On n-in-countable bases

S.A. Peregudov

Abstract. Some results concerning spaces with countably weakly uniform bases are generalized for spaces with n-in-countable ones.

Keywords: weakly uniform base, n-in-countable base, countably compact space, separable space

Classification: Primary 54A25; Secondary 54D30

All spaces in the paper are assumed to be T_1. If τ and λ are cardinals, then one says that a family $\mathcal{B}$ of sets is τ*-in-*λ if for every set A of cardinality τ, $A \subset B$ holds for no more than λ members $B \in \mathcal{B}$. τ*-in-*$< \lambda$ is defined similarly. One says that a family is τ*-in-countable* in place of τ*-in-*ω and that a family is τ*-in-finite* in place of τ*-in-*$< \omega$. A 2*-in-finite* family is called *weakly uniform* and a 2*-in-countable* family is also called *countably weakly uniform*.

We are going to extend results of paper [6] on spaces with *countably weakly uniform bases* to spaces with n*-in-countable bases*, where n is a natural number. A few results concern n*-in-finite bases*.

Lemma 1. *Let $n \geq 1$ be a natural number. An n-in-countable open family in a separable space is countable.*

Proof: It is evident. $\square$

Theorem 1. *Let $n \geq 1$ be a natural number. A regular countably compact space with an n-in-countable T_1-separating open cover is metrizable.*

Proof: Let X be a regular countably compact space and let $\mathcal{B}$ be its n-in-countable T_1-separating open cover. Denote by $\mathcal{L}(X)$ the set of all nonisolated points of X. It is evident that if $x \in \mathcal{L}(X)$ then $ord(x, \mathcal{B}) \leq \min\{|A| : x \textit{ is an accumulation point of } A\}$. Let us consider the set $Z = \{x \in \mathcal{L}(X) : ord(x, \mathcal{B}) \leq \omega\}$. Because X is regular countably compact, $\bar{Z} = \mathcal{L}(X)$. Now, by analogy with the proof of Miscenco's theorem [3] with using Lemma 1 one can prove that there exists a countable set $Y \subset Z$ such that $\bar{Y} = \mathcal{L}(X)$. Hence $|\{B \in \mathcal{B} : B \cap \mathcal{L}(X) \neq \emptyset\}| \leq \omega$. We can assume that if $B \in \mathcal{B}$ and $B \cap \mathcal{L}(X) = \emptyset$ then B is an one-point set. Hence $\mathcal{B}$ is countable. Therefore X is metrizable. $\square$

Corollary 1 ([1])**.** *Let $n \geq 1$ be a natural number. A regular countably compact space with an n-in-countable base is metrizable.*

Later on we will denote by $\mathcal{I}(X)$ the set of all isolated points of a space X.

Lemma 2. *Let $n \geq 1$ be a natural number. Let X be a space with an n-in-countable base $\mathcal{B}$ and let $|\mathcal{I}(X)| = \tau$, where τ is an uncountable cardinal. Then X has an n-in-countable base $\mathcal{B}^\star$ such that for each $x \in \mathcal{I}(X)$, $ord(x, \mathcal{B}^\star) < \tau$.*

Proof: Let $\mathcal{I}(X) = \{x_\xi : \xi < \tau\}$. Let us define $\mathcal{B}_0 = \{B \in \mathcal{B} : x_0 \in B\}$. Let $\eta < \tau$. Assume that $\mathcal{B}_\xi$ are defined for each $\xi < \eta$. Then we define $\mathcal{B}_\eta = \{B \in \mathcal{B} : x_\eta \in B, B \notin \mathcal{B}_\xi$ under $\xi < \eta\}$. Now put $\mathcal{B}^1_\xi = \{B \setminus \{x_\xi\} : B \in \mathcal{B}_\xi\}$ for each $\xi < \tau$ and $\mathcal{B}^1 = \bigcup\{\mathcal{B}^1_\xi : \xi < \tau\}$. It is evident that if $A \subset \mathcal{I}(X)$ and $|A| = n-1$ then $|\{B \in \mathcal{B}^1 : A \subset B\}| < \tau$. Repeating this step $n-1$ times (of course we assume $n \geq 2$), we obtain a family $\mathcal{B}^{(n-1)}$ such that $\mathcal{B}^\star = \mathcal{B}^{(n-1)} \cup \{B \in \mathcal{B} : B \cap \mathcal{I}(X) = \emptyset\} \cup \{\{x\} : x \in \mathcal{I}(X)\}$ is a base with all the required properties. □

Lemma 3. *Let $n \geq 1$ be a natural number. Let X be a space with an n-in-countable base $\mathcal{B}$ and let $|\mathcal{I}(X)| \leq \omega$. Then X has an n-in-countable base $\mathcal{B}^\star$ such that for each $x \in \mathcal{I}(X)$, $ord(x, \mathcal{B}^\star) \leq \omega$.*

Proof: The statement can be proved just as Lemma 2. □

Theorem 2. *Let $n \geq 1$ be a natural number. A regular locally separable space with an n-in-countable base is metrizable.*

Proof: The statement is proved in just the same way as Theorem from [7] with using above-mentioned Lemmas 1, 2, and 3. Here is a sketch of the proof. By Lemma 1 the space is locally metrizable. Hence every its n-in-countable base has a countable order at each nonisolated point. Let us maintain induction on the cardinality of the set of all isolated points. Let a space X satisfy the assumptions of Theorem 2, and let $|\mathcal{I}(X)| \leq \omega$. Then with respect to the above remark and by Lemma 3, X has a point-countable base. Therefore X is divided into a disjoint family of open metrizable subspaces. Consequently, X is metrizable. Now, let τ be an uncountable cardinal. Let us assume that the statement of the theorem is true for each cardinal $\lambda < \tau$. Suppose now that $|\mathcal{I}(X)| = \tau$. We will consider two cases.

Case 1. *τ is regular.* By Lemmas 1 and 2 there is a base $\mathcal{B}$ of X of order $< \tau$ at each point of X. Without loss of generality we may assume that every element of the base is separable. For every $B^\star \in \mathcal{B}$ define a family $\mathcal{E}(B^\star)$ in the following way: $\mathcal{E}_0(\mathcal{B}^\star) = \{B^\star\}$, $\mathcal{E}_1(\mathcal{B}^\star) = \{B \in \mathcal{B} : B \cap B^\star \neq \emptyset\}, \cdots$, $\mathcal{E}_n(B^\star) = \{B \in \mathcal{B} : B \cap (\bigcup \mathcal{E}_{n-1}) \neq \emptyset\}$, etc., $\mathcal{E}(B^\star) = \bigcup\{\mathcal{E}_n(B^\star) : n \in \omega\}$. Then we have $|\mathcal{E}(B^\star)| < \tau$, and $\bigcup \mathcal{E}(B^\star)$ has less than τ isolated points. It follows that $\bigcup \mathcal{E}(B^\star)$ is metrizable, hence X is metrizable too.

Case 2. *τ is singular.* Then there are a cardinal $\lambda < \tau$ and a partition $\{I_\xi : \xi < \lambda\}$ of the set $\mathcal{I}(X)$ such that $|I_\xi| = \tau_\xi < \tau$. Let $\mathcal{B}$ be a base of X with the same properties as in Case 1. Fix an ordinal $\xi < \lambda$. For each point $a \in I_\xi$ put $\mathcal{S}_a = \{B \in \mathcal{B} : a \in B\}$. Put $A_\eta = \cup \cup \{\mathcal{S}_a : |\mathcal{S}_a| < \tau_\eta\}$. Since A_η is an open subspace of X, which has the set of all isolated points of cardinality less than τ, it is metrizable. Hence, the space $G_\xi = \bigcup\{B \in \mathcal{B} : B \cap I_\xi \neq \emptyset\} = \bigcup\{A_\eta : \eta < \lambda\}$

has a base of order no more than λ at each its point. It follows that the space $G = \bigcup\{G_\xi : \xi < \lambda\} = \bigcup\{B \in \mathcal{B} : B \cap \mathcal{I}(X) \neq \emptyset\}$ has a base $\mathcal{U}$ of order no more than λ at each its point. Without loss of generality we may assume that every element of the base is separable. In the same way as in Case 1 it can be proved that G is metrizable. Because $X \setminus G$ is contained in an open subspace of X without isolated points, it is metrizable. Thus, X is the union of two open locally separable metrizable subspaces. Hence X is metrizable. $\square$

Corollary 2. *Let $n \geq 1$ be a natural number. A regular space which admits an n-in-countable cover of open separable metrizable subspaces is metrizable.*

Corollary 3. *Let $n \geq 1$ be a natural number. A regular locally countably compact space with an n-in-countable base is metrizable.*

Corollary 4 ([2])**.** *Let $n \geq 1$ be a natural number. A Hausdorff locally compact space with an n-in-countable base is metrizable.*

Lemma 4 (MA + $\neg$CH)**.** *Let $n \geq 1$ be a natural number. If X is a Čech-complete space with $c(X) = \omega$, then every open n-in-countable family of X is countable.*

PROOF: Under $n = 1$ this is a result of Shapirovskii [9]. Then the statement is proved by induction on n. $\square$

It follows from [8] that the statement of Lemma 4 is false under $\neg$SH even if $n = 1$.

Theorem 3 (MA + $\neg$CH)**.** *Let $n \geq 1$ be a natural number. Suppose that X is a regular space which is locally Čech-complete and locally has the Souslin property. If X has an n-in-countable base, then X is metrizable.*

PROOF: The statement follows from Lemma 4 and Theorem 2. $\square$

Theorem 4. *Let $n \geq 1$ be a natural number. Every pseudocompact space with an n-in-finite base is Čech complete first countable.*

PROOF: Let X be a pseudocompact space and $\mathcal{B}$ be an n-in-finite base for X. For each $B \in \mathcal{B}$ choose an open in βX set B' such that $B = X \cap B'$, and consider the family $\mathcal{B}'$ consisting of such sets. We show that the family $\mathcal{B}'$ is n-in-finite in βX. Let $A \subset \beta X$ and let $|A| = n$. Let $\mathcal{S}$ be an infinite countable subfamily of the family $\mathcal{B}$. Denote by F the set $\bigcap \mathcal{S} \cap A$. By definition, the cardinality of F is less than n. Then $G = \bigcap\{B' \setminus F : B \in \mathcal{S}\} \subset \beta X \setminus X$. Because X is pseudocompact, the set G being a G_δ-set of βX that is contained in $\beta X \setminus X$ is empty by the well known result of Hewitt [5]. Thus $\mathcal{B}'$ is an n-in-finite family in βX. The same arguments show that the family $\mathcal{B}'$ has finite order at each point of $\beta X \setminus X$. Denote by $\mathcal{I}$ the set of all isolated points of X and define $F_m = \{x \in \beta X : ord(x, \mathcal{B}') \leq m\} \setminus \mathcal{I}$. Because $\mathcal{I}$ is open in βX, each F_m is a closed subset of βX. Moreover, $\beta X \setminus X = \bigcup\{F_m : m \in \omega\}$. So X is Čech complete; therefore, X is a k-space. It is evident that a k-space with an n-in-finite base is first countable.

$\square$

Corollary 5. *Let $n \geq 1$ be a natural number. Every submetacompact pseudocompact space with an n-in-finite base is metrizable.*

Theorem 5. *Let $n \geq 2$ be a natural number. Every space X with an n-in-finite base has cardinality at most $\exp_{n-1}(L(X))$, where $L(X)$ is the Lindelöf degree of X.*

PROOF: Let $\mathcal{B}$ be an n-in-finite base of X. Put $\tau = L(X)$. We will use the theorem of Erdös and Rado: $(\exp_{n-1}(\tau))^+ \to (\tau^+)^n_\tau$ ([4]). Let us assume that $|X| > \exp_{n-1}(\tau)$. Consider a mapping $P : [X]^n \to \omega$ defined by the rule: $A \mapsto |\{B \in \mathcal{B} : A \subset B\}|$. There exists a homogeneous with respect to P set H of cardinality τ^+. It is easy to show that the set H is a closed discrete subset of X, a contradiction. $\square$

Let us note that there exists an example of a Hausdorff Lindelöf space with a weakly uniform base which is not first countable ([6]).

References

[1] Arhangel'skii A.V., Just W., Reznichenko E., Szeptyczki P.J., *Sharp bases and weakly uniform bases versus point countable bases*, to appear.

[2] Balogh Z., Gruenhage G., *Base multiplicity in compact and generalized compact spaces*, to appear.

[3] Engelking R., *General Topology*, Warsaw, Heldermann Verlag, Berlin, 1989.

[4] Erdös P., Rado R., *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427–489.

[5] Hewitt E., *Rings of real-valued continuous functions,1*, Trans. Amer. Math. Soc. **64** (1948), 45–99.

[6] Peregudov S.A., *Weakly uniform bases and the first axiom of countability* (in Russian), Math. Zametki **3** (1986), 331–340.

[7] Peregudov S.A., *On metrizability in a class of topological spaces with a weakly uniform base* (in Russian), Bull. Pol. Acad. Nauk **28** (1980), 609–612.

[8] Ponomarev V.I., *On metrizability of finally compact spaces with a point countable base* (in Russian), Dokl. Akad. Nauk SSSR **174** (1967), 1274–1277.

[9] Shapirovskii B.E., *On separability and metrizability of spaces with Souslin condition* (in Russian), Dokl. Akad. Nauk SSSR **207** (1972), 800–803.

State University of Management, Rjazanskij prospekt 99, Moscow, Russia 109 542

E-mail: matem@acman.msk.su

(*Received* March 3, 1999, *revised* July 22, 1999)