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Bol loop actions

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Abstract. The notions of left Bol and Bol-Bruck actions are introduced. A purely algebraic analogue of a Nono family (Lie triple family), the so called Sabinin-Nono family, is given. It is shown that any Sabinin-Nono family is a left Bol-Bruck action.

Finally it is proved that any local Nono family is a local left Bol-Bruck action. On general matters see [L.V. Sabinin 91, 99].

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In this paper we show that any Lie triple family (C^3-smooth Nono family, for short) [Nono 61] is a (local) Bol action. The notion of Bol action is due to L. Sabinin and is formulated in the following way:

1. Definition. Let $\mathcal{Q} = \langle Q, \cdot, \varepsilon \rangle$ be a left Bol loop and $\{f_a : M \to M\}_{a \in Q}$ be a family of maps. We say that this family is a *left Bol loop action* (action of the left Bol loop $\mathcal{Q}$) if $a \mapsto f_a$ is injective and

$$f_a \circ f_b \circ f_a = f_{a \cdot (b \cdot a)}, \quad f_\varepsilon = \mathrm{id}\,. \tag{1}$$

Analogously, one can define a *partial left Bol loop* action.

The notion of a Bol loop action is rather natural since the left translations L_a ($L_a b = a \cdot b$) of a left Bol loop satisfy (1), $L_a \circ L_b \circ L_a = L_{a \cdot (b \cdot a)}$.

Our next purpose is to algebraize, according to L. Sabinin, the notion of Nono family.

2. Definition. We say that a family $\{f_a : M \to M\}_{a \in Q}$ (Q being a set with a selected point $\varepsilon \in Q$) is a *Sabinin-Nono family* if $a \mapsto f_a$ is injective and

$$f_a \circ f_b \circ f_a = f_{q(a,b)}, \; f_\varepsilon = \mathrm{id}, \tag{2}$$

where $q_a \colon b \mapsto q(a,b)$, $g \colon a \mapsto q(a,\varepsilon)$ are invertible.

Analogously one can define a *partial Sabinin-Nono family*.

3. Definition. A left Bol loop, which satisfies the left Bruck identity

$$(a \cdot b)^2 = a \cdot (b^2 \cdot a) \tag{3}$$

is called a *left Bol-Bruck loop*.

4. Proposition. *Any Sabinin-Nono family is a left Bol-Bruck loop action.*

Proof: By (2)

$$(f_a \circ f_b \circ f_a) \circ f_c \circ (f_a \circ f_b \circ f_a) = f_a \circ (f_b \circ (f_a \circ f_c \circ f_a) \circ f_b) \circ f_a$$

implies $f_{[q_{(q_a b)} c]} = f_{(q_a q_b q_a c)}$, and, since $a \mapsto f_a$ is injective, $q_{(q_a b)} c = (q_a q_b q_a c)$. Thus

$$(q_a \circ q_b \circ q_a) = q_{(q_a b)}. \tag{4}$$

Let us introduce

$$L_a = g^{-1} \circ q_a \circ g, \quad a * b \overset{\text{def}}{=} L_a b. \tag{5}$$

It is easily verified that $\langle Q, *, \varepsilon \rangle$ is a loop (because, due to (2), $q_\varepsilon = \mathrm{id}$ which implies $g(\varepsilon) = \varepsilon$).

Further, due to (4), (5),

$$L_a \circ L_b \circ L_a = L_{g(a * g^{-1} b)}. \tag{6}$$

Since at $b = \varepsilon$ (6) gives $L_a \circ L_a = L_{g(a)}$, we have $L_a L_a \varepsilon = L_{g(a)} \varepsilon$, or,

$$a^2 = a * a = g(a). \tag{7}$$

Thus (7) and (6) give

$$L_a \circ L_{b^2} \circ L_a = L_{(a*b)^2}. \tag{8}$$

Applying both parts of (8) to ε we get

$$a * (b^2 * a) = (a * b)^2, \tag{9}$$

that is, the left Bruck property.

Substituting from (9) to (8) and changing b^2 by c, which is correct due to the invertibility of $g\colon a \mapsto a^2$ (see (2), (7)), we get

$$L_a \circ L_c \circ L_a = L_{a*(c*a)},$$

that is, the left Bol property.

As a result, $\langle Q, *, \varepsilon \rangle$ is a left Bol-Bruck left loop with two-sided neutral ε. But it is known [L.V. Sabinin 99] that a left Bol loop with two-sided neutral possesses the right division. Thus $\langle Q, *, \varepsilon \rangle$ is a left Bol-Bruck (two-sided) loop.

Further, by (5)

$$q(a, b) = \left(a * g^{-1} b\right)^2 = a * (b * a) \tag{10}$$

and, due to (2),

$$f_a \circ f_b \circ f_a = f_{a*(b*a)}.$$

This proves the theorem. □

5. Now we are going to consider a local Nono family and to show that it is a Sabinin-Nono (partial) family. First of all we recall the definition [Nono 61].

6. Definition. A family $\{f_a : M \to M\}_{a\in Q}$ of local transformations, defined for $a \in Q$ near fixed $\varepsilon \in Q$ (Q being a set), is called a C^3-*smooth Nono family* (*Lie triple family*) if $a \mapsto f_a$ is injective,

$$f_\varepsilon = \mathrm{id}, \quad f_a \circ f_b \circ f_a = f_{q(a,b)} \tag{11}$$

(if defined) and $(a, b) \mapsto q(a, b)$ is C^3-smooth.

7. Remark. C^3-smoothness is needed for the complete infinitesimal theory.

8. Proposition. *Any local Nono family is a partial Sabinin-Nono family.*

Proof: We should prove that $q_a \colon b \mapsto q(a, b)$ and $g \colon a \mapsto q(a, \varepsilon)$ are locally invertible. For this we use the characteristic differential equation of a local Nono action [Nono 61]:

$$-P^j_\alpha(x)\frac{\partial(bx)^i}{\partial x^j} + 2\Gamma^\lambda_\alpha(b)\frac{\partial(bx)^i}{\partial b^\lambda} = P^i_\alpha(bx)\,, \quad \varepsilon x = x, \tag{12}$$

where $bx = f_b x$,

$$P^j_\alpha(x) = \left[\frac{\partial(ax)^j}{\partial a^\alpha}\right]_{a=\varepsilon}, \quad \Gamma^\lambda_\alpha(b) = \frac{1}{2}\left[\frac{\partial q^\lambda(a,b)}{\partial a^\alpha}\right]_{a=\varepsilon}. \tag{13}$$

Note that $\overline{P}_\alpha(x) = (P^j_\alpha(x))_{j=1\dots n}$ are linearly independent over $\mathbb{R}$ (because of injectivity $a \mapsto f_a$).

Setting $b = \varepsilon$ in (12), we get

$$(\delta^\lambda_\alpha - \Gamma^\lambda_\alpha(\varepsilon))\, P^i_\lambda(x) = 0$$

and, further,

$$\delta^\lambda_\alpha - \Gamma^\lambda_\alpha(\varepsilon) = 0.$$

Thus

$$\left[\frac{\partial\{g(a)\}^\lambda}{\partial a^\alpha}\right]_{a=\varepsilon} = \left[\frac{\partial q^\lambda(a,\varepsilon)}{\partial a^\alpha}\right]_{a=\varepsilon} = \delta^\lambda_\alpha. \tag{14}$$

By the inverse map theorem it means the local existence of g^{-1}.

Further $f_b = f_\varepsilon \circ f_b \circ f_\varepsilon = f_{q_\varepsilon b}$ implies $q_\varepsilon b = b$. Thus

$$\frac{\partial q^\lambda(\varepsilon, b)}{\partial b^\alpha} = \frac{\partial(q_\varepsilon b)^\lambda}{\partial b^\alpha} = \delta^\lambda_\alpha.$$

Since $\partial q(a,b)^\lambda/\partial b^\alpha$ is continuous, the above means that

$$\frac{\partial q^\lambda(a,b)}{\partial b^\alpha} = \frac{\partial (q_a b)^\lambda}{\partial b^\alpha}$$

is an invertible matrix for a near ε.

It means that $q_a\colon b \mapsto q(a,b)$ $(a, b$ being near $\varepsilon)$ is locally invertible.

Thus any local Nono family is a partial Sabinin-Nono family. □

Now, one may repeat the proof of Proposition 4 for a partial Sabinin-Nono family. Thus

9. Proposition. *Any partial Sabinin-Nono family is a left Bol-Bruck loop action.*

Combining Propositions 8 and 9 we come to

10. Proposition. *Any local Nono family is a local left Bol-Bruck action of a left Bol-Bruck loop.*

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