. □

4.2. We use the notation of [GU] and denote by $\widetilde{\mathcal{S}et}$ any small-indexed discrete power of $\mathcal{S}et$.

Theorem. *The following conditions on a category $\mathcal{A}$ are equivalent:*

(i) *$\mathcal{A}$ is multitotal and has a generator,*
(ii) *ΠA is total and has a generator,*
(iii) *$\Pi\mathcal{A}$ admits a solid functor into $\widetilde{\mathcal{S}et}$,*
(iv) *$\mathcal{A}$ admits a multisolid functor into $\widetilde{\mathcal{S}et}$.*

Proof: (i)⇒(ii) by 3.7 and 4.1(1). (ii)⇒(iii) by (2) of Theorem 2.3. (iii)⇒(iv): $\mathcal{A}$ is always multireflective in $\Pi\mathcal{A}$; hence, composition of the solid functor $\Pi\mathcal{A} \to \widetilde{\mathcal{S}et}$ with $J_{\mathcal{A}}$ gives a multisolid functor $\mathcal{A} \to \widetilde{\mathcal{S}et}$. (iv)⇒(i): $\mathcal{A}$ is multitotal by 3.9(1); furthermore, since $\Pi\mathcal{A}$ is solid over $\Pi\widetilde{\mathcal{S}et}$ and since $\Pi\widetilde{\mathcal{S}et}$ has a generator by 4.1(1), also $\Pi\mathcal{A}$ and then $\mathcal{A}$ has a generator, by 4.1(2). □

4.3 Corollary. *The following conditions on a category $\mathcal{A}$ are equivalent:*

(i) *$\mathcal{A}$ is multitotal and has a strong generator,*
(ii) *$\Pi\mathcal{A}$ is total and has a strong generator,*
(iii) *$\Pi\mathcal{A}$ admits a solid, conservative functor into $\widetilde{\mathcal{S}et}$,*
(iv) *$\mathcal{A}$ admits a multisolid, conservative functor into $\widetilde{\mathcal{S}et}$.*

Proof: Thanks to the strong generator part of Lemma 4.1, one can mimic the proof of 4.2. □

4.4. A generator $\mathcal{G}$ of $\mathcal{A}$ is *regular* if the canonical functor $\mathcal{A} \to \mathcal{S}et^{\mathcal{G}}$ reflects regular epimorphisms; equivalently, $f : A \to B$ in $\mathcal{A}$ is a regular epimorphism whenever every map $\mathcal{A}(G, f)$, $G \in \mathcal{G}$, is surjective.

Remark. Every cocomplete category with a regular generator is total, see [4]. These are, by [3], precisely the reflective subcategories of monadic categories over $\widetilde{\mathcal{S}et}$.

Lemma.

(1) *Let $\mathcal{A}$ have a multi-initial object. Then:*
 a. *A morphism $f : A \to B$ in $\Pi\mathcal{A}$ is a regular epimorphism if its components are regular epimorphisms in $\mathcal{A}$ and if Sf is injective; these conditions are also necessary whenever $\mathcal{A}$ has a terminal object.*
 b. *If $\mathcal{G}$ is a regular generator of $\mathcal{A}$, then $\overline{\mathcal{G}}$ is a regular generator of $\Pi\mathcal{A}$.*
(2) *If $\mathcal{H}$ is a regular generator of $\Pi\mathcal{A}$, then $\mathcal{H}^1$ is a regular generator of $\mathcal{A}$.*

Proof: (1)a. Put $\varphi = Sf$. By hypothesis, each f_j is a coequalizer of a pair $p_j, q_j : K_j \to A_{\varphi(j)}$ in $\mathcal{A}$. Put $I = SA$, $J = SB$, and $K = (K_j)_{j\in J}$ and define $p, q : K \times O \times O \to A$ by letting Sp map $I - \varphi(J)$ into the first copy of SO and Sq into the second one. Then f is a coequalizer of p, q in $\Pi\mathcal{A}$. Conversely, assuming

f to be the coequalizer of some pair $g, h : L \to A$ in $\Pi\mathcal{A}$, we first consider j_1, $j_2 \in J$ with $\varphi(j_1) = \varphi(j_2) = i$. With any commutative square

$$\begin{tikzcd} & B_{j_1} \arrow[dr] & \\ A_i \arrow[ur, "f_{j_1}"] \arrow[dr, "f_{j_2}"'] & & C \\ & B_{j_2} \arrow[ur] & \end{tikzcd}$$

in $\mathcal{A}$ (which certainly exists when $\mathcal{A}$ has a terminal object) one obtains morphisms s, $t : B \to C$ in $\Pi\mathcal{A}$ with $sf = tf$, hence $s = t$ and then $j_1 = j_2$. Injectivity of φ now gives that each f_j is a coequalizer of $g_{\varphi(j)}$, $h_{\varphi(j)}$ in $\mathcal{A}$.

(1)b. and (2) follow from an easy analysis of the proof of 4.1. $\square$

Remark. For categories $\mathcal{A}$ which do not have a terminal object the converse implication of (1)a. above is false, in general: suppose that a pair $p, q : A \to B$ in $\mathcal{A}$ has a multicoequalizer $c_k : B \to C_k (k \in K)$ in $\mathcal{A}$. Then the corresponding coequalizer $c : B \to C = (C_k)_{k \in K}$ in $\Pi\mathcal{A}$ is a regular epimorphism and Sc is a constant function.

4.5 Theorem. *The following conditions on a category $\mathcal{A}$ are equivalent:*

(i) *$\mathcal{A}$ is multitotal and has a regular generator;*
(ii) *$\mathcal{A}$ is multicocomplete and has a regular generator;*
(iii) *$\Pi\mathcal{A}$ is cocomplete and has a regular generator;*
(iv) *$\Pi\mathcal{A}$ admits a solid functor into $\widetilde{\mathcal{S}et}$ which reflects regular epimorphisms;*
(v) *$\mathcal{A}$ is equivalent to a multireflective subcategory of a monadic category over $\widetilde{\mathcal{S}et}$;*
(vi) *$\mathcal{A}$ admits a multisolid functor into $\widetilde{\mathcal{S}et}$ which reflects regular epimorphisms.*

Proof: (i)⇒(ii) is trivial, and (ii)⇒(iii) follows from 4.4(2). (iii)⇒(iv): In the presence of a regular generator, the cocomplete category $\Pi\mathcal{A}$ is actually total (see Remark 4.4), so that 2.3(2) becomes applicable. (iv)⇒(v): Since $\Pi\mathcal{A}$ has coequalizers, it is equivalent to a full reflective subcategory of a monadic category over $\widetilde{\mathcal{S}et}$ (see also [3]); statement (v) follows since $\mathcal{A}$ is multireflective in $\Pi\mathcal{A}$. (v)⇒(vi): The forgetful functor of a monadic category over $\widetilde{\mathcal{S}et}$ is solid and reflects regular epimorphisms. One easily shows that the latter statement remains true if we restrict the functor to a multireflective subcategory. (vi)⇒(i): In order to see that $\mathcal{A}$ has a regular generator, one uses 4.4(1) to show that when the multisolid functor $U : \mathcal{A} \to \widetilde{\mathcal{S}et}$ reflects regular epimorphisms, the same is true for the solid functor $\Pi U : \Pi\mathcal{A} \to \Pi\widetilde{\mathcal{S}et}$, so that the existence of a regular generator in $\Pi\widetilde{\mathcal{S}et}$ implies the same for $\Pi\mathcal{A}$; now one applies 4.4(2) again. $\square$

5. Product completions and dense generators

5.1. Recall that a generator $\mathcal{G}$ of $\mathcal{A}$ is said to be *dense* provided that the full subcategory of $\mathcal{A}$ generated by $\mathcal{G}$ (which we denote also by $\mathcal{G}$) is dense in $\mathcal{A}$, that is, every object A of $\mathcal{A}$ is a canonical colimit of the forgetful functor $D_A : (\mathcal{G} \downarrow A) \to \mathcal{A}$.

In the previous section, we related several types of generators of a category $\mathcal{A}$ to the category $\Pi\mathcal{A}$, and this enabled us to obtain results on multitotal categories analogous to the ones of [3] for total categories. Similarly to (2) of Lemma 4.1, in the case of dense generators, we have the following:

Lemma. *If $\mathcal{H}$ is a dense generator of $\Pi\mathcal{A}$, then $\mathcal{H}^1$ is a dense generator of $\mathcal{A}$.*

Proof: Let A be an object in $\mathcal{A}$ and let $D^1 : (\mathcal{H}^1 \downarrow A) \to \mathcal{A}$ and $D : (\mathcal{H} \downarrow A) \to \Pi\mathcal{A}$ be the corresponding canonical diagrams into $\mathcal{A}$ and $\Pi\mathcal{A}$, respectively. If $\gamma : D^1 \to \Delta B$ is a cocone for D^1, define $\overline{\gamma} : D \to \Delta B$ by considering, for each $\Pi\mathcal{A}$-morphism $h : H \to A$ with $H \in \mathcal{H}$, the morphism $\overline{\gamma}_h : H \to B$ such that $\big((\overline{\gamma}_h)_* : H_{(S\overline{\gamma}_h)(*)} \to B\big) = \big(\gamma_{h_*} : H_{(Sh)(*)} \to B\big)$. It is easy to show that $\overline{\gamma}$ is a cocone for D. Thus, there exists a unique morphism $w : A \to B$ such that $w \cdot h = \overline{\gamma}_h$ for every $h : H \to A$ with $H \in \mathcal{H}$. It is now easily verified that $w : A \to B$ is also the unique morphism which fulfills the equality $w \cdot g = \gamma_g$ for every $g : G \to A$ with $G \in \mathcal{H}^1$. $\square$

5.2. However, for dense generators there is no analogous statement to (1) of Lemma 4.1. More precisely, we shall show next that the existence of a dense generator of $\Pi\mathcal{A}$ in the presence of a dense generator of $\mathcal{A}$ depends on the following large-cardinal axiom:

(M) There do not exist arbitrarily large measurable cardinals.

The statement (M) means that we can find a cardinal ρ such that no cardinal larger or equal to ρ is measurable, or, equivalently, every ultrafilter closed under intersections of less then ρ members contains a singleton set (see A.5 in [2]).

Remark. Recall that a functor $F : \mathcal{A} \to \mathcal{B}$ is called *coinitial* (the dual of *cofinal*) provided that for every object B of $\mathcal{B}$ the comma-category $(F \downarrow B)$ is connected; this implies that for every diagram $G : \mathcal{B} \to \mathcal{X}$ we have $\operatorname{colim} G = \operatorname{colim} G \cdot F$ (more precisely, if $(c_B : GB \to C)_{B \in Ob(\mathcal{B})}$ is a colimit of G, then $(c_{FA} : GFA \to C)_{A \in Ob(\mathcal{A})}$ is a colimit of $G \cdot F$).

Theorem. *The following assertions are equivalent:*

(i) *$\Pi\mathcal{A}$ has a dense generator, for every category $\mathcal{A}$ with a dense generator and a multi-initial object;*
(ii) *$\Pi\mathbb{I}$ has a dense generator (with $\mathbb{I}$ the terminal category);*
(iii) *the set-theoretic axiom (M) holds.*

Proof: Since $\Pi\mathbb{I} = \mathcal{S}et^{op}$, we have (ii)$\Leftrightarrow$(iii); this was proved by J. Isbell in [10], by showing that, for any cardinal ρ, the sets of cardinality less than ρ form a codense cogenerator of $\mathcal{S}et$ if and only if no cardinal $\geq \rho$ is measurable. As (i)$\Rightarrow$(ii) is trivial, (iii)$\Rightarrow$(i) remains to be shown. Let $\mathcal{G}$ be a dense generator of $\mathcal{A}$, and let $O = (O_t)_{t\in T}$ be a multi-initial object of $\mathcal{A}$. By hypothesis, there is a cardinal number ρ such that $\hat{\rho} = \{I \mid \operatorname{card} I < \rho\}$ is a codense in $\mathcal{S}et$. Without loss of generality, we may assume $O_t \in \mathcal{G}$ for all $t \in T$ and $\rho > \max\{\operatorname{card} T, \aleph_0\}$. We claim that

$$\mathcal{G}^\rho = \{G \in \Pi\mathcal{A} \mid \operatorname{card}(SG) < \rho \ \text{ and } \ G_i \in \mathcal{G} \ \text{ for all } \ i \in SG\}$$

is dense in $\Pi\mathcal{A}$. Hence, for every $A \in \Pi\mathcal{A}$, we must show that A is a colimit of the canonical diagram $D_A : (\mathcal{G}^\rho \downarrow A) \to \Pi\mathcal{A}$. For that we use Lemma 2.5 and first show that SD_A^{op} has limit SA in $\mathcal{S}et$.

In fact, we have a commutative diagram

$$\begin{tikzcd}
(\mathcal{G}^\rho \downarrow A)^{op} \arrow[r, "D_A^{op}"] \arrow[d, "S_A"'] & (\Pi\mathcal{A})^{op} \arrow[d, "S"] \\
(SA \downarrow \hat{\rho}) \arrow[r, "D_{SA}"] & \mathcal{S}et
\end{tikzcd}$$

with S_A induced by S. Since $\lim D_{SA} = SA$ (canonically), it suffices to show that S_A is coinitial. Hence, for every object $(J, \varphi : SA \to J)$ in $(SA \downarrow \hat{\rho})$ we must show that the comma category $(S_A \downarrow (J, \varphi))$ is connected. In fact, we can define a morphism $c_\varphi : O^J \to A$ by

$$(c_\varphi)_k : O^J \xrightarrow{p_{\varphi(k)}} O \xrightarrow{!_{A_k}} A_k$$

for every $k \in SA$ (where $p_{\varphi(k)}$ is a projection and $!_{A_k}$ is determined by initiality of O in $\Pi\mathcal{A}$). Hence, we have an object $(O^J, c_\varphi) \in (\mathcal{G}^\rho \downarrow A)$, and the diagram

$$\begin{tikzcd}
SO^J = T \times J \arrow[rr, "\pi_2"] & & J \\
& SA \arrow[ul, "Sc_\varphi"] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

commutes. This means that $((O^J, c_\varphi), \pi_2)$ is an object of $(S_A \downarrow (J, \varphi))$; in fact, it is weakly initial in that category, as we shall show next. Given any object $((G, f), \psi)$ in $(S_A \downarrow (J, \varphi))$, so that $\psi : S_A(G, f) \to (J, \varphi)$ is a morphism in $(SA \downarrow \hat{\rho})$, we claim that the morphism $d_\psi : O^J \to G$ with

$$(d_\psi)_i : O^J \xrightarrow{p_{\psi(i)}} O \xrightarrow{!_{G_i}} G_i$$

for all $i \in SG$ makes the following diagrams commute:

$$\begin{tikzcd}
SG \arrow[rr, "Sd_\psi"] \arrow[dr, "\psi"] & & SO^J \arrow[dl, "\pi_2"'] \\
& J & \\
& SA \arrow[uul, "Sf"] \arrow[u, "\varphi"'] \arrow[uur, "Sc_\varphi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[dr, "f"'] & & O^J \arrow[ll, "d_\psi"'] \arrow[dl, "c_\varphi"] \\
& A &
\end{tikzcd}$$

This is obvious for the diagram on the left, while initiality of O in $\Pi\mathcal{A}$ implies commutativity of the right-hand diagram. Consequently we have a morphism $d_\psi : ((O^J, c_\varphi), \pi_2) \to ((G, f), \psi)$ in $(S_A \downarrow (J, \varphi))$. This concludes the proof of $\lim SD_A^{op} = SA$.

According to Lemma 2.5, it now suffices to show that for every $k \in SA$, A_k is the canonical colimit of the diagram $D_k : (\mathcal{G}^\rho \downarrow A) \to \mathcal{A}$, $(G, f) \mapsto G_{(Sf)(k)}$. But D_k factors as

$$\begin{tikzcd}
(\mathcal{G}^\rho \downarrow A) \arrow[rr, "D_k"] \arrow[dr, "F_k"'] & & \mathcal{A} \\
& (\mathcal{G} \downarrow A_k) \arrow[ur, "D_{A_k}"'] &
\end{tikzcd}$$

with $F_k : (G, f) \mapsto (G_{(Sf)(k)}, f_k)$. Since, by hypothesis, A_k is the canonical colimit of D_{A_k}, it suffices to show that F_k is cofinal in order to finish the proof. We show that, given any object $(H, g) \in (\mathcal{G} \downarrow A_k)$, the comma category $((H, g) \downarrow F_k)$ is connected. Let $g^{(k)} : H \times O \to A$ be the $\Pi\mathcal{A}$-morphism with $(g^{(k)})_k = g : H \to A_k$ and $(g^k)_l = !_{A_l} : O \to A_l$ for all $l \neq k$. Then $1_H : (H, g) \to F_k(H \times O, g^{(k)})$ is a morphism in $(\mathcal{G} \downarrow A_k)$, so $((H \times O, g^{(k)}), 1_H)$ is an object in $((H, g) \downarrow F_k)$. Consider another object $((G, f), h)$ in that category; denote $i = (Sf)(k)$. We analyse two cases:

(a) $(Sf)^{-1}(\{i\}) = \{k\}$. Thus one can define a $\Pi\mathcal{A}$-morphism $h^{(k)} : H \times O \to G$ as above, and since $f_k \cdot h = g$ one has $f \cdot h^{(k)} = g^{(k)}$. Consequently, $h^{(k)} : (H \times O, g^{(k)}) \to (G, f)$ is a morphism in $(\mathcal{G}^\rho \downarrow A)$ with $f \cdot F_k h^{(k)} = g$; hence, we have $h^{(k)} : ((H \times O, g^{(k)}), 1_H) \to ((G, f), h)$ in $((H, g) \downarrow F_k)$.

(b) $(Sf)^{-1}(\{i\}) \neq \{k\}$. We show that $((G, f), h)$ belongs to the same connected component of the category $((H, g) \downarrow F_k)$ as an object $((\overline{G}, \overline{f}), \overline{h})$ which is of type (a), so the proof will be complete. Put $\overline{G} = G \times G_i$ and let $\overline{f} : G \times G_i \to A$ be the obvious morphism such that $\overline{f}_l = (f \cdot \pi_1)_l$ for all $l \neq k$, where π_1 is the first projection of $G \times G_i$ and $(S\overline{f})^{-1}(i) = \{k\}$, and put $\overline{h} = h$. Since $\overline{f}_k \cdot \overline{h} = f_k \cdot h = g$, $((\overline{G}, \overline{f}), \overline{h})$ is an object in $((H, g) \downarrow F_k)$. Let $t : G \to \overline{G}$ be the $\Pi\mathcal{A}$-morphism such that St identifies the two copies of i of $S\overline{G}$, and all components of t are identities.

Then we have that $\overline{f} \cdot t = f$, so $t : (G, f) \to (\overline{G}, \overline{f})$ is a morphism in $(\mathcal{G}^\rho \downarrow A)$; furthermore, $F_k t \cdot h = \overline{h}$, thus we have a morphism $t : ((G, f), h) \to ((\overline{G}, \overline{f}), \overline{h})$ in $((H, g) \downarrow F_k)$. □

5.3 Remark. The dense generator of $\Pi\mathcal{A}$ obtained in 5.2 from a dense generator of a category $\mathcal{A}$ with a multi-initial object (in the presence of the axiom (M)) is distinct from the set $\overline{\mathcal{G}} = \{G \times O \mid G \in \mathcal{G}\} \cup \{O \times O\}$ used in Section 4. (We have seen there that $\overline{\mathcal{G}}$ is a (strong or regular) generator of $\Pi\mathcal{A}$ whenever $\mathcal{A}$ has a multi-initial object O and a (strong or regular, respectively) generator $\mathcal{G}$.) Actually, we can prove that the category $\Pi\mathcal{S}et$ does not have a dense generator of the form $1 \times O$ plus O^J, $J \in \mathcal{J}$ (assuming that $\mathcal{J}$ is dense in $\mathcal{S}et^{op}$). Consider $A = (X_1, X_2)$ where $X_1 = X_2 = 1$. The canonical diagram D of A consists of

(a) some copies of O^J;
(b) four copies of $1 \times O$ indexed by the obvious four morphisms $f : 1 \times O \to A$ determined by $Sf : \{1, 2\} \to \{1, 2\}$ uniquely.

Each of the four copies of $1 \times O$ is connected with the objects of type (a) by morphisms $O^J \to 1 \times O$, but there is no morphism in the opposite direction. The colimit of the canonical diagram is, by Lemma 2.5,

$$\operatorname{colim} D = (\operatorname{colim} D_\varphi)_{\varphi \in \lim SD}.$$

If $\mathcal{J}$ were dense in $\mathcal{S}et^{op}$, we would have, for the subdiagram D' of D of all objects O^J, $\lim SD' = \{1, 2\}$ (canonically). Thanks to the morphisms $O^J \to 1 \times O$, we have $\lim SD = \lim SD'$. It is easy to see that the choice of index 1 gives a diagram with two distinct copies of 1 without any morphism between them; thus $\operatorname{colim} D_1$ will have two elements. Analogously, $\operatorname{colim} D_2$ has two elements. Hence $(A_1, A_2) \neq (\operatorname{colim} D_1, \operatorname{colim} D_2)$.

5.4. Although the existence of a dense generator of $\mathcal{A}$ does not guarantee the same for $\Pi\mathcal{A}$, the relationship between dense generators and multitotal categories is similar to the one for totality:

Theorem. *The following statements are equivalent for a category $\mathcal{A}$:*

(i) *$\mathcal{A}$ is multitotal with a dense generator;*
(ii) *$\mathcal{A}$ is multisolid over $\widetilde{\mathcal{S}et}$ and has a dense generator;*
(iii) *$\mathcal{A}$ is equivalent to a full multireflective subcategory of a ranked monadic category over $\widetilde{\mathcal{S}et}$;*
(iv) *$\mathcal{A}$ is a full multireflective subcategory of a locally presentable category;*
(v) *$\mathcal{A}$ is a full multireflective subcategory of a Grothendieck topos.*

Proof: One proceeds similarly as in the total case, see [3]. □

Remark. From the Theorem above and from 6.16 of [2], it follows that multitotal categories with dense generators are precisely the locally multipresentable categories, provided that the set-theoretic Vopěnka Principle holds.

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