A note on copies of c_0 in spaces of weak* measurable functions

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Abstract. If (Ω, Σ, μ) is a finite measure space and X a Banach space, in this note we show that $L^1_{w^*}(\mu, X^*)$, the Banach space of all classes of weak* equivalent X^*-valued weak* measurable functions f defined on Ω such that $\|f(\omega)\| \le g(\omega)$ a.e. for some $g \in L_1(\mu)$ equipped with its usual norm, contains a copy of c_0 if and only if X^* contains a copy of c_0.

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1. Preliminaries

Throughout this paper (Ω, Σ, μ) will be a complete finite measure space and X a real or complex Banach space. We denote by $\mathcal{L}^p_{w^*}(\mu, X^*)$, $1 \le p \le \infty$, the linear space over $\mathbb{K}$ of all weak* measurable functions $f : \Omega \to X^*$ for which there exists a scalar function $g \in \mathcal{L}_p(\mu)$ such that $\|f(\omega)\| \le g(\omega)$ for μ-almost all $\omega \in \Omega$, whereas $L^p_{w^*}(\mu, X^*)$ stands for the quotient space of $\mathcal{L}^p_{w^*}(\mu, X^*)$ via the equivalence relation $\sim^*$ defined by $f_1 \sim^* f_2$ whenever $f_1(\ )x \sim f_2(\ )x$ for each $x \in X$ (here $\sim$ designs the usual equivalence relation in $\mathcal{L}_p(\mu)$). The space $L^p_{w^*}(\mu, X^*)$ is a Banach space when equipped with the norm $\left\|\widehat{f}\right\|_p = \inf \|g\|_{L_p(\mu)}$, the infimum taken over all those functions $g \in \mathcal{L}_p(\mu)$ for which there is some $f \in \widehat{f}$ such that $\|f(\omega)\| \le g(\omega)$ for μ-almost all $\omega \in \Omega$. It can be shown that there is always some $h \in \widehat{f}$ such that $\|h(\ )\| \in \mathcal{L}_p(\mu)$ and $\left\|\widehat{f}\right\|_p = \|\|h(\ )\|\|_{L_p(\mu)}$. We identify $L_p(\mu, X)^*$ with $L^q_{w^*}(\mu, X^*)$, where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, by means of the linear isometry $T : L^q_{w^*}(\mu, X^*) \to L_p(\mu, X)^*$ defined by $\left(T\widehat{f}\right) g = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega)$ for every $f \in \widehat{f}$. A study of $L^p_{w^*}(\mu, X^*)$ can be found in [2, Section 1.5] and [6, Section 3]. When X is separable, $\mathcal{L}^p_{w^*}(\mu, X^*)$ coincides with the space of all weak* measurable functions $f : \Omega \to X^*$ such that $\|f(\ )\| \in \mathcal{L}_p(\mu)$. In this case $L^p_{w^*}(\mu, X^*)$ is the quotient of $\mathcal{L}^p_{w^*}(\mu, X^*)$ via the usual equivalence relation, so $\left\|\widehat{f}\right\|_p = \|\|f(\ )\|\|_{L_p(\mu)}$ for each $f \in \widehat{f}$. We denote by $cabv\,(\Sigma, X^*)$ the Banach space of all X^*-valued countably additive measures F of bounded

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variation defined in Σ, equipped with the variation norm $|F| = |F|(\Omega)$. A result of Kwapień [7] answering a question of Hoffmann-Jørgensen [5] shows that $L_p(\mu, X)$, $1 \leq p < \infty$, contains a copy of c_0 if and only if X does. Since (as Mendoza has proved [8]) $L_p(\mu, X)$, $1 < p < \infty$, contains a complemented copy of ℓ_1 if and only if X contains a complemented copy of ℓ_1, then $L^p_{w^*}(\mu, X^*)$, $1 < p < \infty$, contains a copy of c_0 if and only if X^* does. In this note we show that this is also true for $p = 1$, i.e., that $L^1_{w^*}(\mu, X^*)$ contains a copy of c_0 if and only if X^* does.

2. Copies of c_0 in $L^1_{w^*}(\mu, X^*)$

If X is a separable Banach space, our statement is an easy consequence of an averaging theorem for c_0-sequences due to Bourgain [1] (see also [2, Lemma 2.1.2]). The general case will be derived from Theorem 2.2 below, otherwise well known.

Theorem 2.1. *Assume that X is a separable Banach space. If $L^1_{w^*}(\mu, X^*)$ contains a copy of c_0, then X^* contains a copy of c_0.*

Proof: Let $\left\{\widehat{f_n}\right\}$ be a normalized basic sequence in $L^1_{w^*}(\mu, X^*)$ equivalent to the unit vector basis of c_0. Then $\int_\Omega \|f_n(\omega)\|\, d\mu(\omega) = 1$ for each $n \in \mathbb{N}$ and there is $K > 0$ such that

$$\sup_{n\in\mathbb{N}} \int_\Omega \left\| \sum_{i=1}^{n} \varepsilon_i f_i(\omega) \right\| d\mu(\omega) < K \tag{2.1}$$

for each $f_i \in \widehat{f_i}$, $\varepsilon_i \in \{-1, 1\}$ and $i \in \mathbb{N}$. Setting

$$A_1 = \left\{\omega \in \Omega : \overline{\lim}_{n\to\infty} \|f_n(\omega)\| > 0\right\},$$

we claim that $\mu(A_1) > 0$. Otherwise, $\lim_{n\to\infty} \|f_n(\omega)\| = 0$ for almost all $\omega \in \Omega$ and since the sequence $\{\|f_n(\ )\|\}$ is uniformly integrable (this is essentially contained in the proof of [2, Theorem 2.1.1]), it follows from Vitali's lemma [4, IV.10.9] that $\lim_{n\to\infty} \int_\Omega \|f_n(\omega)\|\, d\mu(\omega) = 0$, a contradiction.

Denoting by Δ the product space $\{-1, 1\}^{\mathbb{N}}$, Γ the σ-algebra of subsets of Δ generated by the n-cylinders of Δ, $n = 1, 2, \ldots$, and ν the probability measure $\otimes_{i=1}^{\infty} \nu_i$ on Γ, where $\nu_i : 2^{\{-1,1\}} \to [0, 1]$ satisfies that $\nu_i(\emptyset) = 0$, $\nu_i(\{-1\}) = \nu_i(\{1\}) = 1/2$ and $\nu_i(\{-1, 1\}) = 1$ for each $i \in \mathbb{N}$, we may consider the μ-measurable map $h_n : \Omega \to \mathbb{R}$ defined by $h_n(\omega) = \int_\Delta \|\sum_{i=1}^{n} \varepsilon_i f_i(\omega)\|\, d\nu(\varepsilon)$ for $n = 1, 2, \ldots$. Since $\{h_n\}$ is a monotone increasing sequence of non negative functions, (2.1) and Fubini's theorem yield $\sup_{n\in\mathbb{N}} \int_\Omega h_n(\omega)\, d\mu(\omega) \leq K$. Hence, by the monotone convergence theorem there exists a μ-null set $A_2 \in \Sigma$ such that $\sup_{n\in\mathbb{N}} h_n(\omega) < \infty$ for each $\omega \in \Omega - A_2$. Considering the set $A := A_1 \cap (\Omega - A_2)$, it is obvious that $\mu(A) > 0$, hence $A \neq \emptyset$. Moreover, $\overline{\lim}_{n\to\infty} \|f_n(\omega)\| > 0$ and $\sup_{n\in\mathbb{N}} \int_\Delta \|\sum_{i=1}^{n} \varepsilon_i f_i(\omega)\|\, d\nu(\varepsilon) < \infty$ for each $\omega \in A$. Choose $\omega_0 \in A$ and a strictly increasing sequence of positive integers $\{n_i\}$ such that $\inf_{i\in\mathbb{N}} \|f_{n_i}(\omega_0)\| > 0$. Setting $x_i^* := f_{n_i}(\omega_0)$ for each $i \in \mathbb{N}$ and using the properties of the measure space we conclude that $\sup_{n\in\mathbb{N}} \int_\Delta \left\|\sum_{i=1}^n \varepsilon_i x_i^*\right\| d\nu(\varepsilon) < \infty$. According to the aforementioned theorem of Bourgain, there is a subsequence $\{z_n^*\}$ of $\{x_n^*\}$ which is a basic sequence in X^* equivalent to the unit vector basis of c_0. □

Theorem 2.2. *If X is an arbitrary Banach space, then $L^1_{w^*}(\mu, X^*)$ is linearly isometric to a subspace of $cabv(\Sigma, X^*)$.*

Proof: Consider the natural map $T : L^1_{w^*}(\mu, X^*) \to cabv(\Sigma, X^*)$ defined by $T\widehat{f} = F$, where

$$F(A)\,x = \int_A f(\omega)\,x\,d\mu(\omega)$$

for each $A \in \Sigma$ and $x \in X$. It is easy to check that F is an X^*-valued μ-continuous countably additive measure, since if $f \in \widehat{f}$ verifies that $\|f(\omega)\| \le g(\omega)$ for μ-almost all $\omega \in \Omega$ and some $g \in L_1(\mu)$, then $\|F(A)\| \le \|\chi_A g\|_{L_1(\mu)}$ for each $A \in \Sigma$. If $\pi(A)$ designs the class of all finite partitions of $A \in \Sigma$ by elements of Σ, then

$$\sum_{E\in\pi(A)} \|F(E)\| \le \sum_{E\in\pi(A)} \int_E g(\omega)\ d\mu(\omega) = \|\chi_A g\|_{L_1(\mu)} \le \|g\|_{L_1(\mu)}$$

which proves that $F \in cabv(\Sigma, X^*)$ and $|F| \le \left\|\widehat{f}\right\|_1$.

According to [2, Theorem 1.5.3] there exists a weak* measurable function $\psi : \Omega \to X^*$ satisfying that $(\omega \to \|\psi(\omega)\|) \in \mathcal{L}_1(\mu)$, $F(A)\,x = \int_A \psi(\omega)\,x\,d\mu(\omega)$ for all $A \in \Sigma$ and $x \in X$, and $|F|(A) = \int_A \|\psi(\omega)\|\ d\mu(\omega)$. Clearly $\psi \in \mathcal{L}^1_{w^*}(\mu, X^*)$ and $\psi \sim^* f$. Consequently,

$$\left\|\widehat{f}\right\|_1 \le \int_\Omega \|\psi(\omega)\|\ d\mu(\omega) = |F|.$$

This shows that $\left|T\widehat{f}\right| = \left\|\widehat{f}\right\|_1$, which concludes the proof. □

Corollary 2.3. *If $L^1_{w^*}(\mu, X^*)$ contains a copy of c_0, then X^* contains a copy of c_0.*

Proof: If $L^1_{w^*}(\mu, X^*)$ contains a copy of c_0, by the previous theorem c_0 embeds into $cabv(\Sigma, X^*)$. So X^* contains a copy of c_0 by virtue of E. and P. Saab's theorem [9] ([2, Theorem 3.1.3]). □

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