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Covering dimension and differential inclusions

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Abstract. In this paper we shall establish a result concerning the covering dimension of a set of the type $\{x \in X : \Phi(x) \cap \Psi(x) \neq \emptyset\}$, where Φ, Ψ are two multifunctions from X into Y and X, Y are real Banach spaces. Moreover, some applications to the differential inclusions will be given.

Keywords: multifunction, Hausdorff distance, convex processes, covering dimension, differential inclusion

Classification: 47H04, 26E25

Introduction

Very recently, in [10], B. Ricceri, improving a theorem of [9], has established the following result:

Theorem A. *Let X, Y be Banach spaces, $\Phi : X \to Y$ a continuous, linear, surjective operator and $\Psi : X \to Y$ a completely continuous operator with bounded range. Then, one has*

$$\dim(\{x \in X : \Phi(x) = \Psi(x)\}) \geq \dim(\Phi^{-}(0)),$$

*where "*dim*" means covering dimension.*

In [9] and [10], he also presented several applications of this result.

The aim of the present paper is to extend Theorem A to the case where both Φ and Ψ are two set-valued operators, dealing with the covering dimension of the set

$$\dim(\{x \in X : \Phi(x) \cap \Psi(x) \neq \emptyset\}).$$

Our main result is Theorem 1, with its variant Theorem 2.

Two applications to differential inclusions are also established.

Basic definitions and preliminary results

Let A, B be two nonempty sets. A multifunction F from A into B (briefly $F : A \to 2^B$) is a function from A into the family of all subsets of B. For every $\Omega \subseteq B$ and every $S \subseteq A$, we put $F^{-}(\Omega) = \{x \in A : F(x) \cap \Omega \neq \emptyset\}$, $F^{+}(\Omega) = \{x \in A : F(x) \subseteq \Omega\}$ and $F(S) = \cup_{x \in C} F(x)$. Further, we put $\mathrm{gr}(F) = \{(x, y) \in A \times B : y \in F(x)\}$ and $\mathrm{gr}(F)$ will be called graph of F.

If A, B are topological spaces and $F : A \to 2^B$ is a multifunction, we say that F is lower semicontinuous (resp. upper semicontinuous) in A when $F^-(\Omega)$ (resp. $F^+(\Omega)$) is open in A for any open $\Omega \subseteq B$. A multifunction $F : A \to 2^B$ is called continuous in A when it is both lower and upper semicontinuous in A.

Let (X, d) be a metric space, for any $X_1, X_2 \subseteq X$, put

$$d_H(X_1, X_2) = \max\{ \sup_{x \in X_1} \inf_{z \in X_2} d(x, z), \sup_{z \in X_2} \inf_{x \in X_1} d(x, z)\}.$$

The number (or eventually the symbol $+\infty$) $d_H(X_1, X_2)$ is called Hausdorff distance between X_1 and X_2. Let (Y, ρ) be another metric space and let F be a multifunction from X into Y with nonempty values. F is called lipschitzean when there exists a real number $k \geq 0$ such that $\rho_H(F(x), F(z)) \leq kd(x, z)$ for any $x, z \in X$. If $k < 1$, F is called multivalued contraction.

Further, given two vector spaces X, Y, we say that a multifunction $F : X \to 2^Y$ is a convex process if it satisfies the following three conditions:

a) $F(x) + F(y) \subset F(x + y)$ for every $x, y \in X$,

b) $F(\lambda x) = \lambda F(x)$ for every $\lambda > 0$ and every $x \in X$,

c) $0 \in F(0)$.

It is easily seen that a convex process is, in particular, a multifunction with convex graph (in fact, its graph is a convex cone).

Finally, for a set S in a Banach space, we denote by $\dim(S)$ its covering dimension ([4, p. 42]). Recall that, when S is a convex set, the covering dimension of S coincides with the algebraic dimension of S, this latter being understood as ∞ if it is not finite ([4, p.57]). Also, $\operatorname{conv}(S)$ will denote the convex hull of S.

Now, we prove some lemmas which will be used in order to prove the main result.

The following lemma is a well known result but we prefer to state and prove it for the sake of clearness and completeness.

Lemma 1. *Let X, Y be topological spaces, let $\Phi : X \to 2^Y$ be a multifunction with closed graph and let $\Psi : X \to 2^Y$ be a multifunction with compact values. Then, one has*

$$\{x \in X : x \in \overline{\Phi^-(\Psi(x))}\} = \{x \in X : \Phi(x) \cap \Psi(x) \neq \emptyset\}.$$

Proof: Let $x \in X$ such that $\Phi(x) \cap \Psi(x) \neq \emptyset$, then $x \in \Phi^-(\Psi(x)) \subseteq \overline{\Phi^-(\Psi(x))}$. Vice-versa, let $x \in \overline{\Phi^-(\Psi(x))}$ and let $\{x_\alpha\}_{\alpha \in D}$ be a net in $\Phi^-(\Psi(x))$ which converges to x. For any $\alpha \in D$, choose $y_\alpha \in \Phi(x_\alpha) \cap \Psi(x)$. Since $\Psi(x)$ is compact, the net $\{y_\alpha\}_{\alpha \in D}$ has a cluster point y which belongs to $\Psi(x)$. Consequently, the net $\{(x_\alpha, y_\alpha)\}_{\alpha \in D}$ lies in $\operatorname{gr}(\Phi)$ and (x, y) is a cluster point of it. Since $\operatorname{gr}(\Phi)$ is closed, it follows that $(x, y) \in \operatorname{gr}(\Phi)$. Hence, $y \in \Phi(x) \cap \Psi(x)$ and so $\Phi(x) \cap \Psi(x) \neq \emptyset$. $\square$

Let X be a real vector space and T be a subset of X. In the sequel, T^* will denote the set:

$\{x \in T$: for any $y \in X$ there exists $r > 0$ such that $x + \rho y \in T$ for any $\rho \in \mathbb{R}$ with $|\rho| < r\}$.

Let Y be another real vector space and let A be a convex subset of $X \times Y$. For each $y \in Y$, we denote by A^y the set $\{x \in X : (x, y) \in A\}$.

Lemma 2. *Let X, Y be real vector spaces and let A be a convex subset in $X \times Y$. Then, for any $y_1, y_2 \in P_Y(A)^*$ one has* $\dim(A^{y_1}) = \dim(A^{y_2})$.

Proof: Fix $y_1, y_2 \in P_Y(A)^*$. Let n be a non negative integer such that $n \leq \dim(A^{y_1})$. Choose $n+1$ affinely-independent points $x_1, \dots, x_{n+1} \in A^{y_1}$ and let r be a positive real number such that, for each $\rho \in \mathbb{R}$ with $|\rho| < r$, one has $y_2 + \rho(y_2 - y_1) \in P_Y(A)$. Since $P_Y(A)$ is convex, then, for each $\lambda \in [0,1]$, we have

$$\lambda y_1 + (1-\lambda)(y_2 + \rho(y_2 - y_1)) \in P_Y(A) \ \text{ for each } \ \rho \in \mathbb{R} \ \text{ with } \ |\rho| < r. \tag{1}$$

Choose $\lambda \in]0,1]$ such that $0 < \frac{2\lambda - \lambda^2}{(1-\lambda)^2} < r$ and put $\rho = \frac{2\lambda-\lambda^2}{(1-\lambda)^2}$. By (1), there exists $x \in Y$ such that

$$(x, \ \lambda y_1 + (1-\lambda)(y_2 + \rho(y_2 - y_1))) \in A.$$

Since A is convex, it follows that

$$\begin{aligned}(\lambda x_i + (1-\lambda)x, \ \lambda y_1 + \lambda(1-\lambda)y_1 + (1-\lambda)^2(y_2 + \rho(y_2 - y_1))) \in A \\ \text{for all } \ i = 1, .., n+1.\end{aligned}$$

By observing that

$$\lambda y_1 + \lambda(1-\lambda)y_1 + (1-\lambda)^2(y_2 + \rho(y_2 - y_1)) = y_2,$$

one has $\lambda x_i + (1-\lambda)x \in A^{y_2}$ for all $i = 1, .., n+1$. Since $\lambda > 0$, the points $\lambda x_1 + (1-\lambda)x, .., \lambda x_{n+1} + (1-\lambda)x$ are affinely independent. Consequently, we have $\dim(A^{y_1}) \leq \dim(A^{y_2})$. By interchanging the roles of y_1 and y_2, it also follows that $\dim(A^{y_1}) \geq \dim(A^{y_2})$. Thus, $\dim(A^{y_1}) = \dim(A^{y_2})$. □

The following lemma gives a characterization of the lower semicontinuous multifunctions.

Lemma 3. *Let X, Y be topological spaces and let $F : X \to 2^Y$ be a multifunction. Then, F is lower semicontinuous in X if and only if, for any subset A of X, one has $F(\overline{A}) \subseteq \overline{F(A)}$.*

Proof: Let F be lower semicontinuous in X and fix $A \subseteq X$. Let $y_0 \in F(\overline{A})$. By absurd, suppose that $y_0 \notin \overline{F(A)}$. Let $x_0 \in \overline{A}$ such that $y_0 \in F(x_0)$. Then, $y_0 \in (Y \setminus \overline{F(A)}) \cap F(x_0)$. Consequently, there exists a neighborhood U of x_0 in

X such that $(Y\backslash\overline{F(A)}) \cap F(x) \neq \emptyset$, for each $x \in U$. Fixing $\overline{x} \in U \cap A$, one has: $\emptyset \neq (Y\backslash\overline{F(A)}) \cap F(\overline{x}) \subseteq (Y\backslash\overline{F(A)}) \cap F(A)$, which is absurd. Vice versa, suppose $F(\overline{A}) \subseteq \overline{F(A)}$ for any subset A of X and prove that, for any open Ω in Y, $F^-(\Omega)$ is open in X. Put $C = Y\backslash\Omega$, we have $F^-(\Omega) = Y\backslash F^+(C)$. Now, if $x \in \overline{F^+(C)}$, one has $F(x) \subseteq F(\overline{F^+(C)}) \subseteq \overline{F(F^+(C))} \subseteq \overline{C} = C$, so $x \in F^+(C)$. Hence, $F^+(C)$ is closed and $F^-(\Omega)$ is open. □

Main result

Before proving our main result, we recall that, if X is a nonempty set and $F : X \to 2^X$ is a multifunction, $x \in X$ is said fixed point of F when $x \in F(x)$. We shall denote by $\mathrm{Fix}(F)$ the set of all fixed points of F.

We point out that the following theorem is an extension of Theorem 1 of [10] where the same result was proved for single valued operator.

Theorem 1. *Let X, Y be real Banach spaces, $\Phi : X \to 2^Y$ a lower semicontinuous convex process with nonempty closed values such that $\Phi(X) = Y$, $\Psi : X \to 2^Y$ be a lower semicontinuous multifunction with nonempty closed convex values such that $\Psi(X)$ is bounded and $\Psi(B)$ is relatively compact for every bounded set $B \subseteq X$. Then, one has*

$$\dim(\{x \in X : \Phi(x) \cap \Psi(x) \neq \emptyset\}) \geq \dim(\Phi^-(0)).$$

Proof: Preliminarily, we suppose that $\dim(\Phi^-(0)) \geq 1$. Thanks to Theorem 2 of [8], the multifunction Φ has closed graph and maps open subsets of X into open subsets of Y. Hence, denoting by $B_X(x, r)$ (resp. $B_Y(y, r)$) the closed ball in X (resp. Y) of center x (resp. y) and radius $r > 0$, there exists $\delta > 0$ such that $B_Y(0, \delta) \subseteq \Phi(B_X(0, 1))$. Moreover, $\overline{\Psi(X)}$ being bounded, there exists $\rho > 0$ such that $\overline{\Psi(X)} \subseteq B_Y(0, \rho)$. Consequently, one has $\overline{\Psi(X)} \subseteq \Phi(B_X(0, \frac{\rho}{\delta}))$. Now, we fix an open convex bounded subset A of X such that $B_X(0, \frac{\rho}{\delta}) \subseteq A$ and put $K = \overline{\Psi(A)}$. By hypotheses, K is compact. Further, we fix a positive integer n such that $n \leq \dim(\Phi^-(0))$ and $z \in K$. Taking into account that $P_Y(\mathrm{gr}(\Phi))^* = Y$, by Lemma 2, we can choose $n + 1$ affinely-independent points $u_{z,1}, .., u_{z,n+1}$ in $\Phi^-(z) \cap A$. By Theorem 2 of [8], the multifunction $y \to \Phi^-(y)$ is lower semicontinuous in Y. So is the multifunction $y \to \overline{\Phi^-(y) \cap A}$. Moreover, its values are convex and closed, and, if $y \in K$, one has $\Phi^-(y) \cap A \neq \emptyset$. Hence, by applying the classical Michael theorem ([6, p. 98]) to the restriction to K of the latter multifunction, we obtain $n+1$ continuous functions $f_{z,1}, .., f_{z,n+1}$ from K into $\overline{A}$ such that, for any $y \in K$ and $i = 1, .., n + 1$, one has

$$\Phi(f_{z,i}(y)) = y \quad \text{and} \quad f_{z,i}(z) = u_{z,i}.$$

Now, for every $i = 1, .., n+1$, fix a neighborhood $U_{z,i}$ of $u_{z,i}$ in A such that, for any choice of points $w_i \in U_{z,i}$, one has that $w_1, .., w_{n+1}$ are affinely independent. Put

$$V_z = \bigcap_{i=1}^{n} f_{z,i}^{-1}(U_{z,i}),$$

V_z is a neighborhood of z in K. Since K is compact, there exist $z_1, .., z_p$ in K such that $K = \cup_{j=1}^{p} V_{z_j}$. At this point, for each $y \in K$, we put

$$F(y) = \operatorname{conv}(\{f_{z,j}(y) \ : \ j = 1, .., p \,; i = 1, .., n+1\}).$$

Since, for each $y \in K$, there exists $j \in \{1, .., p\}$ such that $y \in V_{z_j}$, that is $f_{z,i}(y) \in U_{z_j,i}$ for all $i = 1, .., n+1$, it follows that $F(y)$ is a nonempty convex compact subset of $\Phi^-(y) \cap \overline{A}$, with $\dim(F(y)) \geq n$. Further, F being a continuous multifunction ([6, p. 86 e p. 89]), one has that $F(K)$ is compact. So, put $C = \overline{\operatorname{conv}(F(K))}$, C is compact. Moreover, by Lemma 3, one has $\Psi(\overline{A}) \subseteq \overline{\Psi(A)} = K$. Hence, putting

$$G(x) = \overline{\operatorname{conv}(F(\Psi(x))} \quad \text{for each} \quad x \in C,$$

one has, since $C \subseteq \overline{A}$, that $G(x) \subseteq C$. At this point, by observing that $G : C \to 2^C$ is a lower semicontinuous multifunction with nonempty convex compact values and with $\dim(G(x)) \geq n$ for each $x \in C$, we deduce, by Proposition 2 of [2], that

$$\dim(\{x \in C : x \in G(x)\}) \geq n.$$

Now, if $x \in G(x)$, one has

$$x \in \overline{\operatorname{conv}(F(\Psi(x))} \subseteq \overline{\operatorname{conv}(\Phi^-(\Psi(x)))} \subseteq \overline{\Phi^-(\Psi(x))}.$$

Hence, by Lemma 1, we have $\Phi(x) \cap \Psi(x) \neq \emptyset$. Consequently,

$$\{x \in C : x \in G(x)\} \subseteq \{x \in X : \Phi(x) \cap \Psi(x) \neq \emptyset\}$$

and the conclusion follows from ([4, p. 220]).

If $\dim(\Phi^-(0)) = 0$, by the above proof, we can deduce that $\{x \in X : \Phi(x) \cap \Psi(x) \neq \emptyset\}$ is nonempty, hence the conclusion follows. □

A variant of Theorem 1 is the following:

Theorem 2. *Let X, Y be real Banach spaces, $\Phi : X \to 2^Y$ a lower semicontinuous multifunction with nonempty closed values, with convex graph and such that $\Phi(X) = Y$, and let $\Psi : X \to 2^Y$ be a lower semicontinuous multifunction with nonempty closed convex values and such that $\overline{\Psi(X)}$ is compact. Then, one has*

$$\dim(\{x \in X : \Phi(x) \cap \Psi(x) \neq \emptyset\}) \geq \dim(\Phi^-(0)).$$

Proof: Thanks to Theorem 2 of [8], the multifunction $y \to \Phi^-(y)$ is lower semicontinuous. Moreover, one has

$$\overline{\Psi(X)} \subseteq Y = \Phi(X)$$

and $K = \overline{\Psi(X)}$ is compact.

At this point, the conclusion follows by observing that it is possible to repeat the proof of Theorem 1 taking $A = X$. □

Remark. If Φ is as in Theorem 2 and Ψ as in Theorem 1, it is an open problem to establish if the following condition:

$$\dim(\{x \in X : \Phi(x) \cap \Psi(x) \neq \emptyset\}) \geq \dim(\Phi^{-}(0))$$

holds.

Applications to differential inclusions

Now, we prove two theorems concerning the covering dimension of the solution set of certain differential inclusions. We consider a free problem in Banach spaces. The following result concerns the case of infinite dimensional Banach spaces. It is an extension to differential inclusions of Theorem 2 of [10].

Theorem 3. *Let $I = [0,1]$, E be a infinite dimensional real Banach space, $F : I \times E \to 2^E$ be a lower semicontinuous multifunction, with nonempty closed values and such that:*

1) *there exists L>0 such that $d_H(F(t,x), F(t,y)) \leq L\|x-y\|$ for any $t \in I$, $x, y \in E$;*
2) *$F(t,\cdot)$ is a convex process for every $t \in I$.*

Finally, let $f : I \times E \to E$ be a uniformly continuous function with relatively compact range. Then, one has

$$\dim\{u \in C^1(I,E) : u'(t) \in f(t,u(t)) + F(t,u(t)) \ \ \textit{for each} \ \ t \in I\} = \infty.$$

Proof: Fix $x_0 \in E$, by Theorem 2.1 of [7], the set

$$\{u \in C^1(I,E) : \ u(0) = x_0, \ u'(t) \in F(t,u(t)) \ \text{ for each } \ t \in I\}$$

is nonempty. Then, if $x_1,..,x_n$ are n-linearly independent vectors in E and if $u_1,..,u_n$ are n-function in $C^1(I,E)$ such that

$$u_i(0) = x_i \ \text{ and } \ u_i'(t) \in F(t,u_i(t)) \ \text{ for each } \ t \in I, \quad i = 1,..,n,$$

it follows, in particular, that $u_1,..,u_n$ are n-linearly independent functions in the space $C^1(I,E)$. Consequently, since n is arbitrary, one has that the convex set

$$\{u \in C^1(I,E) : \ u'(t) \in F(t,u(t)) \quad \text{ for each } t \in I\}$$

is infinite-dimensional.

Now, for every $u \in C^1(I,E)$, we put

$$\Phi(u) = \{\varphi \in C^0(I,E) : \ \varphi(t) \in u'(t) - F(t,u(t)) \ \text{ for each } \ t \in I\}.$$

As it has just been seen, one has $\dim(\Phi^{-}(0)) = \infty$. Moreover, by condition 2) we can deduce that $\Phi : C^1(I,E) \to 2^{C^0(I,E)}$ is a convex process. Further, condition 1) assures that $\mathrm{gr}(\Phi)$ is closed in the space $C^1(I,E) \times C^0(I,E)$ equipped with the product topology. Now, if $h \in C^0(I,E)$, by applying once more Theorem 2.1 of [7], we deduce that

$$\Phi^{-}(h) = \{u \in C^1(I,E) :\ u'(t) \in F(t,u(t)) - h(t) \ \text{ for each } \ t \in I\}$$

is nonempty (and infinite-dimensional). Thus, $\Phi(C^1(I,E)) = C^0(I,E)$. Hence, by the Robinson-Ursescu theorem ([1, p. 54]), Φ is lower semicontinuous.

Finally, put $\Psi(u) = f(\cdot, u(\cdot))$ for every $u \in C^1(I,E)$. Thanks to the Ascoli-Arzela theorem, it is easily seen that $\Psi : C^1(I,E) \to C^0(I,E)$ is a continuous function, with bounded range and it maps bounded sets into relatively compact sets. At this point, the conclusion follows by applying Theorem 1 to Φ and Ψ. □

If $E = \mathbb{R}^n$, we obtain the following version of Theorem 3, which is an extension to differential inclusions of Theorem 3 of [10]:

Theorem 4. *Let $I = [0,1]$, $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a lower semicontinuous multifunction, with nonempty closed values and such that:*

1) *there exists $L>0$ such that $d_H(F(t,x),F(t,y)) \le L\|x-y\|$ for any $t \in I$, $x,y \in \mathbb{R}^n$;*
2) *$F(t,\cdot)$ is a convex process for any $t \in I$.*

Finally, let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous bounded function. Then, one has

$$\dim\{u \in C^1(I,\mathbb{R}^n) : u'(t) \in f(t,u(t)) + F(t,u(t)) \ \text{ for each } \ t \in I\} \ge n.$$

PROOF: The proof is omitted since it is similar to the previous one. □

For other works concerning the topological dimension of the solution set of a differential inclusion see also [5] and [3].

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