Zeroes of the Bergman kernel of Hartogs domains

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Abstract. We exhibit a class of bounded, strongly convex Hartogs domains with real-analytic boundary which are not Lu Qi-Keng, i.e. whose Bergman kernel function has a zero.

Keywords: Lu Qi-Keng conjecture, Hartogs domain, Bergman kernel

Classification: Primary 32A07, 32H10

Let Ω be a domain in $\mathbf{C}^n$ and $K_\Omega(z,w)$ its Bergman kernel. It was conjectured by Lu Qi-Keng in [Lu] that if Ω is simply connected, then $K_\Omega(z,w) \neq 0$ for all z and w. This conjecture was shown to be false by Skwarczynski [Skw] who exhibited an unbounded Reinhardt domain in $\mathbf{C}^2$ for which $K_\Omega(z,w)$ has a zero. Later Boas [B1] obtained even a bounded, strongly pseudoconvex counterexample to the Lu Qi-Keng conjecture and showed that the set of domains whose Bergman kernel function has a zero is dense in various topologies [B2], but a possibility still remained that $K_\Omega(z,w)$ is zero-free for all convex domains. Recently Boas, Fu and Straube [BFS] showed that the Bergman kernel function of the domain in $\mathbf{C}^3$ defined by $|z_1|+|z_2|+|z_3|<1$ has a zero. By exhaustion it follows that when $n \geq 3$, there exist bounded, strongly convex domains with real-analytic boundary in $\mathbf{C}^n$ whose Bergman kernel function has a zero. Subsequently Pflug and Youssfi [PY] used the "minimal ball" studied in [OPY] to construct a concrete example of smooth, bounded, strongly convex, algebraic domain in $\mathbf{C}^n$ for any $n \geq 4$ for which the Lu Qi-Keng conjecture fails.

The aim of this short note is to call attention to the fact that there exists a large family of strongly convex domains in $\mathbf{C}^n$, bounded and with smooth (or even real-analytic) boundary, for which the Lu Qi-Keng conjecture fails. In fact, it turns out that in some sense such domains are generic in the class of smoothly bounded, strongly convex domains with a certain circular symmetry. The result is a simple consequence of an earlier result of the author's on the asymptotics of weighted Bergman kernels [E1] and a formula of Ligocka [Lig]. Unfortunately, it gives no information about the dimension n.

More precisely, we will consider the Hartogs domains

$$\widetilde{\Omega}_m = \{(z,t) \in \Omega \times \mathbf{C}^m : \ \|t\|^2 < F(z)\}$$

The research was supported by GA AV ČR grant No. A1019701.

where F is a positive continuous function on some domain $\Omega \subset \mathbf{C}^d$ and $m = 1, 2, \dots$. It is well-known that $\widetilde{\Omega}_m$ is pseudoconvex if and only if Ω is pseudoconvex and $-\log F$ is plurisubharmonic, and convex if and only if Ω is convex and F is concave. Further, it is not difficult to see that $\widetilde{\Omega}_m$ is smoothly (or real-analytically) bounded if Ω is smoothly (real-analytically) bounded and $F \in C^\infty(\overline{\Omega})$ ($F \in C^\omega(\overline{\Omega})$), $F = 0$ on $\partial\Omega$ and $\nabla F \neq 0$ on $\partial\Omega$ (i.e. $-F$ is a smooth resp. a real-analytic defining function for Ω), and in that case it is strongly convex if and only if F is strongly concave.

Let us say that F has *property (K)* if there exists a function $\tilde{F}(z, w)$ on $\Omega \times \Omega$ such that

(i) $\tilde{F}(z, w)$ is holomorphic in z and conjugate-holomorphic in w,
(ii) $\tilde{F}(z, z) = F(z)$,
(iii) $|\tilde{F}(z, w)|^2 \geq \tilde{F}(z, z)\tilde{F}(w, w)$ (the "reverse Schwarz" inequality).

Observe that any function having property (K) is necessarily real-analytic on Ω, and also (iii) and the positivity of F imply that the extension $\tilde{F}$ does not vanish on $\Omega \times \Omega$. Our result is the following.

Theorem. *Let Ω be a bounded domain in $\mathbf{C}^d$, F a bounded positive continuous function on Ω such that $\log F$ is concave. Assume that there exists a sequence of integers $0 < m_1 < m_2 < \dots$ such that for each m_j, $K_{\widetilde{\Omega}_{m_j}}((z, 0), (w, 0)) \neq 0$ $\forall\, z, w \in \Omega$. Then F has property* (K).

Corollary. *Let Ω be a bounded strongly convex domain in $\mathbf{C}^d$ with C^∞ boundary and $-F$ a strongly convex C^∞ defining function for Ω such that F does not have property* (K). *Then there exists an integer m_0 such that $\forall\, m \geq m_0$, $\widetilde{\Omega}_m$ is a bounded, strongly convex domain with C^∞ boundary whose Bergman kernel function has a zero. The same assertion holds with C^∞ replaced by C^ω.*

Observe that a generic C^∞ function is not real-analytic, and, likewise, a generic real-analytic function on Ω fails to have a sesqui-holomorphic extension to all of $\Omega \times \Omega$ (even though such extension always exists in a neighbourhood of the diagonal, by the definition of real-analyticity), i.e. to satisfy the conditions (i) and (ii) above. (Indeed, after making the change of coordinates $z = u + iv$, $w = \overline{u} + i\overline{v}$, the domain $\Omega \times \Omega$ gets transformed into some other domain $U \subset \mathbf{C}^{2d}$, its diagonal into $U \cap \mathbf{R}^{2d}$, and the assertion becomes apparent; cf. Example 2 below.) Thus the functions F to which the last Corollary applies are generic among the strongly concave, C^∞- (resp. C^ω-) smooth positively signed defining functions for Ω.

Proof of the Theorem: According to [Lig, Proposition 0] (cf. also [E2, Proposition 0], and [BFS, Section 2]),

$$K_{\widetilde{\Omega}_m}((z, t), (w, s)) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!\pi^m} K_{\Omega, F^{m+k}}(z, w) \langle t, s \rangle^k$$

where $K_{\Omega,F^{m+k}}$ stands for the Bergman kernel on Ω with respect to the weight $F(z)^{m+k}$, and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbf{C}^m$. In particular,

$$K_{\widetilde{\Omega}_m}((z,0),(w,0)) = \frac{m!}{\pi^m} K_{\Omega,F^m}(z,w).$$

Our hypothesis therefore implies that

$$K_{\Omega,F^{m_j}}(z,w) \neq 0 \qquad \forall\, z,w \in \Omega \quad \forall\, j = 1,2,\dots\ .$$

Note that in view of the boundedness of F and Ω, the function constant 1 belongs to the weighted Bergman spaces $L^2_{\mathrm{hol}}(\Omega, F^\alpha\, d\lambda)$ for any $\alpha > 0$ ($d\lambda$ is the Lebesgue measure). By [E1, Theorem A and Theorem C] (with $G \equiv 1$ and $U = \Omega$), the assertion follows. □

Proof of the Corollary: Immediate from the Theorem, the above remarks concerning (strong) convexity and C^∞- (resp. C^ω-) boundedness of $\widetilde{\Omega}_m$, and the elementary fact that $\log F$ is (strongly) concave whenever F is. □

Example 1. Let f be a strongly convex smooth function on $\mathbf{C}^d$ which satisfies $\lim_{|z|\to\infty} |f(z)| = +\infty$ and which is not real-analytic at some point z_0. Let $c > f(z_0)$ and take $\Omega = \{z:\ f(z) < c\}$ and $F(z) = c - f(z)$. As F is not real-analytic at z_0, it cannot have property (K).

Example 2. Let f be a function holomorphic in a neighbourhood of the interval $[0,1]$ in the complex plane, with $f' < 0, f'' < 0$ on $[0,1]$ and $f(1) = 0$, which cannot be extended holomorphically to the whole unit disc $\mathbf{D}$. (For instance, $f(x) = (\frac{2}{3} - \frac{2}{2x+1}) + 5(1-x)$.) Take $\Omega = \mathbf{D}$, $F(z) = f(|z|^2)$. Then the only candidate for an $\tilde{F}$ satisfying (i) and (ii) is $\tilde{F}(z,w) = f(z\overline{w})$, which however is not defined on all of $\mathbf{D}\times\mathbf{D}$. Hence, F is real-analytic and does not have property (K).

Example 3. Let $\Omega = \mathbf{D}$ and $F(z) = f(|z|^2)$ where $f(x) = (x-1)(x+\frac{3}{4})(x-\frac{11}{4})$. This time $\tilde{F}(z,w) = f(z\overline{w})$ is defined on all of $\Omega\times\Omega$, but (iii) fails since $f(-\frac{3}{4}) = 0$. Consequently, F is a C^ω function on $\mathbf{D}$, even possessing a sesqui-holomorphic extension to $\mathbf{D}\times\mathbf{D}$, which does not have property (K).

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(*Received* May 10, 1999)