Comment.Math.Univ.Carolin. 41,3 (2000)611–617

On weakly bisequential spaces

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Abstract. Weakly bisequential spaces were introduced by A.V. Arhangel'skii [1], in this paper. We discuss the relations between weakly bisequential spaces and metric spaces, countably bisequential spaces, Fréchet-Urysohn spaces.

Keywords: bisequential spaces, filter base, s-map
Classification: 54E99, 54A25

1. Introduction

Let X be a topological space. A filter base (ω-filter base) is defined to be a family ξ of nonempty sets such that if $A, B \in \xi$ (for countable subfamily $\mu \subset \xi$), there is a $C \in \xi$ such that $C \subset A \cap B$ ($C \subset \cap\mu$). A filter base ξ converges to a point x in a space X (accumulates at the point x) if each neighborhood base of x contains an element of ξ (respectively, if $x \in \cap\{\bar{P} : P \in \xi\}$). We say that a filter base ξ meshes with a filter base η if every $A \in \xi$ intersects every $B \in \eta$. A space X is said to be bisequential (countably bisequential, weakly bisequential) at a point $x \in X$ if for any filter base (countable filter base, ω-filter base) in X accumulating at x there is a countable filter base μ in X that converges to x and meshes with ξ. A space is called bisequential (countable bisequential, weakly bisequential) if it is bisequential (countably bisequential, weakly bisequential) at each point.

A space X is called Fréchet-Urysohn if given $A \subset X$, $x \in X$, and $x \in \bar{A}$, there exists a sequence $\{x_n : n \in N\} \subset A$ which converges to x.

A map $f : X \to Y$ is weakly bi-quotient if, whenever $y \in Y$ and $\mathcal{U}$ is a cover of $f^{-1}(y)$ by open subsets of X, then countably many $f(U)$ with $U \in \mathcal{U}$, cover a neighborhood base of y in Y.

Let S_κ be a quotient space of the topological sum of κ many convergent sequences by identifying all limit points to a point. S_ω is called sequential fan.

All the maps in this paper are continuous and onto, spaces are regular T_1. Readers may refer to [1], [2] and [3] for unstated notations and definitions.

The following diagrams indicate the relation between weakly bisequential spaces (bi-quotient maps) and other spaces (maps).

bisequential → weakly bisequential → Fréchet-Urysohn.

bisequential → countably bisequential → Fréchet-Urysohn.

bi-quotient → weakly bi-quotient → pseudo-open.

bi-quotient → countably bi-quotient → pseudo-open.

2. Main results

The following proposition is quite similar to the Proposition 3.2 in [7].

Proposition 2.1. *The following properties of a map $f : X \to Y$ are equivalent:*

(a) *f is weakly bi-quotient;*
(b) *if an ω-filter base $\mathcal{F}$ accumulates at y in Y, then $f^{-1}(\mathcal{F})$ accumulates at some $x \in f^{-1}(y)$.*

PROOF: (a)$\to$(b). Suppose that $f^{-1}(\mathcal{F})$ does not accumulate at any $x \in f^{-1}(y)$. For $x \in f^{-1}(y)$, there is a $F_x \in \mathcal{F}$ and a nbd V_x of x such that $V_x \cap f^{-1}(F_x) = \emptyset$. $\{V_x : x \in f^{-1}(y)\}$ is an open cover for $f^{-1}(y)$. Since f is weakly biquotient, there exists a countable family $\mathcal{U}'=\{V_{x_i} : i \in N\} \subset \{V_x : x \in f^{-1}(y)\}$ such that $y \in intf(\cup\mathcal{U}')$. Let $\{F_{x_i} : i \in N\} \subset \mathcal{F}$ such that $V_{x_i} \cap f^{-1}(F_{x_i}) = \emptyset$ for $i \in N$. So $f(V_{x_i}) \cap F = \emptyset$ for all $i \in N$, where $F \subset \cap\{F_{x_i} : i \in N\}$. Then $f(\cup\mathcal{U}') \cap F = \emptyset$, but $y \in \bar{F}$ and $f(\cup\mathcal{U}')$ is a nbd of y, a contradiction.

(b)$\to$(a). Suppose that f is not weakly biquotient, then there is an open cover $\mathcal{U}$ of $f^{-1}(y)$ for some $y \in Y$ such that for any countable subfamily λ of $\mathcal{U}$, $y \notin intf(\cup\mathcal{U}')$. Let $\mathcal{F} = \{Y - f(\cup\lambda) : \lambda \subset \mathcal{U}, |\lambda| \leq \omega\}$, then $\mathcal{F}$ is an ω-filter base accumulating at y. By (b), $f^{-1}(\mathcal{F})$ accumulates at some $x \in f^{-1}(y)$. Let $U \in \mathcal{U}$ with $x \in U$, let $\lambda = \{U\}$. $U \cap (f^{-1}(Y - f(U))) \neq \emptyset$, hence $f(U) \cap (Y - f(U)) \neq \emptyset$, a contradiction. □

Similar to the proof of Theorem 3.D.2 in [7], we have the following:

Theorem 2.1. *A topological space Y is a weakly bisequential space if and only if it is a weakly bi-quotient image of a metrizable space.*

Corollary 2.1. *A weakly bisequential space is Fréchet-Urysohn* [1].

Theorem 2.2. *A closed image X of a metric space is a closed s-image of a metric space if and only if X is weakly bisequential.*

PROOF: It is easy to see that a closed s-mapping is weakly bi-quotient, so X is weakly bisequential. (In fact, a pseudo-open Lindelöf map is weakly bi-quotient).

Now we prove that a weakly bisequential closed image of a metric space is a closed s-image of a metric space. First, we prove that S_{ω_1} is not weakly bisequential.

We write $S_{\omega_1} = \{\infty\} \cup \{S_\alpha : \alpha < \omega_1\}$, where S_α is a sequence converging to ∞. Let $H_\alpha = \cup\{S_\beta : \beta < \alpha\}$ for $\alpha < \omega_1$, $\infty \in \bar{H_\alpha}$. Suppose S_{ω_1} is weakly bisequential, then there exists a decreasing sequence $\{A_n : n \in N\}$ such that $\{A_n : n \in N\}$ meshes with $\{H_\alpha : \alpha < \omega_1\}$. We may choose $x_n \in A_n \cap S_{\alpha_n} - \{x_1, \dots, x_{n-1}\}$ recursively, then $x_n \to \infty$, a contradiction.

X is a closed image of a metric space, so it is a Fréchet-Urysohn space with a σ-hereditarily closure preserving k-network ([4]). X contains no closed copy of S_{ω_1}, hence X is a Fréchet-Urysohn and $\aleph$-space ([5]), and thus it is a closed s-image of a metric space ([6]). □

Next, we discuss some relations between weakly bisequential spaces and other topological spaces.

From the definition, we know that bisequential spaces are weakly bisequential. Weakly bisequential spaces are Fréchet-Urysohn ([1]). Also, it is well known that countably bisequential spaces are Fréchet-Urysohn. What is the relation between countably bisequential spaces and weakly bisequential spaces? In fact, we have the following examples:

Proposition 2.2. *There exists a weakly bisequential space which is not countably bisequential.*

PROOF: The sequential fan S_ω is such a space, since every countable Fréchet-Urysohn space is weakly bisequential ([1]), so it is weakly bisequential. But it is not countably bisequential. Suppose not, we write $S_\omega = \{\infty\} \cup \{S_n : n \in N\}$, where S_n is a sequence converging to ∞. Let $H_n = \cup\{S_i : i \geq n\}$. Then $\{H_n : n \in N\}$ is a decreasing sequence accumulating at ∞ and we choose a sequence $\{x_k\}$ such that $x_k \in H_k \cap S_{n_k}$ for each $k \in N$ and $\{x_k\}$ converges to ∞, this is a contradiction. □

Proposition 2.3. *There exists a countably bisequential space which is not weakly bisequential.*

PROOF: Let X be the Σ-product of $\{D_\alpha : \alpha < \omega_1\}$, where $D_\alpha = \{0, 1\}$ for each $\alpha < \omega_1$. It is well known that X is countably bisequential. But X is not weakly bisequential ([1]). □

Simon [8] gave an example that the product of two compact Fréchet-Urysohn spaces is not Fréchet-Urysohn. We prove that the spaces in Simon's example are weakly bisequential. So, not every product of compact weakly bisequential spaces is Fréchet-Urysohn.

Let $\mathcal{P}$ be an almost disjoint family in ω, let $\Omega = \omega \cup \{P : P \in \mathcal{P}\}$. Endow Ω with a topology as follow: each singleton in ω is open, for $P \in \mathcal{P}$, a neighborhood base of P is $\{P\} \cup \{P - A : A \in [P]^{<\omega}\}$. Then Ω is a locally compact space. Let Ω' be the one point compactification of Ω, we write $\Omega' {=} \Omega \cup \{\infty\}$.

Theorem 2.3. Ω' *is weakly bisequential if it is Fréchet-Urysohn.*

PROOF: Let $\mathcal{F}$ be an ω-filter base in Ω' accumulating at ∞, let $\mathcal{F}' = \mathcal{F} \cap \omega$, $\mathcal{F}'' = \mathcal{F} \cap \mathcal{P}$.

Case 1. $\mathcal{F}'$ is an ω-filter base in $\{\infty\} \cup \omega$ accumulating at ∞.

By [1, Theorem 6], $\{\infty\} \cup \omega$ is weakly bisequential. So there is a countable decreasing sequence $\{A_n : n \in N\}$ which converges to ∞ and meshes with $\mathcal{F}'$. Hence $\{A_n : n \in N\}$ meshes with $\mathcal{F}$.

Case 2. $\mathcal{F}'$ is not an ω-filter base in $\{\infty\} \cup \omega$ accumulating at ∞.

Then $\mathcal{F}''$ is an ω-filter base in $\{\infty\} \cup \mathcal{P}$ accumulating at ∞. By [7, Example 10.15], $\{\infty\} \cup \mathcal{P}$ is bisequential, so there is a countable decreasing family

$\{A_n : n \in N\}$ which converges to ∞ and meshes with $\mathcal{F}''$. Hence it meshes with $\mathcal{F}$.

So Ω' is weakly bisequential. □

Theorem 2.4. *There are two compact weakly bisequential spaces X and Y such that $X \times Y$ is not Fréchet-Urysohn.*

PROOF: Let X and Y be the spaces in Simon's example ([8]). By the theorem above, X, Y are weakly bisequential, but $X \times Y$ is not Fréchet-Urysohn. □

Proposition 2.4. *There exists a compact, weakly bisequential space which is not bisequential.*

PROOF: In fact, both X and Y in Theorem 2.4 are not bisequential. Suppose one of X and Y is bisequential, then so is α_3 ([2]). So the product $X \times Y$ is Fréchet-Urysohn ([2]), a contradiction. □

Theorem 2.5. *Let X be a discrete space and $X^* = X \cup \{\infty\}$ the one point compactification of X. Then X^* is weakly bisequential if and only if it is bisequential.*

PROOF: We only prove sufficiency. If the cardinality of X is non-measurable then, by [7, Example 10.15], X^* is bisequential. If the cardinality of X is measurable, by [7, Lemma 10.14], there is an ultrafilter $\mathcal{F}$ such that $\cap\mathcal{F} = \emptyset$. But $\cap\mathcal{F}' \in \mathcal{F}$ for every countable $\mathcal{F}' \subset \mathcal{F}$. $\mathcal{F}$ is an ω-filter base accumulating at ∞ [7, Lemma 10.14], then there is a sequence $\{A_n : n \in N\}$ which converge to ∞ and meshes with $\mathcal{F}$. $\{A_n : n \in N\} \subset \mathcal{F}$ because $\mathcal{F}$ is an ultrafilter. $\cap\{A_n : n \in N\} \in \mathcal{F}$, so $\cap\{A_n : n \in N\} \cap X \neq \emptyset$, hence $\{A_n : n \in N\}$ does not converge to ∞, a contradiction. □

Proposition 2.5 ($\exists$ measurable cardinal)**.** *There is a compact, countably bisequential space that is not weakly bisequential.*

PROOF: Let X^* be the space in Example 10.15 in [7]. Then X^* is not bisequential. By Theorem 2.5, X^* is not weakly bisequential. □

A space X is called weakly quasi-first countable ([9]) if for each $i \in N$, there exists a mapping $B^i : N \times X \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the power set of X, such that the following hold:

(i) fix $i \in N$ for each $n \in N$ and $x \in X$, $B^i(n+1, x) \subset B^i(n, x)$, and $\{x\} = \cap\{B^i(n, x) : n \in N\}$; and

(ii) a subset V of X is open if and only if for each $y \in V$ and for each $i \in N$ there exists $n(i)$ with $B^i(n(i), y) \subset V$.

If $B^i = B$ for $i \in N$, then X is called weakly first countable. Obviously, weakly first countable is weakly quasi-first countable.

Theorem 2.6. *A Fréchet-Urysohn, weakly quasi-first countable space X is weakly bisequential.*

Proof: For $x \in X$, let $\mathcal{F}$ be an ω-filter base accumulating at x. Since X is weakly quasi-first countable, there is a family of subsets of X, say, $\{B^i(n,x) : n \in N, i \in N\}$ satisfying (i) and (ii).

Claim 1. *There exists $i_0 \in N$ such that $\{B^{i_0}(n,x) : n \in N\}$ meshes with $\mathcal{F}$.*

Suppose not; then for each $i \in N$, there exist $n(i)$ and $F_i \in \mathcal{F}$ such that $B^i(n(i),x) \cap F_i = \emptyset$. Let $F \in \mathcal{F}$ where $F \subset \cap\{F_i : i \in N\}$. Then $F \cap B^i(n(i),x) = \emptyset$ for all $i \in N$. Since X is Fréchet-Urysohn and x is an accumulating point of F, there is $\{x_n : n \in N\} \subset F$, $x_n \to x$. $\{x_n : n \in N\} \cap B^i(n(i),x) = \emptyset$, it is easy to see that $\{x_n : n \in N\}$ is closed, a contradiction.

So there is $i_0 \in N$ such that $\{B^{i_0}(n,x) : n \in N\}$ converges to x and meshes with $\mathcal{F}$, hence X is weakly bisequential. □

Remark 2.1. It is natural to ask whether every weakly bisequential space is quasi-weakly first countable, the answer is 'No'. The one point compactification of a discrete space Y whose cardinality is 2^ω is such a space. Y is bisequential [7, Example 10.15] but not first countable. So Y is not weakly quasi-first countable because of the following Corollary 2.2.

A space X is called an α_4 space if for every point $x \in X$ and any countable family $\{S_n : n \in N\}$ of sequences converging to x one can find a sequence S converging to x which meets infinitely many S_n.

A subset B of X is called a sequential neighborhood of $x \in X$ if for every sequence converging to x is eventually in B.

Theorem 2.7. *A space X is weakly first countable if and only if X is a weakly quasi-first countable, α_4 space.*

Proof: Necessity is obvious. We only prove sufficiency.

For $x \in X$, let $\mathcal{F}_x$ be the family $\{B^i(n,x) : n \in N, i \in N\}$ that satisfies (i) and (ii) in the definition of weakly quasi-first countable. Let

$\mathcal{B}_x = \{\cup\mathcal{F}' : \mathcal{F}' \subset \mathcal{F}_x, |\mathcal{F}'| < \omega$, and $\cup\mathcal{F}'$ is a sequential neighborhood of $x\}$.

We can see that $\mathcal{B}_x$ is countable, let $\mathcal{B} = \cup\{\mathcal{B}_x : x \in X\}$.

We will prove that $\mathcal{B}$ is a weak base for X.

Let U be a subset of X, for each $x \in U$. If there is a $B \in \mathcal{B}_x$ such that $x \in B \subset U$, then U is open.

In fact, U is a sequential neighborhood for each $x \in U$, hence U is sequential open. But X is a sequential space [9], so U is open.

Let V be an open subset of X, we prove that for $x \in V$, there is $B \in \mathcal{B}_x$ such that $B \subset V$.

Let $\mathcal{P} = \{F \in \mathcal{F}_x : F \subset V\}$, and we rewrite $\mathcal{P} = \{F_n : n \in N\}$.

Claim 2. *There is $m \in N$ such that $\cup\{F_n \in \mathcal{P} : n \leq m\}$ is a sequential neighborhood of x.*

Suppose not, there is a sequence $\{x^{(1)}(n)\}$ with $x^{(1)}(n) \to x$ and $\{x^{(1)}(n)\} \cap F_1 = \emptyset$. Since $F_1 \cup F_2$ is not a sequential neighborhood of x, then there is a sequence $x^{(2)}(n)$ with $x^{(2)}(n) \to x$ and $\{x^{(2)}(n)\} \cap (F_1 \cup F_2) = \emptyset$. continuing this way, we get countably many convergent sequences $\{x^{(i)}(n)\}, (i \in N)$ with $x^{(i)}(n) \to x$ and $\{x^{(i)}(n)\} \cap \cup\{F_j : j \leq i\} = \emptyset$. X is an α_4-space, so there is a sequence $S = \{y_n : n \in N\}$ which converges x and meets infinitely many $\{x^{(i)}(n)\}$. We prove that S is eventually in some finite union of a subfamily of $\mathcal{P}$.

If not, pick $n_1 \in N$ such that $B^1(n_1) \subset U$. Since $B^1(n_1)$ is not a sequential neighborhood of x, there is subsequence $S_1 \subset S$, $S_1 \cap B^1(n_1) = \emptyset$ and $S - S_1$ is eventually in $B^1(n_1)$, choose $y_{m_1} \in S_1$. Pick $n_2 \in N$ such that $B^2(n_2) \subset U$. Since S_1 is not eventually in $B^2(n_2)$, there is a subsequence $S_2 \subset S_1$ such that $S_2 \cap B^2(n_2) = \emptyset$ and $S_1 - S_2$ is eventually in $B^2(n_2)$. Pick $y_{m_2} \in S_2 - \{y_{m_1}\}$. Suppose that $B^i(n_i)$, S_i, y_{m_i} $(i \leq j)$ have been selected in such a way that $S_k \subset S_l$ if $k < l$, S_i is infinite for $i \leq j$. $S_i \cap B^i(n_i) = \emptyset$, $S_{i-1} - S_i$ is eventually in $B^i(n_i)$. Since S is not contained in any finite union of subfamily of $\mathcal{P}$, choose $B^{j+1}(n_{j+1}) \subset U$, S_j is not eventually in $B^{j+1}(n_{j+1})$, there is an infinite subsequence S_{j+1} of S_j such that $S_{j+1} \cap B^{j+1}(n_{j+1}) = \emptyset$. Pick $y_{m_{j+1}} \in S_{j+1} - \{y_{m_i} : i \leq j\}$.

We can get a subsequence $S' = \{y_{m_i}\}$ converging to x. From the construction above, for each $i \in N$, $S' \cap B^i(n_i) = \emptyset$, so it is not difficult to see that S' is closed, a contradiction.

But from the selection of S, S cannot be eventually in any finite union of $\mathcal{P}$. A contradiction. So the claim has been proved.

So the finite union of $\mathcal{P}$ in claim 2 is an element of $\mathcal{B}_x$. Hence $\mathcal{B}$ is a weak base for X, and X is weakly first countable. □

Corollary 2.2. *Let X be a countably bisequential space. Then X is first countable if X is weakly quasi-first countable.*

Proof: Every countably bisequential space is an α_4 space. Thus X is weakly first countable by Theorem 2.7. It is well known that weakly first countable, Fréchet Urysohn spaces are first countable. □

3. Questions

Question 3.1. *Let X and Y be weakly bisequential. Is $X \times Y$ weakly bisequential provided $X \times Y$ is Fréchet-Urysohn?*

Let $\mathcal{P}$ be a cover for X. $\mathcal{P}$ is called a $cs^*-network$ if for any $x \in X$, $x \in U$ with U open and a sequence S converging to x, there is a $P \in \mathcal{P}$ such that $x \in P$, $P \subset U$ and P contains a subsequence of S.

Question 3.2. *Let X be a weakly bisequential space with a point-countable k-network. Does X have a point-countable cs*-network?*

Remark 3.1. If the answer to Question 3.2 is positive, then we can give an affirmative answer to the Question 10.2 in [3].

Question 3.3. *Let X be a Fréchet-Urysohn space with a point-countable k-network. Is X weakly bisequential if it contains no closed copy of S_{ω_1}?*

Question 3.4. *Let X be a Fréchet-Urysohn space with countable network. Is X weakly bisequential?*

Question 3.5. *Is it possible to characterize weak bisequentiality in terms of the Fréchet-Urysohn property?*

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(*Received* November 22, 1999, *revised* March 27, 2000)