Perfect compactifications of functions

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Abstract. We prove that the maximal Hausdorff compactification χf of a T_2-compactifiable mapping f and the maximal Tychonoff compactification βf of a Tychonoff mapping f (see [P]) are perfect. This allows us to give a characterization of all perfect Hausdorff (respectively, all perfect Tychonoff) compactifications of a T_2-compactifiable (respectively, of a Tychonoff) mapping, which is a generalization of two results of Skljarenko [S] for the Hausdorff compactifications of Tychonoff spaces.

Keywords: Hausdorff (Tychonoff) mapping, compactification of a mapping, maximal Hausdorff (Tychonoff) compactification of a mapping, perfect compactification of a mapping

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1. Introduction

In 1961, E.G. Skljarenko introduced the notion of the *perfect compactification* of a Tychonoff space. Given a Tychonoff space X, we say that a compactification γX of X is *perfect* if $cl_{\gamma X}(bd_X(U)) = bd_{\gamma X}(\langle U\rangle_{\gamma X})$ for every open set U of X, where $\langle U\rangle_{\gamma X}$ denotes the *maximal extension* of U relatively to γX, that is the maximal open set of γX whose trace on X is U.

In [S], Skljarenko, using proximal techniques, gave some characterizations of the perfect compactifications and he proved that γX is a perfect compactification of X if and only if the canonical map $\varphi_\gamma : \beta X \to \gamma X$ is monotone (i.e. every its fibre is connected) and so — in particular — that the Stone-Čech compactification βX is a perfect compactification of X.

Further results concerning this class of compactifications were given by Diamond in [D].

Recently, the first author [N] has generalized the notion of perfectness from a Hausdorff compactification of a Tychonoff space to a generic extension of an arbitrary space simplifying the treatment in a more general setting and obtaining several new characterizations.

Since it is clear now what is the compactification of a continuous mapping and since the notion of a topological space is the simplest case of the notion of a

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continuous mapping (a space is its mapping to the one-point space), it is natural to extend to continuous mappings some results concerning compactifications of spaces.

The study of compactifications (= perfect extensions) of a continuous mapping was started in 1953 by Whyburn [W].

In [P], using techniques of partial topological products, Pasynkov described a general way to obtain all Tychonoff (i.e. completely regular, T_0-) compactifications of Tychonoff mappings between arbitrary spaces and he proved that the poset $TK(f)$ of all the Tychonoff compactifications of a Tychonoff mapping $f : X \to Y$ admits the maximal compactification $\beta f : \beta_f X \to Y$ which is the exact analogue of the Stone-Čech compactification of a Tychonoff space (since if $|Y| = 1$, X becomes a Tychonoff space and the domain $\beta_f X$ of βf coincides with βX).

The following similar result is obtained in [BN]:
If a continuous mapping $f : X \to Y$ is T_2-compactifiable (i.e. f has some Hausdorff compactification) then it has the maximal compactification $\chi f : \chi_f X \to Y$ in the poset $HK(f)$ of all Hausdorff compactifications of f.

Let us note in this connection that — unlike the corresponding case for spaces — there exist Hausdorff compact mappings which are not Tychonoff ([HI], [C]). Thus, it is necessary to consider the cases of Tychonoff and T_2-compactifiable mappings separately. It would be interesting
to find wide enough conditions when every Hausdorff compactification of a Tychonoff mapping is Tychonoff.

In this paper, we generalize to continuous mappings two extrinsic characterizations of perfect compactifications of spaces obtained by Skljarenko in [S].

We will prove that:

(1) the maximal Hausdorff (maximal Tychonoff) compactification χf (respectively βf) of a T_2-compactifiable (Tychonoff) mapping f is a perfect extension of f (Theorems 3.1, 3.9);
(2) a Hausdorff (Tychonoff) compactification bf of a T_2-compactifiable (Tychonoff) mapping f is a perfect extension of f if and only if the canonical morphism of χf (respectively βf) to bf is monotone (Theorems 3.6, 3.11).

2. Preliminaries

Throughout the paper, the word "space" will mean "topological space".

If X is a space, $\tau(X)$ will denote the set of all the open subsets of X while $\sigma(X)$ will denote the set of all the closed subsets of X.

As usual, for any pair of spaces X and Y, $C(X, Y)$ denotes the set of all continuous mappings from X to Y and $C^*(X)$ is the set of all continuous real bounded functions on X.

Undefined notions are used as in [E].

Definitions ([N], [S]). Let Y be an extension of a space X, $U \in \tau(X)$ and $x \in Y \backslash X$.

We say that the pair (x, U) is *perfect* if $x \in cl_Y(bd_X(U))$ provided $x \in bd_Y(\langle U \rangle_Y)$, where $\langle U \rangle_Y = \bigcup\{V \in \tau(Y) : V \cap X = U\}$ is the *maximal extension of U in Y*, i.e. the maximum open set of Y whose trace on X is U.

We say that Y is a *perfect extension of X relatively to x* if for every $W \in \tau(X)$ the pair (x, W) is perfect.

We say that Y is a *perfect extension of* X if it is a perfect extension of X relatively to every point of its remainder $Y \backslash X$.

Definition ([N], [S]). Let Y be an extension of X and $x \in Y \backslash X$. We say that $Y \backslash X$ *cuts X at x* if there exists some neighborhood O of x in Y and a pair U, V of disjoint open sets of X such that $O \cap X = U \cup V$ and $x \in cl_Y(U) \cap cl_Y(V)$.

The following characterization is given in [N].

Proposition 2.1. *Let Y be an extension of a space X and $x \in Y \backslash X$. Then Y is a perfect extension of X relatively to x if and only if $Y \backslash X$ does not cut X at x.*

Now, we define our framework.

For any fixed space Y, we consider the category $\mathbf{Top}_Y$, where

$$Ob(\mathbf{Top}_Y) = \{f \in C(X, Y) : \quad X \in Ob(\mathbf{Top})\}$$

is the class of the *objects* and, for every pair $f : X \to Y$, $g : Z \to Y$ of objects,

$$M(f, g) = \{\lambda \in C(X, Z) : \quad g \circ \lambda = f\}$$

is the class of the *morphisms* from f to g, whose generic representant is denoted for short by $\lambda : f \to g$.

A morphism $\lambda : f \to g$ from $f : X \to Y$ to $g : Z \to Y$ will be called *surjective* (resp. *dense*) if $\lambda(X) = Z$ (resp. if $\lambda(X)$ is dense in Z).

If $\lambda : f \to g$ is a surjective morphism, we say that g is the *image* of f (*by the morphism* λ) and we write $g = \lambda(f)$.

Moreover, we say that a morphism $\lambda : f \to g$ from $f : X \to Y$ to $g : Z \to Y$ is an *embedding* (resp. a *homeomorphism*) if the mapping $\lambda : X \to Z$ is an embedding.

A mapping $g : Z \to Y$ is called an *extension* of $f : X \to Y$ if some dense embedding $\lambda : f \to g$ is fixed (as usual X and f are identified with $\lambda(X)$ and $g_{|\lambda(X)}$ respectively).

A morphism $\lambda : g \to h$ between two extensions $g : Z \to Y$ and $h : W \to Y$ of a mapping $f : X \to Y$ will be called *canonical* if $\lambda_{|X} = id_X$.

Now, let us recall some other definitions.

Definitions. A mapping $f : X \to Y$ is said to be T_0 ([P]) if for any $x, x' \in X$ such that $x \neq x'$ and $f(x) = f(x')$ there exist either a neighborhood of x in X which does not contain x' or a neighborhood of x' in X not containing x.

A mapping $f : X \to Y$ is said to be *Hausdorff* (or T_2) [P] if for every $x, x' \in X$ such that $x \neq x'$ and $f(x) = f(x')$ there are disjoint neighborhoods of x and x' in X.

We shall say that $f : X \to Y$ is *compact* if it is perfect (i.e. closed and all its fibres are compact).

A mapping $f : X \to Y$ is said to be *completely regular* [P] if for every $F \in \sigma(X)$ and $x \in X \backslash F$ there exists a neighborhood O of $f(x)$ in Y and a continuous mapping $\varphi : f^{-1}(O) \to [0, 1]$ such that $\varphi(x) = 1$ and $\varphi(F \cap f^{-1}(O)) \subseteq \{0\}$.

A completely regular, T_0 mapping is called *Tychonoff* (or $T_{3\frac{1}{2}}$) [P].

The following lemma is evident.

Lemma 2.2. *Every morphism defined on a Hausdorff mapping is a Hausdorff mapping too.*

The next lemma from [P] will be useful in the following.

Lemma 2.3. *Let $f : X \to Y$ be a Hausdorff mapping, $y \in Y$ and let K_1, K_2 be two disjoint compact subsets of X such that $K_1 \cup K_2 \subseteq f^{-1}(\{y\})$. Then K_1 and K_2 have disjoint neighborhoods in X.*

Corollary 2.4. *If $f : X \to Y$ is a Hausdorff compact mapping, $y \in Y$ and K_1, K_2 are closed disjoint subsets of $f^{-1}(\{y\})$ then K_1 and K_2 have disjoint neighborhoods in X.*

Definition. A restriction $f_{|X'} : X' \to Y$ to $X' \subseteq X$ of a mapping $f : X \to Y$ is called a *closed submapping* of f if X' is a closed subset of X.

Obviously every closed submapping of a compact mapping is compact too.

Many well-known statements which hold in the category **Top** have their analogue (and hence a generalization) in $\mathbf{Top}_Y$. The following properties were given in [P].

Proposition 2.5. *Let λ and μ be morphisms from a mapping $f : X \to Y$ to a Hausdorff mapping $h : Z \to Y$ and D be a dense subset of X. Then, if $\lambda_{|D} = \mu_{|D}$, the morphisms λ and μ coincide.*

Proposition 2.6. *The composition of two compact Hausdorff mappings is compact Hausdorff.*

Proposition 2.7. *Every image $\lambda(k)$ of a compact mapping $k : X \to Y$ (under a morphism λ) is compact.*

Proposition 2.8. *Every compact submapping $h_{|X'} : X' \to Y$ of a Hausdorff mapping $h : X \to Y$ is a closed submapping of h.*

Proposition 2.9. *Every morphism $\lambda : k \to h$ from a compact mapping $k : X \to Y$ to a Hausdorff mapping $h : Z \to Y$ is a perfect mapping.*

Definition. We say that a mapping $c : X^c \to Y$ is a *compactification* of a mapping $f : X \to Y$ if it is a compact extension of f.

Definitions. Let $c : X^c \to Y$ and $d : X^d \to Y$ be compactifications of a mapping $f : X \to Y$. We say that:

- *c is projectively larger than d* (relatively to f) and we write that $c \geq_f d$ (or $c \geq d$, for short) if there exists some canonical morphism $\lambda : c \to d$;
- *c is equivalent to d* (relatively to f) and we write that $c \equiv_f d$ (shortly, $c \equiv d$) if there exists a canonical homeomorphism $\lambda : c \to d$.

In [BN], the following useful result is obtained:

Proposition 2.10. *Let $c : X^c \to Y$ and $d : X^d \to Y$ be Hausdorff compactifications of a mapping $f : X \to Y$. Then $c \equiv_f d$ if and only if $c \geq d$ and $d \geq c$.*

Definition. A Hausdorff mapping $f : X \to Y$ will be called *T_2-compactifiable* (or *Hausdorff compactifiable*) if it has some Hausdorff compactification.

All Hausdorff compactifications of any T_2-compactifiable mapping form a set up to their equivalence (see [BN]).

Definition. If $f : X \to Y$ is a T_2-compactifiable mapping, $HK(f)$ will denote the set of all Hausdorff compactifications of f (up to the equivalence $\equiv_f$).

So, by 2.10, it follows that $(HK(f), \geq)$ is a poset and, for any pair of Hausdorff compactifications $c, d \in HK(f)$, we can write $c = d$ instead of $c \equiv_f d$, that is, we do not distinguish between equivalent Hausdorff compactifications.

In [BN], the following is proved:

Theorem 2.11. *For any T_2-compactifiable mapping $f : X \to Y$, there is in the poset $(HK(f), \geq)$ a* maximal Hausdorff compactification *$\chi f : \chi_f X \to Y$ of f.*

From 2.5 it follows — in particular — that for any Hausdorff compactification $bf : X^b \to Y$ of a T_2-compactifiable mapping $f : X \to Y$ there exists a unique canonical morphism $\lambda_b : \chi f \to bf$.

The following useful property can be found in [P].

Proposition 2.12. *Let $bf : X^b \to Y$ and $bg : Z^b \to Y$ be Hausdorff compactifications of $f : X \to Y$ and $g : Z \to Y$ respectively, $\lambda : f \to g$ be a perfect morphism and $\widetilde{\lambda} : bf \to bg$ be a morphism such that $\widetilde{\lambda}_{|X} = \lambda$. Then $\widetilde{\lambda}(X^b \backslash X) \subseteq Z^b \backslash Z$.*

In [P], Pasynkov proved that any Tychonoff mapping $f : X \to Y$ has a Tychonoff (and hence Hausdorff) compactification.

Definition. For any Tychonoff mapping $f : X \to Y$, we will denote by $TK(f)$ the set of all Tychonoff compactifications of f (up to the equivalence $\equiv_f$).

In [P], it is shown that, for any Tychonoff mapping $f : X \to Y$, there exists in $(TK(f), \geq)$ a *maximal Tychonoff compactification $\beta f : \beta_f X \to Y$* of f.

Definition. For any mapping $g : T \to Y$ and any $U \in \tau(Y)$, let $C^*(U, g) = C^*(g^{-1}(U))$.

The following characterization of βf is given in [P].

Theorem 2.13. *For any Tychonoff compactification $bf : X^b \to Y$ of a Tychonoff mapping $f : X \to Y$, the following conditions are equivalent:*

(1) $bf = \beta f$;
(2) *for every $U \in \tau(Y)$ and $\varphi \in C^*(U, f)$, there exists a unique extension $\widetilde{\varphi} \in C^*(U, bf)$;*
(3) *for every compact Tychonoff mapping $k : Z \to Y$ and every morphism $\lambda : f \to k$ there exists a morphism $\widetilde{\lambda} : bf \to k$ which extends λ.*

Proposition 2.14. ([P]). *For any Tychonoff compactification $bf : X^b \to Y$ of a Tychonoff mapping $f : X \to Y$ there exists a unique (perfect) canonical morphism $\mu_b : \beta f \to bf$ and it results $\mu_b(\beta_f X \setminus X) = X^b \setminus X$.*

3. Perfectness of the maximal compactifications of a mapping

In [S], Skljarenko proved that a compactification γX of a Tychonoff space X is perfect if and only if the canonical map $\varphi_\gamma : \beta X \to \gamma X$ is monotone (that is, every its fibre is connected) and hence — in particular — that the Stone-Čech compactification βX of X is a perfect compactification of X.

In the following we will obtain similar (and more general) results for compactifications of a mapping.

Definition. Let $\widetilde{f} : \widetilde{X} \to Y$ be an extension of a mapping $f : X \to Y$. We say that $\widetilde{f}$ is a *perfect extension* of f if its domain $\widetilde{X}$ is a perfect extension of the space X.

Theorem 3.1. *The maximal Hausdorff compactification $\chi f : \chi_f X \to Y$ of a T_2-compactifiable mapping $f : X \to Y$ is a perfect extension of f.*

PROOF: Suppose by contradiction that χf is not a perfect extension of f. By 2.1, there exists some $x \in \chi_f X \setminus X$ such that $\chi_f X \setminus X$ cuts X at x, i.e. there are a neighborhood U of x in $\chi_f X$ and a pair U_0, U_1 of disjoint open subsets of X such that $x \in cl_{\chi_f X}(U_0) \cap cl_{\chi_f X}(U_1)$ and $U \cap X = U_0 \cup U_1$. Note that $G = cl_U(U_0) \cap cl_U(U_1) \subseteq \chi_f X \setminus X$.

Let X' be the disjoint union of $\chi_f X \setminus U$ and $U_i' = cl_U(U_i)$ (for $i = 0, 1$). The copy of G lying in U_i' will be denoted by G_i and the copy of a point $t \in G$ lying in G_i will be denoted by t_i (for $i = 0, 1$). In particular, we have $x_i \in U_i'$ (for $i = 0, 1$). Set $\lambda(t) = t$ for $t \in X' \setminus (G_0 \cup G_1)$ and $\lambda(t_i) = t$ for $t_i \in G_i$ (for $i = 0, 1$). Hence, $\lambda(x_i) = x$ (for $i = 0, 1$) and $X \subseteq X'$, $\lambda|_X = id_X$.

Let θ consist of inverse images of all open sets of $\chi_f X$ by the mappings λ and $\lambda_i \equiv \lambda|_{U_i'}$ (for $i = 0, 1$). Evidently, θ is a topology on X', U_i' is open in X' (for $i = 0, 1$), λ is continuous and $\lambda : X' \setminus (G_0 \cup G_1) \to \chi_f X \setminus G$ is a homeomorphism.

In particular, $\lambda|_X$ is the identical homeomorphism of X. Since $\lambda^{-1}(\{t\})$ consists of two points for $t \in G$, all fibres of λ are compact.

Since $X' \setminus U_i'$ is closed in X', the corestriction of λ to this set is a homeomorphism and $\lambda(X' \setminus U_i') = (\chi_f X \setminus U) \cup cl_U(U_j)$ (where $j = 1$ when $i = 0$ and $j = 0$ when $i = 1$) is closed in $\chi_f X$ (for $i = 0, 1$), λ is closed and so perfect. Evidently, X is dense in X' and λ is Hausdorff.

Thus, $bf = \chi f \circ \lambda$ is a compact Hausdorff mapping (by 2.6) and $bf_{|X} = f$. So, bf is a Hausdorff compactification of f and λ is a canonical morphism from bf to χf, i.e. $bf \geq \chi f$.

Moreover, λ is not 1–1 because $x = \lambda(x_0) = \lambda(x_1)$. Thus $bf > \chi f$ which is a contradiction to the maximality of χf. □

To obtain an extrinsic characterization of the perfect Hausdorff compactification, we need two lemmas.

Lemma 3.2. *Let Y_1 and Y_2 be extensions of a space X, $x \in Y_1 \backslash X$ and $f : Y_2 \to Y_1$ a continuous mapping closed at x such that $f_{|X} = id_X$ and $f^{-1}(\{x\})$ is connected. Then, if Y_2 is a perfect extension of X relatively to any point of $F = f^{-1}(\{x\})$, Y_1 is a perfect extension of X relatively to x.*

PROOF: First, we observe that $f^{-1}(\{x\}) \neq \emptyset$ as otherwise by the closedness of f at x, there exists some neighborhood N of x such that $f^{-1}(N) \subseteq \emptyset$.

Now, suppose — by contradiction — that Y_1 is not a perfect extension of X relatively to x. By 2.1, $Y_1 \backslash X$ cuts cut X at x, i.e. there exist a neighborhood O of x in Y_1 and disjoint open sets U, V of X such that $O \cap X = U \cup V$ and $x \in cl_{Y_1}(U) \cap cl_{Y_1}(V)$.

We claim that $F \cap cl_{Y_2}(U) \cap cl_{Y_2}(V) = \emptyset$. In fact, if there exists some $t \in F \cap cl_{Y_2}(U) \cap cl_{Y_2}(V)$, by continuity of f, $W = f^{-1}(O)$ is a neighborhood of t in Y_2 and, from $f_{|X} = id_X$ and $O \cap X = U \cup V$, it follows that $W \cap X = U \cup V$. But this means that $Y_2 \backslash X$ cuts X at $t \in F$ and by 2.1, Y_2 is not a perfect extension of X relatively to $t \in F$, which is a contradiction.

Moreover, $x \in O$ implies $F \subseteq W \subseteq cl_{Y_2}(W) = cl_{Y_2}(W \cap X) = cl_{Y_2}(U) \cup cl_{Y_2}(V)$. So, $(cl_{Y_2}(U) \cap F) \cup (cl_{Y_2}(V) \cap F) = F$ and, as F is connected, one of these two closed sets must be empty. Suppose that $cl_{Y_2}(U) \cap F = \emptyset$. Since $f : Y_2 \to Y_1$ is closed at x, there is some neighborhood N of x in Y_1 such that $f^{-1}(N) \subseteq Y_2 \backslash cl_{Y_2}(U)$. So, $cl_{Y_2}(U) \cap f^{-1}(N) = \emptyset$ and $U \cap N = U \cap X \cap N = U \cap f^{-1}(X \cap N) \subseteq cl_{Y_2}(U) \cap f^{-1}(N) = \emptyset$ imply $U \cap N = \emptyset$. This contradicts $x \in cl_{Y_1}(U)$.

Thus, it is proved that Y_1 is a perfect extension of X relatively to x. □

We recall that a mapping is called *monotone* if every its fibre is connected.

Corollary 3.3. *Let Y_1 and Y_2 be extensions of a space X and $f : Y_2 \to Y_1$ be a continuous, closed and monotone mapping such that $f_{|X} = id_X$. Then, if Y_2 is a perfect extension of X, Y_1 is a perfect extension of X too.*

Definition. Let S be a subspace of a space T. We say that S is *normally situated* (strongly normal in the terminology of [A]) in T if every pair of disjoint closed sets of S can be separated by a pair of disjoint open sets of T.

Remark. It follows from Corollary 2.4 that every fibre of a compact Hausdorff mapping is normally situated in its domain.

Lemma 3.4. *Let Y_1 and Y_2 be extensions of X, $x \in Y_1 \setminus X$ and $f : Y_2 \to Y_1$ be a continuous mapping closed at x, such that $F = f^{-1}(\{x\})$ is normally situated in Y_2 and $f_{|X} = id_X$. If Y_1 is a perfect extension of X relatively to x then F is connected.*

Proof: Suppose, by contradiction, that F is not connected, i.e. that there are disjoint non-empty closed sets C_1, C_2 of F such that $C_1 \cup C_2 = F$.

Since F is normally situated in Y_2, there are disjoint open sets U_1, U_2 of Y_2 such that $C_i \subseteq U_i$ (for $i = 1, 2$). So $F \subseteq U_1 \cup U_2$ and, by the closedness of f, there exists an open neighborhood O of x in Y_1 such that $f^{-1}(O) \subseteq U_1 \cup U_2$.

We may suppose that $f^{-1}(O) = U_1 \cup U_2$.

Since X is dense in Y_2, $V_i = U_i \cap X$ for $i = 1, 2$ are non-empty disjoint open sets of X and $O \cap X = f^{-1}(O) \cap X = V_1 \cup V_2$.

On the other hand, $x \in cl_{Y_1}(V_1) \cap cl_{Y_1}(V_2)$ because (for $i = 1, 2$) $U_i \subseteq cl_{Y_2}(U_i) = cl_{Y_2}(U_i \cap X) = cl_{Y_2}(V_i)$ and $x \in f(U_i) \subseteq f(cl_{Y_2}(V_i)) \subseteq cl_{Y_1}(f(V_i)) = cl_{Y_1}(V_i)$.

Thus $Y_1 \setminus X$ cuts X at x. This contradicts that Y_1 is a perfect extension of X relatively to x. Hence, F is connected. □

Corollary 3.5. *Let Y_1 and Y_2 be extensions of X and $f : Y_2 \to Y_1$ be a continuous closed mapping such that $f_{|X} = id_X$, $f^{-1}(X) = X$ and every its fibre is normally situated in Y_2. Then, if Y_1 is a perfect extension of X, the mapping f is monotone.*

Theorem 3.6. *Let $bf : X^b \to Y$ be a Hausdorff compactification of a mapping $f : X \to Y$ and let $\chi f : \chi_f X \to Y$ be the maximal Hausdorff compactification of f. Then bf is a perfect extension of f if and only if the canonical morphism $\lambda_b : \chi f \to bf$ is monotone.*

Proof: Suppose that bf is a perfect compactification of f, i.e. that X^b is a perfect extension of X. From 2.9, λ_b is perfect and, since χf is Hausdorff, by 2.2, λ_b is Hausdorff, too. Hence (see Remark before Lemma 3.4), every fibre of λ_b is normally situated in $\chi_f X$. By Corollary 3.5, λ_b is monotone.

Conversely, suppose that $\lambda_b : \chi_f X \to X^b$ is monotone. Since χf is a perfect extension of f, i.e. $\chi_f X$ if a perfect extension of X, 3.3 implies that X^b is a perfect extension of X. Hence bf is a perfect extension of f. □

If X is a Tychonoff space and $|Y| = 1$, every compactification γX of X corresponds to the (Tychonoff) compactifications $\gamma f : \gamma X \to Y$ of f, the domain

$\chi_f X$ of the maximal Hausdorff compactification of f coincides with the Stone-Čech compactification βX of X, the canonical morphism $\lambda : \chi f \to \gamma f$ becomes the usual canonical map $\varphi_\gamma : \beta X \to \gamma X$ and so the previous theorem gives as corollary the following proposition for spaces proved in [S].

Theorem 3.7. *A compactification γX of a Tychonoff space X is a perfect extension of X if and only if the canonical mapping $\varphi_\gamma : \beta X \to \gamma X$ is monotone.*

Remark. Let us observe that weaker versions of Theorems 3.1 and 3.6 were proved by Mazroa [M] by means of the notion of proximity for mappings (see [No]) only for the particular case of (Tychonoff) compactifications of a surjective (Tychonoff) mapping between T_3-spaces.

Theorem 3.8. *Let $f : X \to Y$ be a Tychonoff mapping, $\beta f : \beta_f X \to Y$ be its maximal Tychonoff compactification and $\chi f : \chi_f X \to Y$ be its maximal Hausdorff compactification. Then the canonical morphism $\lambda : \chi f \to \beta f$ is monotone.*

Proof: Since χf is compact and βf is Hausdorff, by 2.9, λ is perfect. From 2.12 it follows that $\lambda(\chi_f X\backslash X) \subseteq \beta_f X\backslash X$ and as λ is canonical, $\lambda^{-1}(X) = X$ and $\lambda(\chi_f X\backslash X) = \beta_f X\backslash X$.

Now, suppose — by contradiction — that $\lambda : \chi_f X \to \beta_f X$ is not monotone, i.e. that there is some $x \in \beta_f X\backslash X$ such that $\lambda^{-1}(\{x\})$ is not connected. So, there are non-empty disjoint closed sets B, C of $\lambda^{-1}(\{x\})$ such that $B \cup C = \lambda^{-1}(\{x\})$. Since $\lambda^{-1}(\{x\})$ is normally situated in $\chi_f X$ (see Remark before Lemma 3.4), there are disjoint open sets U, V of $\chi_f X$ such that $B \subseteq U$ and $C \subseteq V$. So, $U \cup V$ is an open neighborhood of $\lambda^{-1}(\{x\})$ and as $\lambda : \chi_f X \to \beta_f X$ is closed, there exists an open neighborhood W of x in $\beta_f X$ such that $\lambda^{-1}(W) \subseteq U \cup V$.

Since $\beta_f X\backslash W$ is a closed subset of $\beta_f X$ which does not contain the point x and $\beta f : \beta_f X \to Y$ is a Tychonoff mapping, there exist an open neighborhood H of $\beta f(x)$ in Y and a continuous mapping $\varphi : (\beta f)^{-1}(H) \to [0,1]$ such that $(\beta f)^{-1}(H) \cap (\beta_f X\backslash W) = (\beta f)^{-1}(H)\backslash W \subseteq \varphi^{-1}(\{0\})$ and $\varphi(x) = 1$.

Hence, $W_\beta = W \cap (\beta f)^{-1}(H)$ is an open neighborhood of x in $\beta_f X$ and $W_\chi = \lambda^{-1}(W_\beta)$ is an open set of $\chi_f X$. Obviously, $W_\beta \subseteq W$ and $W_\chi \subseteq U \cup V$.

Let us note that $W_\chi \cap X = \lambda^{-1}(W_\beta) \cap \lambda^{-1}(X) = \lambda^{-1}(W_\beta \cap X) = W_\beta \cap X$.

Now, $W_1 = U \cap W_\chi$ and $W_2 = V \cap W_\chi$ are non-empty disjoint open sets of $\chi_f X$ such that $W_\chi = W_1 \cup W_2$.

Let $O_i = W_i \cap X$ (for $i = 1, 2$). Since X is dense in $\chi_f X$, O_1 and O_2 are non-empty disjoint open sets of X such that $O_1 \cup O_2 = W_\chi \cap X = W_\beta \cap X$, $B \subseteq cl_{\chi_f X}(O_1)$ and $C \subseteq cl_{\chi_f X}(O_2)$.

Moreover, since $\beta f \circ \lambda = \chi f$ and $\chi f_{|X} = f$, we have $O_1 \cup O_2 = W_\chi \cap X = \lambda^{-1}(W_\beta) \cap X \subseteq \lambda^{-1}((\beta f)^{-1}(H)) \cap X = (\chi f)^{-1}(H) \cap X = f^{-1}(H)$.

Since both B and C are contained in the fibre $\lambda^{-1}(\{x\})$, we obtain $x \in \lambda(B) \cap \lambda(C) \subseteq \lambda(cl_{\chi_f X}(O_1)) \cap \lambda(cl_{\chi_f X}(O_2)) \subseteq cl_{\beta_f X}(\lambda(O_1)) \cap cl_{\beta_f X}(\lambda(O_2)) = cl_{\beta_f X}(O_1) \cap cl_{\beta_f X}(O_2)$.

There exists an open neighborhood O of x in $(\beta f)^{-1}(H)$ such that $\varphi(O) \subseteq]\frac{1}{2}, 1]$.
Define the mapping $\psi : f^{-1}(H) \to [-1, 1]$ by setting:

$$\psi(t) = \begin{cases} \varphi(t) & \text{if } t \in f^{-1}(H) \backslash O_2 \\ -\varphi(t) & \text{if } t \in cl_{f^{-1}(H)}(O_2). \end{cases}$$

It is continuous by the *Pasting Theorem* for closed sets because $cl_{f^{-1}(H)}(O_2) \cap (f^{-1}(H) \backslash O_2) = bd_{f^{-1}(H)}(O_2)$ and $O_1 \cap bd_{f^{-1}(H)}(O_2) = \emptyset$, $O_2 \cap bd_{f^{-1}(H)}(O_2) = \emptyset$ imply $(O_1 \cup O_2) \cap bd_{f^{-1}(H)}(O_2) = \emptyset$ and, hence,

$$\begin{aligned} bd_{f^{-1}(H)}(O_2) &\subseteq f^{-1}(H) \backslash (O_1 \cup O_2) \\ &= f^{-1}(H) \backslash (W_\beta \cap X) \\ &\subseteq (\beta f)^{-1}(H) \backslash W_\beta \\ &= (\beta f)^{-1}(H) \backslash W \\ &\subseteq \varphi^{-1}(\{0\}). \end{aligned}$$

Then, by 2.13, there is a continuous extension $\widetilde{\psi} : (\beta f)^{-1}(H) \to [-1, 1]$ of ψ to $(\beta f)^{-1}(H)$. Obviously, it results $\widetilde{\psi}(O_1 \cap O) \subseteq]\frac{1}{2}, 1]$ and $\widetilde{\psi}(O_2 \cap O) \subseteq [-1, -\frac{1}{2}[$.

On the other hand, since $x \in cl_{\beta_f X}(O_1) \cap cl_{\beta_f X}(O_2)$, $O_1 \cup O_2 \subseteq (\beta f)^{-1}(H)$ and $x \in (\beta f)^{-1}(H)$, it follows that $x \in cl_{(\beta f)^{-1}(H)}(O_1) \cap cl_{(\beta f)^{-1}(H)}(O_2)$ and as O is a neighborhood of x in $(\beta f)^{-1}(H)$, $x \in cl_{(\beta f)^{-1}(H)}(O_1 \cap O) \cap cl_{(\beta f)^{-1}(H)}(O_2 \cap O)$. So, by continuity of $\widetilde{\psi}$, we have $\widetilde{\psi}(x) \in \widetilde{\psi}(cl_{(\beta f)^{-1}(H)}(O_1 \cap O)) \cap \widetilde{\psi}(cl_{(\beta f)^{-1}(H)}(O_2 \cap O)) \subseteq cl_{[-1,1]}(\widetilde{\psi}(O_1 \cap O)) \cap cl_{[-1,1]}(\widetilde{\psi}(O_2 \cap O)) = \emptyset$.

A contradiction which proves that the canonical morphism $\lambda : \chi f \to \beta f$ is monotone. □

Theorems 3.6 and 3.8 allow us to obtain immediately the following:

Theorem 3.9. *The maximal Tychonoff compactification $\beta f : \beta_f X \to Y$ of a Tychonoff mapping $f : X \to Y$ is a perfect extension of f.*

Remark. If X is a Tychonoff space and $|Y| = 1$ then, for the maximal Tychonoff compactification $\beta f : \beta_f X \to Y$ and for the maximal Hausdorff compactification $\chi f : \chi_f X \to Y$, $\beta_f X$ and $\chi_f X$ coincide with the Stone-Čech compactification βX of X and so Theorems 3.1 and 3.9 give us as simple corollary the following proposition for spaces proved in [S].

Theorem 3.10. *The Stone-Čech compactification of a Tychonoff space X is a perfect extension of X.*

Theorems 3.6, 3.9 and Corollary 3.3 imply

Theorem 3.11. *A Tychonoff compactification bf of a Tychonoff mapping f is perfect if and only if the canonical morphism $\mu_b : \beta f \to bf$ is monotone.*

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