Hereditarily normal Katětov spaces and extending of usco mappings

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Abstract. Several classes of hereditarily normal spaces are characterized in terms of extending upper semi-continuous compact-valued mappings. The case of controlled extensions is considered as well. Applications are obtained for real-valued semi-continuous functions.

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1. Introduction

Let X and Y be topological spaces and let 2^Y stand for the family of non-empty subsets of Y. We will need the following special subsets of 2^Y:

$$\mathcal{F}(Y) = \{S \in 2^Y \,:\, S \text{ is closed}\}, \quad C(Y) = \{S \in \mathcal{F}(Y) \,:\, S \text{ is compact}\},$$

and

$$C'(Y) = C(Y) \cup \{Y\}.$$

In case Y is a linear space, we will need also the following one:

$$C_c(Y) = \{S \in C(Y) \,:\, S \text{ is convex}\}.$$

A set-valued mapping $\Phi : X \to 2^Y$ is *lower semi-continuous* (resp., *upper semi-continuous*), or *l.s.c.* (resp., *u.s.c.*), if the set

$$\Phi^{-1}(U) = \{x \in X \,:\, \Phi(x) \cap U \neq \emptyset\} \quad (\text{resp.}, \Phi^{\#}(U) = \{x \in X \,:\, \Phi(X) \subset U\})$$

is open in X for every open $U \subset Y$. A set-valued mapping $\varphi : X \to 2^Y$ is *usco* provided it is u.s.c. and compact-valued, simultaneously. A set-valued mapping $\varphi : X \to 2^Y$ is a *selection* for $\Phi : X \to 2^Y$ if $\varphi(x) \subset \Phi(x)$ for every $x \in X$.

A real-valued function $f : X \to \mathbb{R}$ is *lower semi-continuous* (resp., *upper semi-continuous*), or *lsc* (resp., *usc*), if the set $f^{-1}((r, +\infty))$ (resp., $f^{-1}((-\infty, r))$) is open in X for every $r \in \mathbb{R}$.

Throughout this paper, the topological weight of Y, denoted by $w(Y)$, is the smallest infinite cardinal τ such that Y has a base of cardinality less than or equal to τ.

Also, we shall use all conventional notations, such as $\overline{A}$ to denote the closure of a subset A of a topological space X, $\varphi|A$ to denote the restriction of a mapping φ on A, etc.

A starting point of the present paper is given by the following two well-known results concerning u.s.c. selections of l.s.c. mappings.

Theorem 1.1 (Michael [10], Choban [1]). *A T_1-space X is paracompact if and only if for every completely metrizable space Y, every l.s.c. mapping $\Phi : X \to \mathcal{F}(Y)$ admits a u.s.c. selection $\varphi : X \to C(Y)$.*

Theorem 1.2 (Nedev and Choban [2]). *A T_1-space X is collectionwise normal if and only if for every completely metrizable space Y, every l.s.c. mapping $\Phi : X \to C'(Y)$ admits a u.s.c. selection $\varphi : X \to C(Y)$.*

As a rule, the selection theorems are analogues and in most respects generalizations of ordinary extension theorems [M1]. In contrast to this, the above mentioned theorems do not generalize any extension result. As it is shown below, neither the assumptions of Theorem 1.1 nor that of Theorem 1.2 are sufficient for the "extending of a partial (defined on a closed subset) usco mapping to an usco mapping".

Proposition 1.3 (Gutev [5]). *Let $Y = \{0, 1\}$ be the discrete two-point space, and let X be a T_1-space such that, whenever $A \subset X$ is closed, every u.s.c. mapping $\theta : A \to C(Y)$ can be extended to a u.s.c. mapping $\varphi : X \to C(Y)$ so that $\varphi|A = \theta$. Then X is hereditarily normal.*

In this paper we establish the necessary and sufficient conditions for the extendability of both partial usco mappings and partial usco selections. In fact, this is incorporated in our first principal result which is a "selection-extension" analogue of Theorem 1.2.

Theorem 1.4. *For a T_1-space X the following conditions are equivalent:*

(a) *X is hereditarily normal and Katětov;*
(b) *whenever Y is a completely metrizable space, $A \subset X$ is closed, $\Phi : X \to C'(Y)$ is l.s.c. and $\theta : A \to C(Y)$ is a u.s.c. selection for $\Phi|A$, there exists a u.s.c. selection $\varphi : X \to C(Y)$ for Φ such that $\varphi|A = \theta$;*
(c) *whenever Y is a completely metrizable space, $A \subset X$ is closed and $\theta : A \to C(Y)$ is u.s.c., there exists a u.s.c. $\varphi : X \to C(Y)$ such that $\varphi|A = \theta$.*

It should be said that the *Katětov spaces* lie strictly between the collectionwise normal spaces and the paracompact ones. For more details and the right definition of these spaces we refer the reader to the beginning of the next section.

The second main result of the paper deals with the same problem but now in the situation of Theorem 1.1. Also, it answers to a question raised by Gutev [5].

Theorem 1.5. *For a T_1-space X the following conditions are equivalent:*

(a) *X is hereditarily normal and paracompact;*

(b) *whenever Y is a completely metrizable space, $A \subset X$ is closed, $\Phi : X \to \mathcal{F}(Y)$ is l.s.c. and $\theta : A \to C(Y)$ is a u.s.c. selection for $\Phi|A$, there exists a u.s.c. selection $\varphi : X \to C(Y)$ for Φ such that $\varphi|A = \theta$.*

In conclusion, let us also mention that the paper contains some natural generalizations of Theorems 1.4 and 1.5 (see Theorem 2.2 and 3.1, respectively). These generalizations provide the link between the "degree" of Katětov's property (resp., paracompactness) of X and the topological weight of Y. As a result, they lead us to the following characterizations of hereditary normality.

Theorem 1.6. *For a T_1-space X the following conditions are equivalent:*

(a) *X is hereditarily normal;*

(b) *whenever Y is a completely metrizable separable space, $A \subset X$ is closed and $\theta : A \to C(Y)$ is u.s.c., there exists a u.s.c. $\varphi : X \to C(Y)$ such that $\varphi|A = \theta$.*

Theorem 1.7. *For a T_1-space X the following conditions are equivalent:*

(a) *X is hereditarily normal;*

(b) *if $A \subset X$ is closed and the functions $l, u : A \to \mathbb{R}$ are respectively lsc and usc such that $l(x) \leq u(x)$ for every $x \in A$, then there exist functions $l', u' : X \to \mathbb{R}$, respectively lsc and usc, such that $l'|A = l$, $u'|A = u$ and $l'(x) \leq u'(x)$ for every $x \in X$.*

Let us observe the analogy of Theorem 1.7 with the famous Tietze-Urysohn's theorem [14]. Namely, according to Tietze-Urysohn's theorem, X is normal if and only if all inequalities in Theorem 1.7 turn into equalities. Similar analogy can be done also with some other classic characterizations of normality, such as Katětov's [6], [7] and Tong's [13].

The paper is arranged as follows. A proof of Theorem 1.4 is obtained in the next Section 2. In particular, Section 2 contains also the necessary preparation for proving Theorem 1.5. In Section 3, the proof of Theorem 1.5 will be finally accomplished. The last Section 4 is devoted to applications and contains, in particular, the proofs of Theorems 1.6 and 1.7.

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2. Katětov spaces and extending of usco mappings

The so called (in [12]) *Katětov spaces* are introduced by Katětov [8]. A normal space X is *Katětov* if for every closed $A \subset X$ and every locally finite and open (in A) cover $\{O_\alpha : \alpha \in \mathcal{A}\}$ of A there exists an open and locally finite cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of X such that $U_\alpha \cap A = O_\alpha$ for every $\alpha \in A$. Let us especially mention that every Katětov space is certainly collectionwise normal while the converse is not true ([12, Example 3]). On the other hand, every countably paracompact collectionwise normal space is Katětov, the converse fails again ([12, Example 1, (V=L)]).

If τ is an infinite cardinal and the above property holds provided the cardinality $|\mathcal{A}|$ of $\mathcal{A}$ is less than or equal to τ, then X is called a *τ-Katětov space.*

The considerations of this section are inspired by the following necessary condition for extending usco mappings with values in discrete spaces (by analogy with Proposition 1.3).

Lemma 2.1. *Let τ be an infinite cardinal number and let X be a T_1-space such that for every closed $A \subset X$, discrete space Y with $|Y| \leq \tau$ and u.s.c. $\theta : A \to C(Y)$ there exists a u.s.c. $\varphi : X \to C(Y)$ such that $\varphi|A = \theta$. Then X is τ-Katětov.*

Proof: Let $A \subset X$ be closed and $\mathcal{U} = \{U_y\}_{y \in Y}$ be a locally finite and open in A cover of A with $|Y| \leq \tau$. By [12, Remark 1], it suffices to find an open locally finite in X family $\{W_y\}_{y \in Y}$ such that $W_y \cap A = U_y$ for every $y \in Y$. Towards this end, let $s(x) = \{y \in Y : x \in U_y\}$ for every $x \in A$. Also, for every $s \in C(Y)$ we set $O_s = \{x \in A : s(x) = s\}$. The family $\mathcal{O} = \{O_s : s \in C(Y)\}$ is locally finite. Indeed, let $x \in X$ and P_x be a neighborhood of x which intersects only a finite number of members of $\mathcal{U}$. Let $L(x) = \{y \in Y : P_x \cap U_y \neq \emptyset\}$. Then, the set $\{O_s : s \subset L(x)\}$ is finite and P_x does not intersect any element of $\mathcal{O} \setminus \{O_s : s \subset L(x)\}$. In what follows, we consider $C(Y)$ as a discrete space. Since $\{\overline{O_s} : s \in C(Y)\}$ is clearly a locally finite closed cover of A, we can define a u.s.c. mapping $\theta : A \to C(C(Y))$ by $\theta(x) = \{s \in C(Y) : x \in \overline{O_s}\}$, $x \in X$. Indeed, the set $\theta^{-1}(K) = \bigcup\{\overline{O_s} : s \in K\}$ is closed for every $K \subset C(Y)$.

Now we shall prove the following property:

$$(*) \qquad y \in \bigcap \theta(x) \iff x \in U_y.$$

Suppose $y \in \bigcap \theta(x)$. Since $x \in O_{s(x)}$, we have $s(x) \in \theta(x)$. Then $y \in s(x)$ and therefore $x \in U_y$. Now suppose $x \in U_y$. Take an $l \in C(Y)$ such that $y \notin l$. Since $O_l \cap U_y = \emptyset$ and U_y is open in A, it follows that $\overline{O_l} \cap U_y = \emptyset$. Hence $x \notin \overline{O_l}$ and therefore $l \notin \theta(x)$. So, each $l \in \theta(x)$ contains y, i.e. $y \in \bigcap \theta(x)$. Thus, the verification of $(*)$ completes.

The next property of θ is a consequence from $(*)$.

$$(**) \qquad \theta^{\#}(\{s \in C(Y) : y \in s\}) = U_y.$$

Indeed, if $x \in \theta^{\#}(\{s \in C(Y) : y \in s\})$, then $\theta(x) \subset \{s \in C(Y) : y \in s\}$ and therefore $y \in \bigcap \theta(x)$. Hence, by $(*)$, $x \in U_y$. The inverse inclusion is simple and is left to the reader.

Finally, note that $|C(Y)| \leq |Y| \cdot \aleph_0 \leq \tau \cdot \aleph_0 = \tau$. Then, by assumption, there exists a u.s.c. $\varphi : X \to C(C(Y))$ such that $\varphi|A = \theta$. Set $W_y = \varphi^{\#}(\{s \in C(Y) : y \in s\})$ for every $y \in Y$. Since φ is u.s.c., W_y is open in X and, by $(**)$, $W_y \cap A = U_y$. It remains only to show that $\{W_y : y \in Y\}$ is locally finite. Pick an $x \in X$ and then consider the neighborhood $\varphi^{\#}(\varphi(x))$ of x. If $W_y \cap \varphi^{\#}(\varphi(x)) \neq \emptyset$

for some $y \in Y$, then $y \in \bigcup \varphi(x)$. It now follows from $|\bigcup \varphi(x)| < \aleph_0$ that $\varphi^{\#}(\varphi(x))$ intersects only finitely many elements of $\{W_y : y \in Y\}$. □

The main purpose of the rest part of this section is to show that the condition "hereditarily normal and τ-Katětov" is also sufficient for extending usco mappings with values in arbitrary completely metrizable Y with $w(Y) \leq \tau$. Namely, the following slight generalization of Theorem 1.4 will be proved.

Theorem 2.2. *For a T_1-space X and an infinite cardinal τ the following conditions are equivalent:*

(a) *X is hereditarily normal and τ-Katětov;*
(b) *whenever $A \subset X$ is closed, Y is completely metrizable with $w(Y) \leq \tau$, $\Phi : X \to C'(Y)$ is l.s.c. and $\theta : A \to C(Y)$ is a u.s.c. selection for $\Phi|A$ there exists a u.s.c. selection $\varphi : X \to C(Y)$ for Φ such that $\varphi|A = \theta$;*
(c) *whenever $A \subset X$ is closed, Y is completely metrizable with $w(Y) \leq \tau$ and $\theta : A \to C(Y)$ is u.s.c. there exists a u.s.c. $\varphi : X \to C(Y)$ such that $\varphi|A = \theta$.*

Before turning to the proof of Theorem 2.2, it should be mentioned that this result is as natural as a "working" generalization of Theorem 1.4. This is illustrated in Section 4. Here, let us turn the reader's attention only to the following consequence of Theorem 2.2 presenting a covering-type characterization of the hereditarily normal τ-Katětov spaces.

Corollary 2.3. *Let τ be an infinite cardinal. A T_1-space X is hereditarily normal and τ-Katětov if and only if for every closed $A \subset X$ and every closed locally finite cover $\{F_\alpha : \alpha \in \mathcal{A}\}$ of A with $|\mathcal{A}| \leq \tau$ there exists a closed locally finite cover $\{M_\alpha : \alpha \in \mathcal{A}\}$ of X such that $M_\alpha \cap A = F_\alpha$ for every $\alpha \in \mathcal{A}$.*

Proof: Let Z be a topological space and $\{B_\alpha : \alpha \in \mathcal{A}\}$ be a cover of Z. Considering $\mathcal{A}$ as a discrete space, we define a set-valued mapping $\psi : Z \to 2^{\mathcal{A}}$ by $\psi(z) = \{\alpha \in \mathcal{A} : z \in B_\alpha\}$, $z \in Z$. As it is well known, $\{B_\alpha : \alpha \in \mathcal{A}\}$ is closed and locally finite if and only if ψ is usco. By the help of this remark and implication (a) $\Rightarrow$ (c) of Theorem 2.2, we get immediately that every hereditarily normal τ-Katětov space has the property of interest. The converse follows from Proposition 1.3 and Lemma 2.1. □

To prepare for the proof of Theorem 2.2 we need the following property of hereditary normality which is maybe known in some quarters.

Lemma 2.4. *Let X be a hereditarily normal space, $A \subset X$ closed and let $O \subset A$ be open in A. Then there exists an open subset U of X such that $U \cap A = O$ and $\overline{U} \cap A = \overline{O}$.*

Proof: Let $A_0 = O$ and $A_1 = A \setminus \overline{O}$. Then $\overline{A_i} \cap A_{1-i} = \emptyset$ for $i = 0, 1$. Hence, by condition, there exist open $U_0, U_1 \subset X$ such that $A_i \subset U_i$, $i = 0, 1$ and $U_0 \cap U_1 = \emptyset$. Since both A_0 and A_1 are open in A we may assume that $U_i \cap A = A_i$, $i = 0, 1$. Finally, U_0 is as required because $\overline{U_0} \cap A \subset A \setminus (U_1 \cap A) = \overline{O}$. □

We now state the principal scheme of proving Theorem 2.2: the implication (b) $\Rightarrow$ (c) is simply obtained by letting $\Phi(x) = Y$ for all $x \in X$; the implication (c) $\Rightarrow$ (a) follows from Proposition 1.3 and Lemma 2.1. So, we must verify only (a) $\Rightarrow$ (b). To make the arguments clear, we subdivide the proof into two steps. The first is the special case of a discrete space Y. The second one is the reduction from "arbitrary completely metrizable Y" to "discrete Y". This is what we shall do in the remaining part of this section.

Proof of Theorem 2.2 in the case of discrete spaces.

A central position in this step of the proof occupies the following lemma.

Lemma 2.5. *Let τ be an infinite cardinal, X a hereditarily normal τ-Katětov space, $A \subset X$ closed, Y a discrete space with $|Y| \leq \tau$, $\Phi : X \to 2^Y$ l.s.c. and let $\theta : A \to C(Y)$ be a u.s.c. selection for $\Phi|A$. Then there exists a neighborhood W of A and a u.s.c. selection $\psi : W \to C(Y)$ for $\Phi|W$ such that $\psi|A = \theta$.*

Proof: If we forget Φ, to construct these W and ψ it will be sufficient to find an open in X family $\{W_s : s \in C(Y)\}$ with the following properties:

$$(\star) \qquad W_s \cap A = \theta^{\#}(s) \quad \text{for every } s \in C(Y),$$

and

$$(\star\star) \qquad \bigcap_{i=1}^{n} W_{s_i} = \emptyset \quad \text{whenever} \quad \bigcap_{i=1}^{n} s_i = \emptyset.$$

Indeed, let us set $W = \bigcup\{W_s : s \in C(Y)\}$ and then let us define $\psi : W \to C(Y) \cup \{\emptyset\}$ by $\psi(x) = \bigcap\{s \in C(Y) : x \in W_s\}, \quad x \in W$. If we suppose that the $\psi(x)$ is empty for some $x \in W$, then there has to exist a finite number of compacts $s_1, s_2, \ldots, s_n$ of $\{s \in C(Y) : x \in W_s\}$ with $\bigcap_{i=1}^n s_i = \emptyset$. However, by $(\star\star)$, this will imply that $\bigcap_{i=1}^n W_{s_i} = \emptyset$ which is impossible because $x \in W_{s_i}, i = 1, 2, \ldots, n$. So, $\psi : W \to C(Y)$. Next, it follows from $(\star)$ that $\psi(x) = \bigcap\{s \in C(Y) : x \in \theta^{\#}(s)\} = \theta(x)$ for every $x \in A$. Finally, ψ is also u.s.c. Indeed, let $L \subset Y$ be arbitrary and $x \in \psi^{\#}(L)$. Then there exists a finite number of elements $s_1, s_2, \ldots, s_n$ of $\{s \in C(Y) : x \in W_s\}$ such that $\bigcap_{i=1}^n s_i \subset L$. Therefore $\bigcap_{i=1}^n W_{s_i}$ is a neighborhood of x which lies in $\psi^{\#}(L)$. Then $\psi^{\#}(L)$ is open, i.e. ψ is u.s.c.

The manner of constructing $\{W_s : s \in C(Y)\}$ will give us that the corresponding ψ above is also a selection for $\Phi|W$. So, we proceed to the construction of this family. We set $O_s = \{x \in A : \theta(x) = s\}$ for all $s \in C(Y)$. Also, we set $\mathcal{S}_n = \{s \in C(Y) : |s| = n\}$, $\mathcal{O}_n = \{O_s : s \in \mathcal{S}_n\}$ and $A_n = \{x \in A : |\theta(x)| \geq n\}$ for all $n \in \mathbb{N}$. Note that $A \backslash A_n = \bigcup\{\theta^{\#}(s) : |s| < n\}$ and hence A_n is closed, $A_n \supset A_{n+1}$ for all $n \in \mathbb{N}$ and $A_1 = A$. We shall establish some simple properties.

(i) *Let $l, s \in C(Y)$ and $\theta^{\#}(s) \cap O_l \neq \emptyset$. Then $l \subset s$.*

This is obvious.

(ii) *Let $l, s \in C(Y)$ and l is not contained in s. Then $\overline{O_l} \cap O_s = \emptyset$.*

It follows from (i) that $\theta^{\#}(s) \cap O_l = \emptyset$. So, $\theta^{\#}(s) \cap \overline{O_l} = \emptyset$ because $\theta^{\#}(s)$ is open in A. Therefore $O_s \cap \overline{O_l} = \emptyset$ because $O_s \subset \theta^{\#}(s)$.

(iii) *$\{O_s : s \in C(Y)\}$ is a locally finite cover of A.*

Let $x \in A$. Then $\theta^{\#}(\theta(x))$ is a neighborhood of x in A. By (i), $\theta^{\#}(\theta(x)) \cap O_s \neq \emptyset$ implies $s \subset \theta(x)$. Hence, $\theta^{\#}(\theta(x))$ intersects only finitely many elements of $\{O_s : s \in C(Y)\}$ because $\theta(x)$ is finite.

(iv) *$\mathcal{O}_n$ is locally finite and open in A_n.*

Since $O_s = \theta^{\#}(s) \cap A_n$ for every $s \in \mathcal{S}_n$, each O_s is open in A_n. The local finiteness follows from (iii).

(v) *$\bigcap_{y \in s} \Phi^{-1}(\{y\})$ is a neighborhood of O_s in X for every $s \in C(Y)$.*

The intersection $\bigcap_{y \in s} \Phi^{-1}(\{y\})$ is open as a finite intersection of open sets. Next, $O_s \subset \bigcap_{y \in s} \Phi^{-1}(\{y\})$ because θ is a selection for $\Phi|A$.

Now we shall construct an open in X family $\mathcal{V} = \{V_s : s \in C(Y)\}$ so that the following holds for every $s \in C(Y)$.

(1) $V_s \cap A_n = O_s$ if $|s| = n$,
(2) $V_s \cap V_l = \emptyset$ for every $l \in C(Y)$ such that $|l| \leq |s|$ and l is not contained in s,
(3) $V_s \subset \bigcap_{y \in s} \Phi^{-1}(\{y\})$.

First, note that $|\mathcal{S}_n| \leq |C(Y)| \leq |Y| \cdot \aleph_0 \leq \tau \cdot \aleph_0 = \tau$. Since X is τ-Katětov, by (iv) and [12, Remark 1], for each n we can find a locally finite open in X family $\mathcal{U}_n = \{U_s : s \in \mathcal{S}_n\}$ such that $U_s \cap A_n = O_s$ for each $s \in \mathcal{S}_n$. By Lemma 2.4, we may assume that $\overline{U_s} \cap A_n = \overline{O_s}$ for each $s \in \mathcal{S}_n$, and by (v), that $U_s \subset \bigcap_{y \in s} \Phi^{-1}(\{y\})$. Finally, we let

$$V_s = U_s \setminus \bigcup\{\overline{U_l} : l \in C(Y),\ |l| \leq |s| \text{ and } l \text{ is not contained in } s\}$$

for every $s \in C(Y)$. We will show that $O_s \subset V_s$. Let $l \in C(Y)$, $|l| \leq |s|$ and let s do not contain l. Also, let $|l| = m$. Then $O_s, O_l \subset A_m$ and, by (ii), we obtain

$$O_s \cap \overline{U_l} = O_s \cap \overline{U_l} \cap A_m = O_s \cap \overline{O_l} = \emptyset,$$

whence $O_s \subset V_s$. Since every union of finitely many locally finite families is also locally finite, the set $\bigcup\{\overline{U_l} : l \in C(Y),\ |l| \leq |s|$ *and l is not contained in s*$\}$ is closed. Thus, V_s is open and $O_s \subset V_s \subset U_s \subset \bigcap_{y \in s} \Phi^{-1}(\{y\})$. Now (1), (2) and (3) follow easily.

Finally, we define

$$W_s = \bigcup_{l \subset s} V_l, \quad s \in C(Y).$$

Let us observe that $\{O_s : s \in C(Y)\}$ consists of pairwise disjoint sets and $A_n = \bigcup\{O_s : |s| \geq n\}$. This fact together with (1) gives us that $|s| < n$ implies

$$V_s \cap A_n = V_s \cap (A_{|s|} \cap A_n) = O_s \cap A_n = \emptyset.$$

Then, $V_{s_1} \cap O_{s_2} \neq \emptyset$ implies $|s_1| \geq |s_2|$ and hence, by virtue of (2), $s_1 \supset s_2$. This shows that

$$W_s \cap A = (\bigcup_{l \subset s} V_l) \cap (\bigcup\{O_s : s \in C(Y)\}) = \bigcup_{l \subset s} O_l = \theta^{\#}(s),$$

i.e., $(\star)$ holds.

Let $\bigcap_{i=1}^{n} l_i = \emptyset$, where $l_i \in C(Y)$ and $i = 1, 2, \ldots, n$. Without loss of generality, we may assume that $|l_1| \leq |l_i|$ for $i = 1, 2, \ldots, n$. Since $\bigcap_{i=1}^{n} l_i = \emptyset$, there is $k \in \{2, \ldots, n\}$ such that l_1 is not contained in l_k. Then, by (2), $V_{l_1} \cap V_{l_k} = \emptyset$ and hence $\bigcap_{i=1}^{n} V_{l_i} = \emptyset$. So, if $\bigcap_{i=1}^{n} s_i = \emptyset$ for $s_i \in C(Y)$, $i = 1, 2, \ldots, n$, then we obtain

$$\bigcap_{i=1}^{n} W_{s_i} = \bigcap_{i=1}^{n} (\bigcup_{l \subset s_i} V_l) = \bigcup\{\bigcap_{i=1}^{n} V_{l_i} : l_i \subset s_i\} = \emptyset,$$

i.e., $(\star\star)$ holds.

To finish the proof, it only remains to check that the mapping ψ, associated with the family $\{W_s : s \in C(Y)\}$, is a selection for $\Phi|W$. On the one hand, $x \in W$ implies $x \in V_s$ for some $s \in C(Y)$ and, by (3), $\Phi(x) \supset s$. On the other hand, $x \in W_s$ and, by definition, $\psi(x) \subset s$. Therefore, $\psi(x) \subset \Phi(x)$. □

Corollary 2.6. *Let Y be a discrete space, X a hereditarily normal $w(Y)$-Katětov space, $A \subset X$ closed, $\Phi : X \to C'(Y)$ l.s.c. and $\theta : A \to C(Y)$ a u.s.c. selection for $\Phi|A$. Then there exists a u.s.c. selection $\varphi : X \to C(Y)$ for Φ such that $\varphi|A = \theta$.*

Proof: Applying Lemma 2.5, we get a neighborhood W of A and an usco selection ψ for $\Phi|W$ such that $\psi|A = \theta$. Let us take a neighborhood V of A such that $A \subset V \subset \overline{V} \subset W$. Since $X\backslash V$ is $w(Y)$-collectionwise normal, by a result of [2] (see also [11]), there exists an usco selection ϕ for $\Phi|X\backslash V$. Finally we define $\varphi : X \to C(Y)$ by the formula

$$\varphi(x) = \begin{cases} \psi(x), & x \in V \\ \psi(x) \cup \phi(x), & x \in \overline{V}\backslash V \\ \phi(x), & x \in X\backslash\overline{V}. \end{cases}$$

This φ is clearly a selection for Φ and $\varphi|A = \theta$. In order to check that φ is u.s.c., take an $L \subset Y$. Since ψ and ϕ are u.s.c., the set

$$\varphi^{-1}(L) = \left[\overline{V} \cap \psi^{-1}(L)\right] \bigcup \left[(X \setminus V) \cap \phi^{-1}(L)\right]$$

is closed, i.e. φ is u.s.c. □

Approximate representations of usco selections.

Here we have collected the needed technique for the step from "discrete Y" to "completely metrizable Y". It is based on an approach in [5] (see also [3], [4]) for approximating usco selections.

Let X be a topological space, (Y, d) a metric space and $\Phi : X \to \mathcal{F}(Y)$. Also, let $\mathcal{A}$ be a set, $p : \mathcal{A} \to \mathcal{F}(X) \cup \{\emptyset\}$ and $t : \mathcal{A} \to \mathcal{T}(Y)$, where $\mathcal{T}(Y)$ denotes the topology of Y. We shall say that the triple $(p, t; \mathcal{A})$ is a $t(\mathcal{A})$-approximate selection for Φ ([5]), if

(1) $\{p(\alpha) | \alpha \in \mathcal{A}\}$ is a locally finite cover of X,
(2) $\{t(\alpha) | \alpha \in \mathcal{A}\}$ is a locally finite cover of Y,
(3) $p(\alpha) \subset \Phi^{-1}(t(\alpha))$ for every $\alpha \in \mathcal{A}$.

We consider the set

$$\Omega(\Phi) = \{(p, t; \mathcal{A}) : (p, t; \mathcal{A}) \text{ is a } t(\mathcal{A})\text{-approximate selection for } \Phi\}.$$

Suppose $(p, t; \mathcal{A}), (q, l; \mathcal{B}) \in \Omega(\Phi)$. We need the following definitions:

(4) *Relation* $\ll$ of a partial order in $\Omega(\Phi)$ by $(p, t; \mathcal{A}) \ll (q, l; \mathcal{B})$ if and only if there exists a map $\pi : \mathcal{A} \to \mathcal{B}$ such that, for every $\beta \in \mathcal{B}$, $q(\beta) = \bigcup\{p(\alpha) : \alpha \in \pi^{-1}(\beta)\}$ and $l(\beta) = \bigcup\{t(\alpha) : \alpha \in \pi^{-1}(\beta)\}$.
(5) *Mesh* of $(p, t; \mathcal{A})$ by $\mathrm{mesh}(p, t; \mathcal{A}) = \sup\{\mathrm{diam}(t(\alpha)) : \alpha \in \mathcal{A}\}$.

Finally, with every sequence $\{(p_k, t_k; \mathcal{A}_k)\}_{k\in\mathbb{N}}$ in $\Omega(\Phi)$ we associate a set-valued mapping $(p_\infty, t_\infty; \mathcal{A}_\infty) : X \to 2^Y \cup \{\emptyset\}$ defined by:

$$(p_\infty, t_\infty; \mathcal{A}_\infty)(x) = \bigcap \left\{ \bigcup \{\overline{t_k(\alpha)} : \alpha \in \mathcal{A}_k,\ x \in p_k(\alpha)\} : k \in \mathbb{N} \right\}.$$

The key step in the reduction from "completely metrizable Y" to "discrete Y" is based on the following characterization of usco selections.

Lemma 2.7 ([5]). *Let (Y, d) be a complete metric space and $\Phi : X \to \mathcal{F}(Y)$. For a mapping $\theta : X \to 2^Y \cup \{\emptyset\}$ the following conditions are equivalent:*

(i) *θ is an usco selection for Φ;*
(ii) *there exists a decreasing sequence $\{(p_k, t_k; \mathcal{A}_k)\}_{k\in\mathbb{N}}$ in $\Omega(\Phi)$ such that $\theta = (p_\infty, t_\infty; \mathcal{A}_\infty)$ and* $\lim_{k\to\infty} \mathrm{mesh}(p_k, t_k; \mathcal{A}_k) = 0$.

In order to use Lemma 2.7, we need the following property of set-valued mappings with hereditarily normal Katětov domain.

Lemma 2.8. *Let Y be a topological space, X a hereditarily normal $w(Y)$-Katětov space, $\Phi : X \to C'(Y)$ l.s.c. and $A \subset X$ closed. Also, let*

(1) *$\{t(\alpha) : \alpha \in \mathcal{A}\}$ be an open locally finite family in Y such that $\{\Phi^{-1}(t(\alpha)) : \alpha \in \mathcal{A}\}$ covers X,*
(2) *$\{q(\alpha) : \alpha \in \mathcal{A}\}$ be a closed locally finite cover of A such that $q(\alpha) \subset \Phi^{-1}(t(\alpha)) \cap A$ for every $\alpha \in \mathcal{A}$.*

Then there exists a closed locally finite cover $\{p(\alpha) : \alpha \in \mathcal{A}\}$ of X such that $p(\alpha) \cap A = q(\alpha)$ and $p(\alpha) \subset \Phi^{-1}(t(\alpha))$ for every $\alpha \in \mathcal{A}$.

Proof: Let $\mathcal{Y} = \{\alpha \in \mathcal{A} : t(\alpha) \neq \emptyset\}$. Considering $\mathcal{Y}$ as a discrete space, we define a u.s.c. $\theta : A \to C(\mathcal{Y})$ by

$$\theta(x) = \{\gamma \in \mathcal{Y} : x \in q(\gamma)\}.$$

We also define an l.s.c. $\Psi : X \to C'(\mathcal{Y})$ by

$$\Psi(x) = \{\gamma \in \mathcal{Y} : x \in \Phi^{-1}(t(\gamma))\}.$$

Since Φ is l.s.c., the set $\Psi^{-1}(L) = \bigcup\{\Phi^{-1}(t(\gamma)) : \gamma \in L\}$ is open for every $L \subset \mathcal{Y}$, i.e. Ψ is l.s.c. Now let us pick an $x \in X$. Since $\Phi(x) \in C'(Y)$, either $\Phi(x) = Y$ or $\Phi(x) \cap t(\alpha) \neq \emptyset$ for finitely many $\alpha \in \mathcal{A}$. In the first case $\Psi(x) = \mathcal{Y}$, and in the second one $|\Psi(x)| < \aleph_0$. That is, $\Psi : X \to C'(\mathcal{Y})$.

Since $q(\gamma) \subset \Phi^{-1}(t(\gamma)) = \Psi^{-1}(\gamma)$, the mapping θ is a selection for $\Psi|A$. Note that $|\mathcal{Y}| \leq w(Y) \cdot \aleph_0 = w(Y)$ because $\{t(\gamma) : \gamma \in \mathcal{Y}\}$ is an open locally finite family of non-empty sets. Then, applying Corollary 2.6 to Ψ and θ, we get a u.s.c. selection $\psi : X \to C(\mathcal{Y})$ for Ψ such that $\psi|A = \theta$. We let $p(\gamma) = \psi^{-1}(\gamma)$ for every $\gamma \in \mathcal{Y}$ and $p(\alpha) = \emptyset$ for every $\alpha \in \mathcal{A}\backslash\mathcal{Y}$. Now $\psi|A = \theta$ implies $p(\alpha) \cap A = q(\alpha)$. The inclusion $p(\alpha) \subset \Phi^{-1}(t(\alpha))$ holds because ψ is a selection for Ψ and $\Psi^{-1}(\gamma) = \Phi^{-1}(t(\gamma))$. Since ψ is usco, $\{p(\alpha) : \alpha \in \mathcal{A}\}$ is a closed pointwise finite and closure-preserving (i.e., a closed locally finite) cover of X.

□

Proof of Theorem 2.2 in the general case.

In this last part of Section 2 we shall complete the proof of Theorem 2.2 by accomplishing the implication (a) ⇒ (b). To this end, let Y, A, Φ and θ be as in (b) of Theorem 2.2. Let d be a compatible complete metric of Y. According to Lemma 2.7, there exists a decreasing sequence $\{(q_k, t_k; \mathcal{A}_k)\}_{k\in\mathbb{N}}$ in $\Omega(\Phi|A)$ such that $\lim_{k\to\infty} \mathrm{mesh}(q_k, t_k; \mathcal{A}_k) = 0$ and $\theta = (q_\infty, t_\infty; \mathcal{A}_\infty)$. Applying once again Lemma 2.7, we shall achieve our aim if we construct a sequence of covers $\{\{p_k(\alpha) : \alpha \in \mathcal{A}_k\}\}_{k\in\mathbb{N}}$ of X such that $\{(p_k, t_k; \mathcal{A}_k)\}_{k\in\mathbb{N}}$ is a decreasing sequence in $\Omega(\Phi)$ and $p_k(\alpha) \cap A = q_k(\alpha)$ for all α. We proceed by induction. Applying Lemma 2.8 with $\mathcal{A} = \mathcal{A}_1$, $t(\alpha) = t_1(\alpha)$ and $q(\alpha) = q_1(\alpha)$ for every $\alpha \in \mathcal{A}_1$, we find a closed locally finite cover $\{p_1(\alpha) : \alpha \in \mathcal{A}_1\}$ of X such that $p_1(\alpha) \cap A = q_1(\alpha)$ and $p_1(\alpha) \subset \Phi^{-1}(t_1(\alpha))$ for every $\alpha \in \mathcal{A}_1$.

Suppose now that we have already constructed $\{p_n(\alpha) : \alpha \in \mathcal{A}_n\}$ such that $(p_n, t_n; \mathcal{A}_n) \in \Omega(\Phi)$ and $p_n(\alpha) \cap A = q_n(\alpha)$ for each $\alpha \in \mathcal{A}_n$. We will construct $\{p_{n+1}(\alpha) : \alpha \in \mathcal{A}_{n+1}\}$ such that $(p_{n+1}, t_{n+1}; \mathcal{A}_{n+1}) \in \Omega(\Phi)$, $p_{n+1}(\alpha) \cap A = q_{n+1}(\alpha)$ for each $\alpha \in \mathcal{A}_{n+1}$ and $(p_{n+1}, t_{n+1}; \mathcal{A}_{n+1}) \ll (p_n, t_n; \mathcal{A}_n)$. Towards this end, let $\beta \in \mathcal{A}_n$ and let us apply Lemma 2.8 with $p_n(\beta)$ instead of X, $q_n(\beta)$ instead of A, $\{t_{n+1}(\alpha) : \alpha \in \pi_n^{-1}(\beta)\}$ instead of $\{t(\alpha) : \alpha \in \mathcal{A}\}$ and $\{q_{n+1}(\alpha) : \alpha \in \pi_n^{-1}(\beta)\}$ instead of $\{q(\alpha) : \alpha \in \mathcal{A}\}$. To be correct, let us note that every closed subspace of a τ-Katětov space is also τ-Katětov. In addition, we note that $\{\Phi^{-1}(t_{n+1}(\alpha)) : \alpha \in \pi_n^{-1}(\beta)\}$ is a cover of $p_n(\beta)$. This merely holds because $p_n(\beta) \subset \Phi^{-1}(t_n(\beta)) = \bigcup\{\Phi^{-1}(t_{n+1}(\alpha)) : \alpha \in \pi_n^{-1}(\beta)\}$. So, by Lemma 2.8, there exists a closed locally finite cover $\{p_{n+1}(\alpha) : \alpha \in \pi_n^{-1}(\beta)\}$ of $p_n(\beta)$ such that

$$q_{n+1}(\alpha) = p_{n+1}(\alpha) \cap q_n(\beta) = p_{n+1}(\alpha) \cap p_n(\beta) \cap A = p_{n+1}(\alpha) \cap A,$$

and $p_{n+1}(\alpha) \subset \Phi^{-1}(t_{n+1}(\alpha))$ for every $\alpha \in \pi_n^{-1}(\beta)$.

It only remains to show that the cover of X defined by

$$\{p_{n+1}(\alpha) : \alpha \in \mathcal{A}_{n+1}\} = \bigcup_{\beta \in \mathcal{A}_n} \{p_{n+1}(\alpha) : \alpha \in \pi_n^{-1}(\beta)\},$$

is locally finite. To this end pick an $x \in X$ and let U be a neighborhood of x such that $\{\beta \in \mathcal{A}_n : U \cap p_n(\beta) \neq \emptyset\}$ is finite, say $\{\beta \in \mathcal{A}_n : U \cap p_n(\beta) \neq \emptyset\} = \{\beta_i\}_{i=1}^k$. For each $i \in \{1, 2 \dots, k\}$, the point x has a neighborhood U_i such that $\{\alpha \in \pi_n^{-1}(\beta_i) : U_i \cap p_{n+1}(\alpha) \neq \emptyset\}$ is finite. Then $U \cap U_1 \cdots \cap U_k$ is a neighborhood of x which meets only finitely many members of $\{p_{n+1}(\alpha) : \alpha \in \mathcal{A}_{n+1}\}$. Thus, the proof is complete.

3. Paracompact spaces and extending of usco selections

The technique developed in the previous section allows us to obtain the following characterization of hereditarily normal τ-paracompact spaces, which is a slight generalization of Theorem 1.5.

Theorem 3.1. *For an infinite cardinal τ and a T_1-space X the following conditions are equivalent:*

(a) *X is hereditarily normal and τ-paracompact;*
(b) *whenever $A \subset X$ is closed, Y completely metrizable with $w(Y) \leq \tau$, $\Phi : X \to \mathcal{F}(Y)$ l.s.c. and $\theta : A \to C(Y)$ a u.s.c. selection for $\Phi|A$ there exists a u.s.c. selection $\varphi : X \to C(Y)$ for Φ such that $\varphi|A = \theta$.*

The proof of the implication (a) $\Rightarrow$ (b) of Theorem 3.1 repeats that of the implication (a) $\Rightarrow$ (b) of Theorem 2.2, with Corollary 2.6 and Lemma 2.8 replaced respectively by Corollary 3.2 and Lemma 3.3 below.

Corollary 3.2. *Let Y be a discrete space, X a hereditarily normal $w(Y)$-paracompact space, $A \subset X$ closed, $\Phi : X \to 2^Y$ l.s.c. and $\theta : A \to C(Y)$ a u.s.c. selection for $\Phi|A$. Then there exists a u.s.c. selection $\varphi : X \to C(Y)$ for Φ such that $\varphi|A = \theta$.*

Proof: Following the proof of Corollary 2.6, it suffices to show how in this case we obtain an usco selection ϕ for $\Phi|X\backslash V$. Since $X\backslash V$ is $w(Y)$-paracompact, by results of [10] and [1], such ϕ certainly exists. □

Lemma 3.3. *Let Y be a topological space, X a hereditarily normal $w(Y)$-paracompact space, $\Phi : X \to 2^Y$ l.s.c. and $A \subset X$ closed. Also, let*

(1) *$\{t(\alpha) : \alpha \in \mathcal{A}\}$ be an open locally finite family in Y such that $\{\Phi^{-1}(t(\alpha)) : \alpha \in \mathcal{A}\}$ covers X,*
(2) *$\{q(\alpha) : \alpha \in \mathcal{A}\}$ be a closed locally finite cover of A such that $q(\alpha) \subset \Phi^{-1}(t(\alpha)) \cap A$ for every $\alpha \in \mathcal{A}$.*

Then there exists a closed locally finite cover $\{p(\alpha) : \alpha \in \mathcal{A}\}$ of X such that $p(\alpha) \cap A = q(\alpha)$ and $p(\alpha) \subset \Phi^{-1}(t(\alpha))$ for every $\alpha \in \mathcal{A}$.

Proof: With Corollary 3.2 instead of Corollary 2.6, the proof repeats precisely that of Lemma 2.8. □

To finish the proof of Theorem 3.1, it only remains to verify the implication (b) $\Rightarrow$ (a). To this end, let us observe that (b) implies the following property of X: For every completely metrizable Y with $w(Y) \le \tau$, every l.s.c. $\Phi : X \to \mathcal{F}(Y)$ admits a u.s.c. selection $\varphi : X \to C(Y)$. Indeed, letting A to be a singleton, it suffices to extend an arbitrary selection $\theta : A \to C(Y)$ for $\Phi|A$ to an usco selection φ for Φ. This property of X together with a result of [1] imply the τ-paracompactness of X. Now using Proposition 1.3, we finally obtain (a).

4. Hereditarily normal spaces and extending of usco mappings

The considerations in this last section are inspired by the following known property of hereditary normality.

Lemma 4.1 ([12]). *Every hereditarily normal space is countably Katětov.*

After Lemma 4.1 and the results of preceding sections, the following characterization of hereditarily normality sounds naturally.

Theorem 4.2. *For a T_1-space X the following conditions are equivalent:*

(a) *X is hereditarily normal;*
(b) *whenever $A \subset X$ is closed, Y a completely metrizable separable space and $\theta : A \to C(Y)$ u.s.c. there exists a u.s.c. $\varphi : X \to C(Y)$ such that $\varphi|A = \theta$;*
(c) *whenever $A \subset X$ is closed and $\theta : A \to C_c(\mathbb{R})$ is u.s.c. there exists a u.s.c. $\varphi : X \to C_c(\mathbb{R})$ such that $\varphi|A = \theta$.*

Proof: The implication (a) $\Rightarrow$ (b) is a consequence of Lemma 4.1 and the special case $\tau = \aleph_0$ of Theorem 2.2. As for (b) $\Rightarrow$ (c), note that if $\phi : X \to C(\mathbb{R})$ is u.s.c., then the mapping which assigns to each $x \in X$ the convex hull of $\phi(x)$ is also u.s.c.

Let us prove (c) $\Rightarrow$ (a). Suppose $A_0, A_1 \subset X$ are such that $\overline{A_i} \cap A_{1-i} = \emptyset$ for $i = 0, 1$. Put $A = \overline{A_0} \cup \overline{A_1}$ and then define a u.s.c. $\theta : A \to C_c(\mathbb{R})$ by

$$\theta(a) = \begin{cases} \{0\}, & a \in A \backslash \overline{A_1} \\ \{1\}, & a \in A \backslash \overline{A_0} \\ [0,1], & a \in \overline{A_0} \cap \overline{A_1}\,. \end{cases}$$

By (c), there exists an u.s.c. $\varphi : X \to C_c(\mathbb{R})$ extending θ. Put $O_i = \varphi^{\#}((i - 1/2, i+1/2))$, $i = 0, 1$, and then observe that O_i, $i = 0, 1$, are disjoint open subsets of X such that $A_i \subset O_i$, $i = 0, 1$. □

Let us now pay attention to the following (maybe known) relation between set-valued and single-valued mappings.

Lemma 4.3. *Let X be a topological space and $\theta : X \to C_c(\mathbb{R})$. Then the following conditions are equivalent:*

(a) *θ is u.s.c.;*
(b) *the functions $l, u : X \to \mathbb{R}$ defined by $l(x) = \min(\theta(x))$ and $u(x) = \max(\theta(x))$ for $x \in X$, are respectively lsc and usc.*

Proof: It follows immediately from the equalities that $\theta^{\#}((r, +\infty)) = l^{-1}((r, +\infty))$ and $\theta^{\#}((-\infty, r)) = u^{-1}((-\infty, r))$ for each $r \in \mathbb{R}$. □

Now, Theorem 1.7 from the introduction becomes a simple consequence of the above lemma and Theorem 4.2.

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