Tower extension of topological constructs

Dexue Zhang

Abstract. Let L be a completely distributive lattice and $\mathbf{C}$ a topological construct; a process is given in this paper to obtain a topological construct $\mathbf{C}(L)$, called the tower extension of $\mathbf{C}$ (indexed by L). This process contains the constructions of probabilistic topological spaces, probabilistic pretopological spaces, probabilistic pseudotopological spaces, limit tower spaces, pretopological approach spaces and pseudotopological approach spaces, etc, as special cases. It is proved that this process has a lot of nice properties, for example, it preserves concrete reflectivity, concrete coreflectivity, and it preserves convenient hulls of topological construct, i.e., the extensional topological hulls (ETH), the cartesian closed topological hulls (CCTH) and the topological universe hulls (TUH) of topological constructs.

Keywords: topological construct, extensionality, cartesian closedness, tower extension, completely distributive lattice

Classification: 54B30, 18B15, 18B30

0. Introduction

In 1989 R. Lowen [20] introduced the category of approach spaces which can be characterized as chains (indexed by $[0,\infty]$) of closure operators satisfying certain coherence axioms; in [17] E. & R. Lowen introduced pretopological approach spaces, pseudotopological approach spaces which can be described as chains (indexed by $[0,\infty]$) of pretopologies (pseudotopologies) with certain conditions. Richardson and Kent [25], Brock and Kent [6] introduced probabilistic pretopological spaces, probabilistic pseudotopological spaces, as chains (indexed by $[0,1]$) of pretopologies, pseudotopologies respectively with certain conditions. And the probabilistic topologies in [6] (or equivalently the F-diagonal probabilistic convergence structures with respect to the triangular norm $\wedge$ on $[0,1]$ in [25], [15]) are nothing but chains of topologies. The approach uniformities in [22] are defined to be chains (indexed by $[0,\infty]$) of semiuniformities with certain coherence axioms.

It is easy to see that the underlying ideas of the above constructs are similar. It is the purpose of this paper to give a unifying construction of these constructs. Precisely, for a completely distributive lattice L and a topological construct $\mathbf{C}$, another topological construct $\mathbf{C}(L)$ is constructed, called the tower extension

This work is supported by the Natural Science Foundation of China, the Foundation for Fellows Returned From Abroad and the Mathematical Center of the Education Ministry of China.

(indexed by L) of $\mathbf{C}$. $\mathbf{C}(L)$ contains $\mathbf{C}$ as a simultaneously bireflective and bicoreflective construct; and it is proved that the process of tower extension has many nice properties, for example, it preserves concrete reflectivity, concrete coreflectivity and various convenient hulls, namely, the extensional topological hulls, the cartesian closed topological hulls and the topological universe hulls of topological constructs. Hence the related results in [17], [15] about the extensional topological hulls and topological universe hulls of approach spaces and probabilistic topological spaces are special cases of the results in this paper.

1. Preliminaries

Let a, b be elements in a complete lattice L, we say a is wedge below b, in symbols, $a \lhd b$ or $b \rhd a$, if for every subset $D \subseteq L$, $\vee D \geq b$ implies $a \leq d$ for some $d \in D$. L is called completely distributive if every element $a \in L$ is the supremum of all the elements wedge below it. For an equational definition of complete distributivity we refer to [8].

If L is a complete chain, then $a \lhd b$ if $a < b$ or $a = b$ and b has a direct predecessor. Hence every complete chain is completely distributive.

Proposition 1.1 ([8]). *Let L be a complete lattice, then the following are equivalent:*

(1) *L is completely distributive;*

(2) *the supremum operation* sup : $Low(L) \longrightarrow L$ *has a left adjoint β, where $Low(L)$ is the lattice of all the lower sets in L with respect to the partial order of inclusion, and in this case $\beta(a) = \{b \in L \mid b \lhd a\}$ for all $a \in L$.* □

From (2) in the above proposition we see that for all $a \in L$, $\beta(a) = \bigcup_{b \lhd a} \beta(b) = \bigcup_{b \in \beta(a)} \beta(b)$. Hence the wedge below relation in a completely distributive lattice has the interpolation property, this means $a \lhd b$ in a completely distributive lattice L implies that there is some $c \in L$ such that $a \lhd c \lhd b$.

For basic results about completely distributive lattices we refer to the *Compendium* [8], and for categorical notions we refer to [1], [16], [24]. Subcategories are always supposed to be full in this paper. And for two objects X, Y in a category $\mathbf{A}$, the set of all the $\mathbf{A}$-morphisms is denoted $\mathrm{Hom}_{\mathbf{A}}(X, Y)$, and sometimes we simply write $\mathrm{Hom}(X, Y)$ if no confusion will arise.

2. Tower extension of topological constructs

A construct is a concrete category over $\mathbf{SET}$, i.e., a pair $(\mathbf{C}, U)$, where U is a forgetful functor from the category $\mathbf{C}$ to $\mathbf{SET}$. For each $\mathbf{C}$-object A, $U(A)$ is called the underlying set of A and for each $\mathbf{C}$-morphism f, $U(f)$ is called the underlying map of f.

We can consider a construct $\mathbf{C}$ as a category, with objects structured sets, i.e. pairs (X, ξ) where X is the underlying set and ξ is called the $\mathbf{C}$-structure, and with morphisms suitable maps between X and Y. Sometimes we identify a $\mathbf{C}$-object with its underlying set, this is purely notational, it should be clear from the context what is meant.

Definition 2.1. A construct $\mathbf{C}$ is called a *topological construct* if it satisfies the following conditions:

(TC1) *Existence of Initial Structures*: For any set X, any class T, and family $((X_t,\xi_t))_{t\in T}$ of $\mathbf{C}$-objects and any family $(f_t : X \longrightarrow X_t)_{t\in T}$ of maps, there exists a unique $\mathbf{C}$-structure ξ on X which is initial with respect to the source $(f_t : X \longrightarrow (X_t,\xi_t))_{t\in T}$, this means that for a $\mathbf{C}$-object (Y,η), a map $g : (Y,\eta) \longrightarrow (X,\xi)$ is a $\mathbf{C}$-morphism iff for all $t \in T$, $f_t \circ g : (Y,\eta) \longrightarrow (X_t,\xi_t)$ is a $\mathbf{C}$-morphism.

(TC2) *Fibre-smallness*: For any set X, the $\mathbf{C}$-fibre of X, i.e. the class of all $\mathbf{C}$-structure on X, which we denote by $\mathbf{C}(X)$, is a set.

(TC3) *Terminal Separator Property*: For any set with cardinality at most 1 there is exactly one $\mathbf{C}$-structure on it.

Note. Topological constructs defined in the above definition are called well-fibred topological constructs by some authors, and they call a construct with (TC1) and (TC2) a topological construct.

A nice property of topological constructs is that they have final structures, precisely, let $\{U(X_t,\xi_t) \xrightarrow{f_t} X\}_{t\in T}$ be a sink, then there exists a unique $\mathbf{C}$-structure ξ on X such that the sink $\{(X_t,\xi_t) \xrightarrow{f_t} (X,\xi)\}_{t\in T}$ is final. And it is well-known that every bireflective and every bicoreflective subconstruct of a topological construct is topological.

Let $\mathbf{C}$ be a topological construct, and let X be a set, the initial $\mathbf{C}$-structure ξ_d (the final $\mathbf{C}$-structure ξ_{in}) on X with respect to the empty source (the empty sink) is called the *discrete structure* (*indiscrete structure*) on X. Let ξ and η be $\mathbf{C}$-structures on X, ξ is called coarser than η, in symbols $\xi \le \eta$, if $id_X : (X,\eta) \longrightarrow (X,\xi)$ is a $\mathbf{C}$-morphism. It is easy to see that set $\mathbf{C}(X)$ of $\mathbf{C}$-structures on X is a complete lattice with respect to the partial order $\le$, with the indiscrete structure being the bottom element and the discrete structure the top element.

Definition 2.2. Let $\mathbf{C}$ be a topological construct, and let L be a completely distributive lattice, a tower indexed by L (or simply a tower) in $\mathbf{C}$ is a pair (X,Γ) where X is a set and $\Gamma : L \longrightarrow \mathbf{C}(X)$ is a map such that for any $a \in L$, the source $\{(X,\Gamma(a)) \xrightarrow{id_X} (X,\Gamma(b))\}_{b\lhd a}$ is initial, or equivalently Γ is a join-preserving map from L to the complete lattice $\mathbf{C}(X)$. A morphism between two towers (X,Γ) and (Y,Ξ) is a map $f : X \longrightarrow Y$ such that for all $a \in L$, $f : (X,\Gamma(a)) \longrightarrow (Y,\Xi(a))$ is a $\mathbf{C}$-morphism. The construct of all towers in $\mathbf{C}$ is denoted $\mathbf{C}(L)$, called the tower extension of $\mathbf{C}$ (with respect to L).

Note. The word 'tower' is taken from [20] where it is used to denote a tower of closure operators indexed by $[0,\infty]$ with certain coherence conditions.

Proposition 2.3. $\mathbf{C}(L)$ *is a topological construct.*

Proof: (TC2) and (TC3) are trivial for $\mathbf{C}(L)$, as for (TC3) let $\{X \xrightarrow{f_t} (X_t,\Gamma_t)\}_{t\in T}$ be a source, for each $a \in L$, let $\Gamma(a)$ be the $\mathbf{C}$-structure on X such that $\{(X,\Gamma(a)) \xrightarrow{f_t} (X_t,\Gamma_t(a))\}_{t\in T}$ is initial, then it is easy to see that (X,Γ) is a tower in $\mathbf{C}$, and Γ is the initial structure on X with respect to the given source. $\square$

Hence a source $\{f_t : (X,\Gamma) \longrightarrow (X_t,\Gamma_t)\}_{t\in T}$ in $\mathbf{C}(L)$ is initial iff for all $a \in L$, $\{f_t : (X,\Gamma(a)) \longrightarrow (X_t,\Gamma_t(a))\}_{t\in T}$ is initial. Meanwhile the final structure for a sink $(f_t : (X_t,\Gamma_t) \longrightarrow X)_{t\in T}$ can be obtained as follows: for each $a \in L$, let $\Gamma^*(a)$ be the final structure on X with respect to the sink $(f_t : (X_t,\Gamma_t(a)) \longrightarrow X)_{t\in T}$ in $\mathbf{C}$, trivially we have that for all $a, b \in L$, $a \le b$ implies $\Gamma^*(a) \le \Gamma^*(b)$, now let $\Gamma(a)$ be the initial structure on X with respect to the source $\{X \xrightarrow{id_X} (X,\Gamma^*(b))\}_{b\lhd a}$, then Γ is the final structure on X with respect to the given sink.

Examples. (1) Let $L = [0,1]$, $\mathbf{C} = \mathbf{Top}$ ($\mathbf{PrTop}$, $\mathbf{PsTop}$), then $\mathbf{C}(L)$ is the construct of probabilistic topological spaces (probabilistic pretopological spaces, probabilistic pseudotopological spaces respectively) in [6].

(2) Let $\mathbf{C}$ be the construct of limit spaces, and $L = [0,\infty]^{op}$, then $\mathbf{C}(L)$ is the construct of limit tower spaces in [6].

(3) Let $\mathbf{C} = \mathbf{PrTop}$ ($\mathbf{PsTop}$), $L = [0,\infty]^{op}$, then $\mathbf{C}(L)$ is the construct of pretopological approach spaces (pseudotopological approach spaces) in [17]. The construct of approach spaces ([20]) is a subconstruct of $\mathbf{PrTop}(L)$. And $\mathbf{Top}(L)$ is the construct of topological approach spaces in [6].

(4) The construct of fuzzy uniform spaces in the sense of Lowen [18] is concretely isomorphic to $\mathbf{Unif}([0,1])$ [7]; The construct of fuzzy neighbourhood spaces in the sense of Lowen [19] is concretely isomorphic to $\mathbf{Top}([0,1]^{op}) = \mathbf{Top}([0,1])$ [19], [26]; and the construct of fuzzy neighbourhood convergence spaces in the sense of Blasco and Lowen [4] is concretely isomorphic to $\mathbf{PrTop}([0,1])$.

(5) Let $\mathbf{C}$ be the construct of semiuniform spaces, then the construct $\mathbf{AUnif}$ of approach uniform spaces ([22]) is a subconstruct of $\mathbf{C}([0,\infty]^{op})$.

Given an object (X,ξ) in a topological construct $\mathbf{C}$ let $\omega_L(\xi) : L \longrightarrow \mathbf{C}(X)$ be defined as follows:

$$\omega_L(\xi)(a) = \begin{cases} \xi, & a > 0; \\ \text{the indiscrete structure}, & a = 0. \end{cases}$$

It is easy to see that ω_L induces a concrete embedding of $\mathbf{C}$ in $\mathbf{C}(L)$, and we have more.

Proposition 2.4. *The embedding functor $\omega_L : \mathbf{C} \longrightarrow \mathbf{C}(L)$ has a concrete left adjoint and a concrete right adjoint, hence $\mathbf{C}$ can be embedded in $\mathbf{C}(L)$ as a both bireflective and bicoreflective subcategory.*

Proof: Give an object (X,Γ) in $\mathbf{C}(L)$, its $\mathbf{C}$-reflection is given by

$$id_X : (X,\Gamma) \longrightarrow (X,\omega_L(\gamma))$$

where γ is the $\mathbf{C}$-structure on X such that the sink $\{(X,\Gamma(a)) \xrightarrow{id_X} (X,\gamma)\}_{a>0}$ is final. And its $\mathbf{C}$-coreflection is given by

$$id_X : (X, \omega_L(\gamma^*)) \longrightarrow (X, \Gamma)$$

where γ^* is the $\mathbf{C}$-structure on X such that the source $\{(X,\gamma^*) \xrightarrow{id_X} (X,\Gamma(a))\}_{a>0}$ is initial. □

For each $a \in L$, we define two functors $\omega_a, \omega_a^* : \mathbf{C} \longrightarrow \mathbf{C}(L)$ as follows: for each $\mathbf{C}$-object (X,γ),

$$\omega_a(\gamma)(b) = \begin{cases} \gamma, & \text{if } a \lhd b, \\ \text{the indiscrete structure}, & \text{otherwise}, \end{cases}$$

and

$$\omega_a^*(\gamma)(b) = \begin{cases} \text{the indiscrete structure}, & b = 0, \\ \gamma, & b \leq a \text{ and } b \neq 0, \\ \text{the discrete structure}, & \text{otherwise}. \end{cases}$$

These two functors will be employed in the sequel, trivially ω_a and ω_a^* preserve initial sources and final sinks.

The following proposition shows that the process of tower extension preserves concrete reflectivity and concrete coreflectivity.

Proposition 2.5. *Let* $\mathbf{A}$ *be a concretely reflective* (*concretely coreflective*) *subconstruct of a topological construct* $\mathbf{C}$ *then* $\mathbf{A}(L)$ *is concretely reflective* (*concretely coreflective*) *in* $\mathbf{C}(L)$.

Proof: (1) Suppose $\mathbf{A}$ is concretely reflective in $\mathbf{C}$, given an object (X,Γ) in $\mathbf{C}(L)$, for each $a \in L$, let $(X,\gamma(a))$ be the $\mathbf{A}$-reflection of $(X,\Gamma(a))$, it is easy to see that for all $a \leq b$ in L, $\gamma(a)$ is coarser than $\gamma(b)$. Now for each $a \in L$, let $\Gamma^r(a)$ be the $\mathbf{C}$-structure on X such that the source $\{(X,\Gamma^r(a)) \xrightarrow{id_X} (X,\gamma(b))\}_{b \lhd a}$ is initial, or equivalently Γ^r is the biggest join-preserving map smaller than γ. Since $\mathbf{A}$ is concretely reflective in $\mathbf{C}$, $\mathbf{A}$ is initially closed in $\mathbf{C}$, hence (X,Γ^r) is in $\mathbf{A}(L)$, and it is trivially the $\mathbf{A}(L)$-reflection of (X,Γ).

(2) Suppose $\mathbf{A}$ is concretely coreflective in $\mathbf{C}$, given an object (X,Γ) in $\mathbf{C}(L)$, for each $a \in L$, let $(X,\Gamma^c(a))$ be the $\mathbf{A}$-coreflection of $(X,\Gamma(a))$. Since concrete right adjoints preserve initial sources, it is easy to check that (X,Γ^c) is an object in $\mathbf{A}(L)$, and it is the $\mathbf{A}(L)$-coreflection of (X,Γ). □

Hence $\mathbf{Top}(L)$ is concretely reflective in $\mathbf{PrTop}(L)$ and $\mathbf{PrTop}(L)$ is concretely reflective in $\mathbf{PsTop}(L)$.

3. Tower extension preserves ETH

Definition 3.1 ([12], [13]). Let $\mathbf{C}$ be a topological construct;

(1) A *partial morphism* from X to Y is a morphism $f : Z \longrightarrow Y$ whose domain Z is a subspace of X.

(2) *Partial morphisms to* Y *are representable* provided Y can be embedded via the addition of a single point ∞ into an object $Y^{\#}$ with the property that for every partial morphism $f : Z \longrightarrow Y$ from X to Y, the map $f^X : X \longrightarrow Y^{\#}$, defined by $f^X(x) = f(x)$ if $x \in Z$, $f^X(x) = \infty$ if $x \notin Z$, is a morphism. The object $Y^{\#}$ is called the *one-point extension* of Y, and if the $\mathbf{C}$-structure on Y is ξ, the $\mathbf{C}$-structure on $Y^{\#}$ is denoted $\xi^{\#}$.

(3) $\mathbf{C}$ is extensional if partial morphisms to all $\mathbf{C}$-object are representable.

Lemma 3.2. *In an extensional topological construct, the one-point extension process preserves initial sources, precisely, let* $(f_t : X \longrightarrow X_t)_{t\in T}$ *be an initial source, then* $(f_t^{\#} : X^{\#} \longrightarrow X_t^{\#})_{t\in T}$ *is an initial source, where* $f_t^{\#}$ *is defined by* $f_t^{\#}(x) = f_t(x)$ *if* $x \neq \infty$ *and* $f_t^{\#}(\infty) = \infty$.

Proof: Straightforward verifications. □

Definition 3.3. An extensional topological construct $\mathbf{A}$ is called an *extensional topological hull* of topological construct $\mathbf{C}$ if $\mathbf{A}$ is a finally dense extension of $\mathbf{C}$ with the property that any finally dense embedding of $\mathbf{C}$ into an extensional topological construct can be uniquely extended to $\mathbf{A}$.

The extensional topological hull of a topological construct always exists, and it is unique up to isomorphisms ([12]).

Theorem 3.4 ([12], [13]). *The extensional topological hull* $\mathbf{A}$ *of a topological construct* $\mathbf{C}$ *is characterized by the following properties:*

(1) $\mathbf{A}$ *is an extensional topological construct;*
(2) $\mathbf{C}$ *is finally dense in* $\mathbf{A}$*;*
(3) $\{Y^{\#} \mid Y \in \mathbf{C}\}$ *is initially dense in* $\mathbf{A}$. □

The construct **Top** of topological spaces is not extensional and its extensional topological hull is the construct **PrTop** of pretopological spaces ([16]).

Theorem 3.5. *Tower extensions preserve extensional topological hull, this means if* $\mathbf{A}$ *is the extensional topological hull of a topological construct* $\mathbf{C}$, *then* $\mathbf{A}(L)$ *is the extensional topological hull of* $\mathbf{C}(L)$.

Proof: (1) $\mathbf{A}(L)$ is extensional.

Given an object (X, Γ) in $\mathbf{A}(L)$, for each $a \in L$, let $(X^{\#}, \Gamma(a)^{\#})$ be the one point extension of $(X, \Gamma(a))$, by the above lemma it is easy to see that $(X^{\#}, \Gamma^{\#})$ is an object in $\mathbf{A}(L)$, and it is the one point extension of (X, Γ).

(2) $\mathbf{C}(L)$ is finally dense in $\mathbf{A}(L)$.

Suppose (X,Γ) is a tower in $\mathbf{A}$, since $\mathbf{C}$ is finally dense in $\mathbf{A}$, for each $a \in L$, there is a final sink $(f_t : (X_t,\gamma_t) \longrightarrow (X,\Gamma(a)))_{t\in T_a}$ with (X_t,γ_t) in $\mathbf{C}$ for all $t \in T_a$.

At first for all $a \in L$ it is easy to check that $f_t : (X_t, \omega_a^*(\gamma_t)) \longrightarrow (X,\Gamma)$ is continuous for all $t \in T_a$.

Next we assert that for all $a, b \in L$ and $t \in T_b$, $f_t : (X_t, \omega_b^*(\gamma_t)(a)) \longrightarrow (X,\Gamma(a))$ is continuous.

Indeed the conclusion is trivial for $a = 0$; if $0 < a \leq b$, then $\omega_b^*(\gamma_t)(a) = \omega_a^*(\gamma_t)(b) = \gamma_t$, thus the conclusion follows from the continuity of $f_t : (X_t,\gamma_t) \longrightarrow (X,\Gamma(b))$; finally if $a \not\leq b$, then $\omega_b^*(\gamma_t)(a)$ is the discrete structure, thus our conclusion follows.

Therefore for all $a \in L$, the sink

$$\{f_t : (X_t, \omega_b^*(\gamma_t)(a)) \longrightarrow (X,\Gamma(a)) \mid t \in T_b\}_{b\in L}$$

is final. Now it is easy to see that the sink

$$\{f_t : (X_t, \omega_a^*(\gamma_t)) \longrightarrow (X,\Gamma) \mid t \in T_a\}_{a\in L}$$

is final since for all $a \in L$, $\{(X,\Gamma(a)) \xrightarrow{id_X} (X,\Gamma(b))\}_{b\lhd a}$ is initial. Trivially all the objects $(X_t, \omega_a^*(\gamma_t))$ are in $\mathbf{C}(L)$, hence $\mathbf{C}(L)$ is finally dense in $\mathbf{A}(L)$.

(3) $\{(X^{\#},\Gamma^{\#}) \mid (X,\Gamma) \in \mathbf{C}(L)\}$ is initially dense in $\mathbf{A}(L)$.

Given an object (X,Γ) in $\mathbf{A}(L)$, for each $a \in L$, there is an initial source $\{f_t : (X,\Gamma(a)) \longrightarrow (X_t^{\#}, \gamma_t^{\#})\}_{t\in T_a}$ with all (X_t,γ_t) in $\mathbf{C}$ since $\mathbf{A}$ is the extensional topological hull of $\mathbf{C}$. It is easy to check that $(X_t^{\#}, \omega_a(\gamma_t^{\#})) = (X_t^{\#}, (\omega_a(\gamma_t))^{\#})$ for all $t \in T_a$, $a \in L$, this is to say the functor ω_a preserves one point extensions. Now as in (2) it can be checked that the source

$$\{f_t : (X,\Gamma) \longrightarrow (X_t^{\#}, \omega_a(\gamma_t^{\#})) \mid t \in T_a\}_{a\in L}$$

is initial. Therefore $\mathbf{A}(L)$ is the extensional topological hull of $\mathbf{C}(L)$. □

By the above theorem ETH($\mathbf{Top}(L)$) = $\mathbf{PrTop}(L)$ and ETH($\mathbf{AP}$) = $\mathbf{PrTop}([0,\infty]^{op})$ since $\mathbf{Top}([0,\infty]^{op}) \subset \mathbf{AP} \subset \mathbf{PrTop}([0,\infty]^{op})$.

4. Tower extension preserves CCTH and TUH

A category $\mathbf{A}$ with finite products is *cartesian closed* if for each object A in $\mathbf{A}$, the functor $- \times A : \mathbf{A} \longrightarrow \mathbf{A}$ has a right adjoint, denoted by $[A,-]$.

For topological constructs, cartesian closedness is characterized by *canonical function spaces*, i.e. $[X,Y]$ is given by the set $\mathrm{Hom}(X,Y)$ endowed with a structure fulfilling condition (2) of the following theorem, and the counit of this adjunction is the usual evaluation map.

Theorem 4.1 ([9]). *For a topological construct* $\mathbf{C}$, *the following are equivalent:*

(1) $\mathbf{C}$ *is cartesian closed;*

(2) *for each pair* X, Y *of* $\mathbf{C}$*-objects, the set* $\mathrm{Hom}(X, Y)$ *can be endowed with a* $\mathbf{C}$*-structure* α *such that*

  (a) *the evaluation map* $\mathrm{ev} : (\mathrm{Hom}(X, Y), \alpha) \times X \longrightarrow Y$, $\mathrm{ev}(f, x) = f(x)$, *is a morphism, and*

  (b) *for each* $\mathbf{C}$*-object* W *and each morphism* $h : W \times X \longrightarrow Y$, *the map* $h^* : W \longrightarrow (\mathrm{Hom}(x, Y), \alpha)$ *defined by* $h^*(w)(x) = h(w, x)$ *is a morphism.* □

Definition 4.2. A cartesian closed topological construct $\mathbf{A}$ is called a *cartesian closed topological hull* of a topological construct $\mathbf{C}$ if $\mathbf{A}$ is a finally dense extension of $\mathbf{C}$ with the property that any finally dense embedding of $\mathbf{C}$ into a cartesian closed topological construct can be uniquely extended to $\mathbf{A}$.

The cartesian closed topological hull (CCTH) of a topological construct is unique up to isomorphism (but it does not always exist [2]), and it can be characterized in terms of function spaces.

Theorem 4.3 ([14], [16]). *The cartesian closed topological hull* $\mathbf{A}$ *of a topological construct* $\mathbf{C}$ *is characterized by the following properties:*

(1) $\mathbf{A}$ *is a cartesian closed topological construct;*

(2) $\mathbf{C}$ *is finally dense in* $\mathbf{A}$*;*

(3) $\{[X, Y] \mid X, Y \in \mathbf{C}\}$ *is initially dense in* $\mathbf{A}$. □

The purpose of this section is to show that the process of tower extension preserves cartesian closedness and cartesian closed topological hulls.

Theorem 4.4. *Suppose* $\mathbf{A}$ *is the cartesian closed topological hull of a topological construct* $\mathbf{C}$, *then* $\mathbf{A}(L)$ *is the cartesian closed topological hull of* $\mathbf{C}(L)$.

Proof: We prove the conclusion by the characterization of cartesian closed topological hull in the above theorem.

(1) $\mathbf{A}(L)$ is a cartesian closed topological construct.

Given objects (X, Γ), (Y, Ξ) in $\mathbf{A}(L)$, we write $\mathrm{Hom}(X, Y)$ for the set of all the morphisms between (X, Γ) and (Y, Ξ) in $\mathbf{A}(L)$, then trivially we have

$$\mathrm{Hom}(X, Y) = \bigcap_{a \in L} \mathrm{Hom}_{\mathbf{A}}(X_a, Y_a)$$

where $X_a = (X, \Gamma(a))$ and $Y_a = (Y, \Xi(a))$. Let $\Upsilon^*(a)$ denote the $\mathbf{A}$-structure on $\mathrm{Hom}(X, Y)$ inherited from the function space structure on $[X_a, Y_a]$, and let $\Upsilon(a)$ be the $\mathbf{A}$-structure on $\mathrm{Hom}(X, Y)$ such that the source $\{(X, \Upsilon(a)) \xrightarrow{id_X} (X, \Upsilon^*(b))\}_{b \lhd a}$ is initial. Now we prove that $[X, Y] = (\mathrm{Hom}(X, Y), \Upsilon)$ satisfies condition (2) in Theorem 4.1.

(a) The evaluation map $\mathrm{ev} : (\mathrm{Hom}(X,Y), \Upsilon) \times (X, \Gamma) \longrightarrow (Y, \Xi)$ is continuous. It is sufficient to prove that for all $a \in L$,

$$\mathrm{ev} : (\mathrm{Hom}(X,Y), \Upsilon(a)) \times (X, \Gamma(a)) \longrightarrow (Y, \Xi(b))$$

is continuous for all $b \lhd a$, and this follows from the continuity of

$$\mathrm{ev} : (\mathrm{Hom}(X,Y), \Upsilon^*(b)) \times (X, \Gamma(b)) \longrightarrow (Y, \Xi(b))$$

and the fact that $\Upsilon(a)$ is finer than $\Upsilon^*(b)$ for all $b \lhd a$.

(b) Suppose $h : (W, \Psi) \times (X, \Gamma) \longrightarrow (Y, \Xi)$ is a continuous map in $\mathbf{A}(L)$, we say that the function $h^* : (W, \Psi) \longrightarrow (\mathrm{Hom}(X,Y), \Upsilon)$ defined by $h^*(w)(x) = h(w,x)$ is continuous. It suffices to show that for each $a \in L$, $h^* : (W, \Psi(a)) \longrightarrow (\mathrm{Hom}(X,Y), \Upsilon^*(b))$ is continuous for all $b \lhd a$, this follows from the continuity of $\mathrm{ev} : (W, \Psi(b)) \longrightarrow (\mathrm{Hom}(X,Y), \Upsilon^*(b))$ and that $\Psi(a)$ is finer than $\Psi(b)$.

Hence $\mathbf{A}(L)$ is a cartesian closed topological construct.

(2) $\mathbf{C}(L)$ is finally dense in $\mathbf{A}(L)$, this follows from the final density of $\mathbf{C}$ in $\mathbf{A}$.

(3) $\{[(X,\Gamma), (Y,\Xi)] \mid (X,\Gamma), (Y,\Xi) \in \mathbf{C}(L)\}$ is initially dense in $\mathbf{A}(L)$.

At first we observe that for objects $(X,\gamma), (Y,\xi)$ in $\mathbf{A}$ and $a \in L$, we have

$$\mathrm{Hom}_{\mathbf{A}}((X,\gamma), (Y,\xi)) = \mathrm{Hom}((X, \omega_a(\gamma)), (Y, \omega_a(\xi)))$$

and the function space structure on $\mathrm{Hom}((X, \omega_a(\gamma)), (Y, \omega_a(\xi)))$ is $\omega_a(\alpha)$, where α is the function space structure on $\mathrm{Hom}_{\mathbf{A}}((X,\gamma), (Y,\xi))$ in $\mathbf{A}$, that is to say

$$[(X, \omega_a(\gamma)), (Y, \omega_a(\xi))] = \omega_a([(X,\gamma), (Y,\xi)])$$

or equivalently the functor ω_a preserves function spaces.

Suppose (X,Γ) is an object in $\mathbf{A}(L)$, for each $a \in L$, since $\mathbf{A}$ is the cartesian closed topological hull of $\mathbf{C}$ there is an initial source $\{f_t : (X, \Gamma(a)) \longrightarrow [(X_t, \gamma_t), (Y_t, \xi_t)]\}_{t \in T_a}$ with all $(X_t, \gamma_t), (Y_t, \xi_t)$ in $\mathbf{C}$. Now it can be checked that

$$\{f_t : (X, \Gamma) \longrightarrow [(X_t, \omega_a(\gamma_t)), (Y_t, \omega_a(\xi_t))] \mid t \in T_a\}_{a \in L}$$

is an initial source with all $(X_t, \omega_a(\gamma_t)), (Y_t, \omega_a(\xi_t))$ in $\mathbf{C}(L)$, hence $\{[(X,\Gamma), (Y,\Xi)] \mid (X,\Gamma), (Y,\Xi) \in \mathbf{C}(L)\}$ is initially dense in $\mathbf{A}(L)$.

Therefore $\mathbf{A}(L)$ is the cartesian closed topological hull of $\mathbf{C}(L)$. □

By the above theorem we know that the cartesian closed topological hull of the construct of probabilistic topological spaces is $\mathbf{Ant}(I)$ where $\mathbf{Ant}$ is the cartesian closed topological hull of $\mathbf{Top}$, that is to say it is the construct of Antoine spaces ([3], [5]). Hence an answer to the problem in [15] is given with respect to the triangular norm $\wedge$. And the cartesian closed topological hull of $\mathbf{PrTop}(L)$ is given by $\mathbf{PsTop}(L)$.

Definition 4.5. A topological construct is called a topological universe if it both extensional and cartesian closed, and it is also called a topological quasitopos ([28]).

For convenient properties of topological universes we refer to [10], [11], [28].

Definition 4.6. A topological universe **A** is called a topological universe hull of a topological construct **C** if **A** is a finally dense extension of **C** and every finally dense embedding of **C** into any topological universe can be uniquely extended to **A**.

The topological universe hull of **Top** is **PsTop** ([27], [5]). And we refer to [1], [16] for characterizations of topological universe hulls.

The topological universe hull of a topological construct does not always exit, but when it exits it is the cartesian closed topological hull of its extensional topological hull ([23]). And conversely if the cartesian closed topological hull of the extensional topological hull of a topological construct **C** is still extensional, then it is the topological universe hull of **C**. Hence by Theorem 3.5 and Theorem 4.4 we have

Theorem 4.7. *Tower extension preserves topological universe hulls.* □

Hence the topological universe hull of **AP** is $\mathbf{PsTop}([0,\infty]^{op})$ since $\mathrm{TUH}(\mathbf{Top}([0,\infty]^{op})) = \mathbf{PsTop}([0,\infty]^{op})$ and $\mathbf{Top}([0,\infty]^{op}) \subset \mathbf{AP} \subset \mathbf{PsTop}([0,\infty]^{op})$.

References

[1] Adámek J., Herrlich H., Strecker G.E., *Abstract and Concrete Categories*, Wiley, New York, 1990.

[2] Adámek J., Koubek V., *Cartesian closed initial completions*, Topology Appl. **11** (1980), 1–16.

[3] Antoine P., *Étude élémentaire des catégories d'ensembles structrés*, Bull. Soc. Math. Belgique **18** (1960), 142–164, 387–414.

[4] Blasco N., Lowen R., *Fuzzy neighbourhood convergence spaces*, Fuzzy Sets and Systems **76** (1995), 395–406.

[5] Bourdaud G., *Some cartesian closed topological categories of convergence spaces*, in E. Binz, H. Herrlich (eds.), Categorical Topology, (Proc. Mannheim, 1975), Lecture Notes in Mathematics, 540, Springer, Berlin, 1976, pp. 93–108.

[6] Brock P., Kent D.C., *Approach spaces, limit tower spaces and probabilistic convergence spaces*, Applied Categorical Structures **5** (1997), 99–110.

[7] Burton M.H., *The relationship between a fuzzy uniformity and its family of α-level uniformities*, Fuzzy Sets and Systems **54** (1993), 311–315.

[8] Gierz G., et al, *A Compendium of Continuous Lattices*, Springer, Berlin, 1980.

[9] Herrlich H., *Cartesian closed topological categories*, Math. Colloq. Univ. Cape Town **9** (1974), 1–16.

[10] Herrlich H., *Are there convenient subcategories of Top?*, Topology Appl. **15** (1983), 263–271.

[11] Herrlich H., *Topological improvements of categories of structured sets*, Topology Appl. **27** (1987), 145–155.

[12] Herrlich H., *Hereditary topological constructs*, in Z. Frolík (ed.), General Topology and its relations to Modern Analysis and Algebra VI, Proc. Sixth Prague Topological Symposium, Heldermann Verlag, Berlin, 1988, pp. 240–262.
[13] Herrlich H., *On the representability of partial morphisms in Top and in related constructs*, in F. Borceux (ed.), Categorical Algebra and its Applications, (Proc. Louvain-La-Neuve, 1987), Lecture Notes in Mathematics, 1348, Springer, Berlin, 1988, pp. 143–153.
[14] Herrlich H., Nel L.D., *Cartesian closed topological hulls*, Proc. Amer. Math. Soc. **62** (1977), 215–222.
[15] Herrlich H., Zhang D., *Categorical properties of probabilistic convergence spaces*, Applied Categorical Structures **6** (1998), 495–513.
[16] Herrlich H., Lowen-Colebunders E., Schwarz F., *Improving Top: PrTop and PsTop*, in H. Herrlich, H.E. Porst (eds.), Category Theory at Work, Heldermann Verlag, Berlin, 1991, pp. 21–34.
[17] Lowen E., Lowen R., *Topological quasitopos hulls of categories containing topological and metric objects*, Cahiers Top. Géom. Diff. Cat. **30** (1989), 213–228.
[18] Lowen R., *Fuzzy uniform spaces*, J. Math. Anal. Appl. **82** (1981), 370–385.
[19] Lowen R., *Fuzzy neighbourhood spaces*, Fuzzy Sets and Systems **7** (1982), 165–189.
[20] Lowen R., *Approach spaces, a common supercategory of TOP and MET*, Math. Nachr. **141** (1989), 183–226.
[21] Lowen R., *Approach Spaces, the missing link in the Topology-Uniformity-Metric triad*, Oxford Mathematical Monographs, Oxford University Press, 1997.
[22] Lowen R., Windels B., *AUnif: A common supercategory of pMet and Unif*, Internat. J. Math. Math. Sci. **21** (1998), 1–18.
[23] Schwarz F., *Description of topological universes*, in H. Ehrig et al, (eds.), Categorical Methods in Computor Science with Aspects from Topology, (Proc. Berlin, 1988), Lecture Notes in Computor Science, 393, Springer, Berlin, 1989, pp. 325–332.
[24] Preuss G., *Theory of Topological Structures, an Approach to Categorical Topology*, D. Reidel Publishing Company, Dordrecht, 1988.
[25] Richardson G.D., Kent D.C., *Probabilistic convergence spaces*, J. Austral. Math. Soc., (series A) **61** (1996), 400–420.
[26] Wuyts P., *On the determination of fuzzy topological spaces, and fuzzy neighbourhood spaces by their level topologies*, Fuzzy Sets and Systems **12** (1984), 71–85.
[27] Wyler O., *Are there topoi in topology*, in E. Binz, H. Herrlich, (eds.), Categorical Topology, (Proc. Mannheim, 1975), Lecture Notes in Mathematics, 540, Springer, Berlin, 1976, pp. 699–719.
[28] Wyler O., *Lecture Notes on Topoi and Quasitopoi*, World Scientific, Singapore, 1991.

Department of Mathematics, Sichuan University, Chengdu 610064, China

(*Received* April 7, 1999, *revised* September 20, 1999)