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Simple multilinear algebras and hermitian operators

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Abstract. The paper studies multilinear algebras, known as comtrans algebras, that are determined by so-called T-Hermitian matrices over an arbitrary field. The main result of this paper shows that the comtrans algebra of n-dimensional T-Hermitian matrices furnishes a simple comtrans algebra.

Keywords: comtrans algebras, T-Hermitian matrices, simple algebras

Classification: 15A69

1. Introduction

A *comtrans algebra* E over a commutative ring R with unit is a unital R-module E equipped with two trilinear operations, a *commutator* $[x, y, z]$ and a *translator* $\langle x, y, z\rangle$, such that the commutator is *left alternative*:

$$[x, x, z] = 0, \tag{1.1}$$

the translator satisfies the *Jacobi identity*:

$$\langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle = 0, \tag{1.2}$$

and together the commutator and translator satisfy the *comtrans identity*:

$$[x, y, x] = \langle x, y, x\rangle. \tag{1.3}$$

Comtrans algebras were introduced ([9]) as part of the algebraic structure in the tangent bundle corresponding to the coordinate n-ary loop of an $(n+1)$-web ([1], [2]). Thus their relationship with smooth n-ary loops is analogous to the relationship of Lie algebras with Lie groups. In fact the theory of comtrans algebras is modelled on (and to some extent may subsume) the theory of Lie algebras. The present paper is part of a continuing program (cf. [5], [6], [8]) of classifying simple comtrans algebras. Previously known simple comtrans algebras have arisen from rectangular matrices ([5]), simple Lie algebras ([5]), spaces equipped with a bilinear form having trivial radical ([6]), and comtrans algebras on spaces of Hermitian operators over a field L with minimal polynomial $x^2 + 1$ ([8]). This paper extends the results of [8] to a determination of simple comtrans algebras of Hermitian operators over an arbitrary field. Theorem 4.2 below, the main result of the current paper, shows that the vector space $H_n(T; L, F)$ of so-called n-dimensional T-Hermitian matrices furnishes a simple comtrans algebra. Comtrans algebra structure on spaces of Hermitian operators was introduced in [10], where applications to quantum mechanics were discussed.

Section 2 covers the necessary algebraic fundamentals of comtrans algebras: ideals, simplicity, and the enveloping algebras. Section 3 gives the definition of the comtrans algebras on spaces of Hermitian operators over an arbitrary field, and introduces notations for the special elements that feature in the proof of simplicity. Section 4 derives the main simplicity result from the somewhat technical Slimming Lemma (Proposition 4.1) whose proof comprises the final Section 5.

2. Ideals and enveloping algebras

For elements x, y of a comtrans algebra E over the ring R, there are R-module endomorphisms of E defined by

$$K(x,y) : E \longrightarrow E; z \mapsto [z,x,y] \tag{2.1}$$

$$R(x,y) : E \longrightarrow E; z \mapsto \langle z,x,y \rangle \tag{2.2}$$

$$L(x,y) : E \longrightarrow E; z \mapsto \langle y,x,z \rangle. \tag{2.3}$$

The *enveloping algebra* $M(E)$ of the comtrans algebra E is the subalgebra of the R-module endomorphism algebra $End_R E$ generated by

$$\{K(x,y), R(x,y), L(x,y) | x, y \in E\}$$

([7]). An *ideal* J of the comtrans algebra E is a submodule of the $M(E)$-module E. The ideals of E are precisely the kernels of comtrans algebra homomorphisms with domain E ([5, Proposition 3.1]). A comtrans algebra is *abelian* if the commutators and translators are all zero. A non-abelian comtrans algebra E is *simple* if it is irreducible as an $M(E)$-module, i.e. if it has no proper non-trivial ideals. For example, a Lie algebra L furnishes a comtrans algebra $CT(L)$ whose commutator and translator are both equal to repeated Lie-commutator $[[z,x],y]$ in the Lie-algebra. Thus $K(x,y) = Ad(x)Ad(y)$ and $R(x,y) = Ad(x)Ad(y)$. Then simplicity of the Lie algebra L is equivalent to simplicity of the comtrans algebra $CT(L)$ ([5, Theorem 3.2]).

3. Hermitian operators

Let L be a field of odd or zero characteristic. Define an involutory automorphism $\tau : L \longrightarrow L$, having a fixed field F. For $x \in L$, $x = \frac{1}{2}(x+x^\tau)+\frac{1}{2}(x-x^\tau) = x_0+x_1$, with $x_0 \in F$, i.e. $x_0^\tau = x_0$, and $x_1^\tau = -x_1$. So $L = F \bigoplus F_1$ is a vector space over F, with $F_1 = \{y \in L \mid y^\tau = -y\}$. Let $G = \{1, \tau\}$. Then G is a group of automorphisms of L, and F is the fixed field of G in L. Moreover, $dim_F L = 2$. If

$q \in F_1 - \{0\}$, then $q^2 = k$ for some $k \in F$. This follows because $(q^2)^\tau = q^\tau q^\tau = (-q)(-q) = q^2$, thus $q^2 \in F$. In general, elements of F in L are called *real*, while elements of qF in L are called *imaginary*. Fix an invertible diagonal $n \times n$ matrix T over F. Let $H_n(T; L, F)$ be the F-vector space of $n \times n$ T-Hermitian matrices over L, i.e. $n \times n$ matrices A over L satisfying $T^{-1}A^tT = A^\tau$. Choose $q \in L - F$, so $q^2 = k$ for some $k \in K$. Then $H_n(T; L, F)$ is closed under the Lie product

$$[X, Y] = q(XY - YX) \tag{3.1}$$

and the Jordan product

$$X \circ Y = \frac{1}{2}(XY + YX). \tag{3.2}$$

These products are connected by the identities

$$\frac{1}{4}[Y, [X, Z]] = k^{-1}(X \circ (Y \circ Z) - (X \circ Y) \circ Z) \tag{3.3}$$

and

$$[X \circ Y, Z] = X \circ [Y, Z] + Y \circ [X, Z] \tag{3.4}$$

(cf. [4, §2.2]). Set

$$[X, Y, Z]_1 = X \circ (Y \circ Z) - Y \circ (Z \circ X), \tag{3.5}$$

$$\langle X, Y, Z\rangle_1 = Z \circ (X \circ Y) - Y \circ (Z \circ X), \tag{3.6}$$

$$[X, Y, Z]_2 = [X, Y] \circ Z, \tag{3.7}$$

$$\langle X, Y, Z\rangle_2 = [X, Y \circ Z]. \tag{3.8}$$

Clearly (1.1)–(1.3) hold. Therefore the algebras $(H_n(T; L, F), [,,]_1, \langle,,\rangle_1)$ and $(H_n(T; L, F), [,,]_2, \langle,,\rangle_2)$ are comtrans algebras. A linear combination of these two comtrans algebras defines a new comtrans algebra on $H_n(T; L, F)$ by taking

$$[X, Y, Z] = a[X, Y, Z]_1 + b[X, Y, Z]_2 \tag{3.9}$$

as the commutator and

$$\langle X, Y, Z\rangle = a\langle X, Y, Z\rangle_1 + b\langle X, Y, Z\rangle_2 \tag{3.10}$$

as the translator. For concreteness in the following, we focus on the particular choice $a = 2$ and $b = 1$, obtaining a comtrans algebra with

$$[X, Y, Z] = 2[X, Y, Z]_1 + [X, Y, Z]_2 \tag{3.11}$$

and

$$\langle X,Y,Z\rangle = 2\langle X,Y,Z\rangle_1 + \langle X,Y,Z\rangle_2. \tag{3.12}$$

(This particular choice of a and b goes back to the quantum mechanical application, in particular the solution to the Heisenberg equation formulated using this comtrans algebra ([10, 6.11]).)

It is useful to have some notation available for the proof of the simplicity of the comtrans algebra $H_n(T;L,F)$ of $n\times n$ T-Hermitian matrices defined in (3.11) and (3.12). Let E^{ij} denote the $n\times n$ matrix with zero everywhere except for a one located at the intersection of row i and column j. Let $T=\sum_{i=1}^n a_iE^{ii}$. Let B^{ij} denote the symmetric matrix $a_iE^{ij}+a_jE^{ji}$. Let C^{ij} denote the skew-symmetric matrix $q(a_iE^{ji}-a_jE^{ij})$. Let D^{ij} denote the traceless diagonal matrix $E^{ii}-E^{jj}$. Note that the matrices I, B^{ij}, C^{ij}, and D^{ij} are all T-Hermitian, and that $H_n(T;L,F)$ has the n^2 elements B^{ij} $(1\le i\le j\le n)$ and C^{ij} $(1\le i<j\le n)$ as a basis (cf. (4.1), (4.2) below).

Remark 3.1. In the case $F=R$, $L=C$, $T=I$, the matrices B^{12}, C^{12} and D^{12} in $H_2(I;C,R)$ are observables representing the three components σ_x, σ_y and σ_z of electron spin ([3, §37]).

The following formulas record the Lie and Jordan products in $H_n(T;L,F)$ for pairs of matrices B^{ij}, C^{ij}, D^{ij}:

$$\begin{aligned}
[B^{ij},B^{kl}] &= -\{\delta_{jl}a_jC^{ik}+\delta_{jk}a_jC^{il}+\delta_{il}a_iC^{jk}+\delta_{ik}a_iC^{jl}\}\\
[B^{ij},C^{kl}] &= k\{\delta_{jl}a_jB^{ik}-\delta_{jk}a_jB^{il}+\delta_{il}a_iB^{jk}-\delta_{ik}a_iB^{jl}\}\\
[B^{ij},D^{kl}] &= -\{\delta_{jk}C^{ik}-\delta_{jl}C^{il}+\delta_{ik}C^{jk}-\delta_{il}C^{jl}\}\\
[C^{ij},C^{kl}] &= k\{\delta_{jl}a_jC^{ik}-\delta_{jk}a_jC^{il}-\delta_{il}a_iC^{jk}+\delta_{ik}a_iC^{jl}\}\\
[C^{ij},D^{kl}] &= k\{-\delta_{jk}B^{ik}+\delta_{jl}B^{il}+\delta_{ik}B^{jk}-\delta_{il}B^{jl}\}
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
B^{ij}\circ B^{kl} &= \frac{1}{2}\{\delta_{jl}a_jB^{ik}+\delta_{jk}a_jB^{il}+\delta_{il}a_iB^{jk}+\delta_{ik}a_iB^{jl}\}\\
B^{ij}\circ C^{kl} &= \frac{1}{2}\{-\delta_{jl}a_jC^{ik}+\delta_{jk}a_jC^{il}-\delta_{il}a_iC^{jk}+\delta_{ik}a_iC^{jl}\}\\
B^{ij}\circ D^{kl} &= \frac{1}{2}\{\delta_{jk}B^{ik}-\delta_{jl}B^{il}+\delta_{ik}B^{jk}-\delta_{il}B^{jl}\}\\
C^{ij}\circ C^{kl} &= \frac{k}{2}\{-\delta_{jl}a_jB^{ik}+\delta_{jk}a_jB^{il}+\delta_{il}a_iB^{jk}-\delta_{ik}a_iB^{jl}\}\\
C^{ij}\circ D^{kl} &= \frac{1}{2}\{\delta_{jk}C^{ik}-\delta_{jl}C^{il}-\delta_{ik}C^{jk}+\delta_{il}C^{jl}\}.
\end{aligned} \tag{3.14}$$

A typical application of (3.13) in the subsequent sections is to determine $AK(A^{'},I)=[A,A^{'}]$ for various T-Hermitian matrices A, A'.

The next two sections are devoted to the proof of the simplicity of the comtrans algebra $H_n(T;L,F)$ of T-Hermitian operators defined in (3.11) and (3.12). We will simply call it the comtrans algebra $H_n(T;L,F)$.

4. Simplicity and Slimming Lemma

The sets

$$\{B^{ij}|1 \leq i \leq j \leq n\} \tag{4.1}$$

of *real basis elements* and

$$\{C^{st}|1 \leq s < t \leq n\} \tag{4.2}$$

of *imaginary basis elements* together comprise a basis of the F-space $H_n(T;L,F)$. Consider an element

$$A = \sum_{1\leq i\leq j\leq n} b_{ij}B^{ij} + \sum_{1\leq s<t\leq n} c_{st}C^{st} \tag{4.3}$$

of $H_n(T;L,F)$. It is said to have *real weight* $|\{b_{ij}|b_{ij} \neq 0\}|$ and *imaginary weight* $|\{c_{st}|c_{st} \neq 0\}|$. Its (*total*) *weight* is the sum of its real and imaginary weights. The following result, whose proof is relegated to the final section, is known as the *Slimming Lemma.*

Proposition 4.1. *An Hermitian operator* (4.3) *of weight bigger than one may be reduced to an operator of strictly smaller positive weight by the action of the enveloping algebra.*

The Slimming Lemma is the key result yielding the simplicity of the comtrans algebras $H_n(T;L,F)$.

Theorem 4.2. *For $n > 1$, the comtrans algebra $H_n(T;L,F)$ of T-Hermitian operators is simple.*

PROOF: Let J be non-zero ideal of $H_n(T;L,F)$. Recall that J is invariant under the action of the enveloping algebra. Let A be non-zero element of J. By successive application of the Slimming Lemma (and possibly a scalar multiplication), it follows that a real or imaginary basis element is an image of A under the action of the enveloping algebra, and thus an element of J. In fact, since $\frac{1}{2}k^{-1}a_t^{-1}C^{st}K(B^{tt},I) = B^{st}$, this image may be taken to be B^{ij} for some $i \leq j$. For each of an exhaustive set of three cases, it will be shown that $B^{ij} \in J$ entails containment of all the basis elements within J, so that J is improper and $H_n(T;L,F)$ is simple.

Case 1. $i = j$. For $k \neq i = j$, one has

$$\frac{1}{2}k^{-1}a_i^{-1}B^{ij}K(C^{kj},-I) = B^{ik} \tag{4.4}$$

in J. Then $B^{kk} = a_k a_i^{-1}[B^{ii} + k^{-1} a_k^{-1} B^{ik} K(C^{ik}, I)] \in J$. Using (4.4) again, all the real basis elements lie in J. Finally, $-\frac{1}{2} B^{st} K(D^{st}, I) = C^{st} \in J$ for $s < t$.

Case 2. $i \neq j$, $n > 2$. For $i \neq k \neq j, k^{-1} a_j^{-1} B^{ij} K(C^{kj}, -I) = B^{ik}$, so $B^{ik} \in J$. Similarly, one obtains $B^{st} \in J$ for $s \neq t$.
Then $-k^{-1} B^{st} K(C^{st}, I) = -a_s B^{tt} + a_t B^{ss} \in J$ and $-\frac{1}{2} B^{st} K(D^{st}, I) = C^{st} \in J$. Also $\frac{1}{2} a_t^{-1}[-a_s B^{tt} + a_t B^{ss} + k^{-1} B^{st} K(D^{st}, C^{st})] = B^{ss}$, so $B^{ss} \in J$ for $1 \leq s \leq n$.

Case 3. $i = 1, j = 2$, $n = 2$. First, $-k^{-1} B^{12} K(C^{12}, I) = -a_1 B^{22} + a_2 B^{11} \in J$. Then $\frac{1}{2} a_2^{-1}[-a_1 B^{22} + a_2 B^{11} + k^{-1} B^{12} K(D^{12}, C^{12})] = B^{11} \in J$ and $-\frac{1}{2} a_1^{-1}[-a_1 B^{22} + a_2 B^{11} - k^{-1} B^{12} K(D^{12}, C^{12})] = B^{22} \in J$. Therefore $I \in J$. Also $-\frac{1}{2} B^{12} K(D^{12}, I) = C^{12} \in J$. □

5. Proof of the Slimming Lemma

In this section, the Slimming Lemma (Proposition 4.1) is proved. The proof is divided into three lemmas corresponding to an exhaustive set of distinct cases.

Lemma 5.1. *The Slimming Lemma holds for T-Hermitian operators*

$$A = bB^{ij} + cC^{st} \tag{5.1}$$

of real and imaginary weight one.

Proof: If the index sets $\{i, j\}$ and $\{s, t\}$ are distinct, or if $i \neq s < t = j$, or if $j > s < t = i$, then

$$AK(B^{ss}, I) = -2cka_s B^{st} \tag{5.2}$$

has a positive weight strictly smaller than that of A. If $i = s < t = j$, consider

$$AK(B^{ij}, I)^2 = ck(a_j B^{ii} - a_i B^{jj}) K(B^{ij}, I) = 4cka_i a_j C^{ij}. \tag{5.3}$$

Otherwise, i.e. if $i = s < t \neq j$ or $j = s < t > i$,

$$AK(B^{tt}, I) = 2cka_t B^{st}. \tag{5.4}$$

□

Lemma 5.2. *The Slimming Lemma holds for T-Hermitian operators* (4.3) *of imaginary weight bigger than one.*

Proof: If (4.3) comprises non-zero coefficients c_{st} and $c_{s't'}$ with distinct index sets $\{s, t\}$ and $\{s', t'\}$, or with $t = t', s \neq s'$, or with $s < t = s' < t'$, then it may be written in the form

$$A = c_{st} C^{st} + c_{s't'} C^{s't'} + \sum_{s<q\neq t} c_{sq} C^{sq} + \sum_{p<s<t} c_{ps} C^{ps} + \sum_{s\neq p<q\neq s,(p,q)\neq(s',t')} c_{pq} C^{pq}$$
$$+ b_{ss} B^{ss} + \sum_{s<q} b_{sq} B^{sq} + \sum_{p<s} b_{ps} B^{ps} + \sum_{s\neq p\leq q\neq s} b_{pq} B^{pq}$$

For an operator of positive weight strictly smaller than that of A, one may then take

$$AK(B^{ss},I) = -2c_{st}a_s k B^{st} - 2a_s k \sum_{s<q\neq t} c_{sq}B^{sq} + 2a_s k \sum_{p<s<t} c_{ps}B^{ps}$$
$$-2a_s \sum_{s<q} b_{sq}C^{sq} + 2a_s \sum_{p<s} b_{ps}C^{ps}.$$

Otherwise, i.e. if $s = s' < t' \neq t$, (4.3) may be written in the form

$$A = c_{st}C^{st} + c_{s't'}C^{s't'} + \sum_{s\neq p<t} c_{pt}C^{pt} + \sum_{s<t<q} c_{tq}C^{tq} + \sum_{t\neq p<q\neq t,(p,q)\neq(s',t')} c_{pq}C^{pq}$$
$$+b_{tt}B^{tt} + \sum_{p<t} b_{pt}B^{pt} + \sum_{t<q} b_{tq}B^{tq} + \sum_{t\neq p\leq q\neq t} b_{pq}B^{pq}.$$

For an operator of positive weight strictly smaller than that of A, one may then take

$$AK(B^{tt},I) = 2c_{st}a_t k B^{st} + 2a_t k \sum_{s\neq p<t} c_{pt}B^{pt}$$
$$- 2a_t k \sum_{s<t<q} c_{tq}B^{tq} + 2a_t \sum_{p<t} b_{pt}C^{pt} - 2a_t \sum_{t<q} b_{tq}C^{tq}.$$

□

Lemma 5.3. *The Slimming Lemma holds for T-Hermitian operators* (4.3) *of real weight bigger than one and imaginary weight one or zero.*

Proof: The Hermitian operator (4.3) may be written in the form

$$A = cC^{st} + \sum_{1\leq i\leq j\leq n} b_{ij}B^{ij} \quad \text{with } s < t. \tag{5.5}$$

The proof breaks up into two distinct cases.

Case 1. $\exists\,(i,j) \neq (i',j')$, $b_{ij} \neq 0 \neq b_{i'j'}$, $\{i,j\} \cap \{i',j'\} \neq \emptyset$. Without loss of generality, $i < j$. If $j \in \{i',j'\}$, then A may be written in the form

$$A = b_{ij}B^{ij} + b_{i'j'}B^{i'j'} + \sum_{i\leq q\neq j} b_{iq}B^{iq} + \sum_{p\leq i} b_{pi}B^{pi} + \sum_{i\neq p\leq q\neq i} b_{pq}B^{pq} + cC^{st}.$$

For an operator of positive weight strictly smaller than that of A, one may take

$$AK(B^{ii},I) = -2b_{ij}a_i C^{ij} - 2a_i \sum_{i<q\neq j} b_{iq}C^{iq}$$
$$+ 2a_i \sum_{p<i} b_{pi}C^{pi} + 2cka_i\delta_{it}B^{is} - 2cka_i\delta_{is}B^{it}.$$

If $i \in \{i', j'\}$, then A may be written in the form

$$A = b_{ij}B^{ij} + b_{i'j'}B^{i'j'} + \sum_{i \neq p \leq j} b_{pj}B^{pj} + \sum_{j \leq q} b_{jq}B^{jq} + \sum_{j \neq p \leq q \neq j} b_{pq}B^{pq} + cC^{st}.$$

For an operator of positive weight strictly smaller than that of A, one may take

$$\begin{aligned} AK(B^{jj}, I) = 2b_{ij}a_jC^{ij} + 2a_j \sum_{i \neq p < j} b_{pj}C^{pj} - 2a_j \sum_{j < q} b_{jq}C^{jq} \\ + 2cka_j\delta_{jt}B^{js} - 2cka_j\delta_{js}B^{jt}. \end{aligned}$$

Case 2. $\forall (i,j) \neq (i',j')$, $b_{ij} \neq 0 \neq b_{i'j'} \Rightarrow \{i,j\} \cap \{i',j'\} = \emptyset$. If there is a non-zero b_{ij} with $i \neq j$, then A may be written in the form

$$A = b_{ij}B^{ij} + \sum_{p \leq q, p \notin \{i,j\}, q \notin \{i,j\}} b_{pq}B^{pq} + cC^{st}.$$

For an operator of positive weight strictly smaller than that of A, one may take

$$AK(B^{ii}, I) = -2b_{ij}a_iC^{ij} + 2cka_i\delta_{it}B^{is} - 2cka_i\delta_{is}B^{it}.$$

Otherwise, the real basis elements having non-zero coefficients in A are all diagonal.

The remaining possibility within Case 2 is where A has the form

$$A = cC^{st} + \sum_{i=1}^{n} b_{ii}B^{ii}. \tag{5.6}$$

If $c \neq 0$, then (5.2) holds. If $c = 0$, then

$$A = \sum_{i=1}^{n} b_{ii}E^{ii}, \quad \text{and} \quad b_{ii} \neq 0 \neq b_{jj} \quad \text{for some } i, j.$$

If $a_ib_{ii} - a_jb_{jj} \neq 0$, then

$$AK(B^{ij}, I) = -b_{ii}[B^{ii}, B^{ij}] - b_{jj}[B^{jj}, B^{ij}] = 2(a_ib_{ii} - a_jb_{jj})C^{ij} \neq 0.$$

If $a_ib_{ii} - a_jb_{jj} = 0$, then $a_ib_{ii} + a_jb_{jj} \neq 0$. Therefore

$$\begin{aligned} AL(C^{ij}, D^{ij}) &= \langle D^{ij}, C^{ij}, A \rangle \\ &= \langle D^{ij}, C^{ij}, b_{ii}B^{ii} \rangle + \langle D^{ij}, C^{ij}, b_{jj}B^{jj} \rangle \\ &= 2(b_{jj}a_j - b_{ii}a_i)C^{ij} - 2k(b_{jj}a_j + b_{ii}a_i)B^{ij} \\ &= -2k(b_{jj}a_j + b_{ii}a_i)B^{ij} \neq 0. \end{aligned}$$

□

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