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# A construction of simplicial objects

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*Abstract.* We construct a simplicial locally convex algebra, whose weak dual is the standard cosimplicial topological space. The construction is carried out in a purely categorical way, so that it can be used to construct (co)simplicial objects in a variety of categories — in particular, the standard cosimplicial topological space can be produced.

*Keywords:* simplicial object, locally convex algebra, topological space

*Classification:* 18, 55

Simplicial sets play the central part in the combinatorial homotopy theory. The bridge between the category of simplicial sets (denoted by $\Delta^o\mathrm{SET}$) and other categories is maintained by a pair of adjoint functors: Let $E$ be an arbitrary covariant functor of the category $\Delta$ (objects of $\Delta$ are the sets $[n] = \{0, \cdots, n\}$, $n < \aleph_0$, and morphisms the nondecreasing maps) into a category $\mathcal{T}$. The *singular functor* associated with $E$ is the covariant functor $E^{\blacktriangledown} : \mathcal{T} \longrightarrow \Delta^o\mathrm{SET}$ given by

$$E^{\blacktriangledown}(P) = \mathcal{T}(E(-), P).$$

The *realization functor* $E^{\blacktriangle} : \Delta^o\mathrm{SET} \longrightarrow \mathcal{T}$ is the left adjoint to $E^{\blacktriangledown}$ — it exists whenever $\mathcal{T}$ is cocomplete.

Thus we see that the singular and realization functors depend on — and, in fact, are uniquely determined by — the *base functor* $E$.

Sometimes there is a "natural" choice for $E$ — this is the case with the geometric realization, where $E = \Delta^*$ is the standard cosimplicial topological space (as defined in [1], f.g.). In [2] Besser finds a kind of justification for this choice, but the geometric realization stays tightly connected with the closed unit interval in his work. However, we will see (cf. Remark 1.3) that an arbitrary locally compact Hausdorff monoid with an annihilating element can be taken instead of the closed unit interval — then a cosimplicial topological space $E$ can be constructed in such a way that $E_1$ is homeomorphic to the monoid.

Another time there is no choice for $E$, which would be commonly accepted as the "right" choice and various (co)simplicial objects $E$ come in useful — in Cartan Theorem (see [3]; here contravariant functors $E$, $E^{\blacktriangledown}$ and $E^{\blacktriangle}$ are used instead of the covariant ones — take the opposite category of $\mathcal{A}$ to obtain this version) simplicial DGAs are studied. In order to define a suitable simplicial DGA, the Cenkl-Porter construction can be used (cf. [4]).

Finally, the realization functors need not exist at all, as it is the case with the category of smooth manifolds; nonexistence of realization functors is due to nonexistence of certain colimits. The category of locally convex algebras (denoted by $\mathrm{L^{C}A}$) is a good substitute for the category of smooth manifolds: The category $\mathrm{L^{C}A}$ is complete, so that the realization functor exists for each simplicial locally convex algebra $E$ (note, that the transition from a smooth manifold to the locally convex algebra of the smooth functions defined on the manifold is contravariant!).

The original purpose of this work was to find a simplicial locally convex algebra with weak dual the standard cosimplicial topological space, which would not contain "welds" arising when the Cenkl-Porter construction is used. It turned out that the construction can be carried out in a purely categorical way — this is how the construction is presented.

## 1. Construction itself

The essential principle of the construction assumes that we are given a method (formally represented by a functor $T$) corresponding to taking the Cartesian product of a "geometric" object with the unit interval. The simplicial object is then constructed recursively: when the $n$-dimensional simplex has been constructed, the $(n+1)$-dimensional one is obtained by dilating the former by $T$ and collapsing one of the faces (see Figure 1.1).

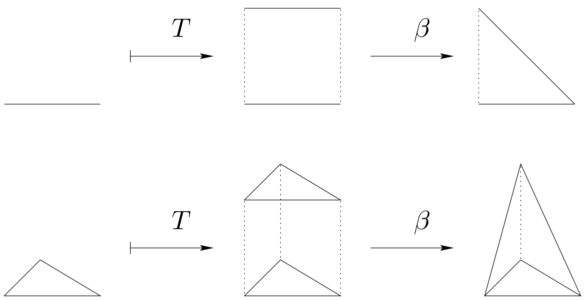


Figure 1.1. The principle of the recursive definition of (co)simplicial objects. The collapse mapping of the faces is indicated by $\beta$, $T$ stands for the dilation.

Some of the face and degeneracy operators are lifted from the preceding step, some of them are newly defined in the induction step. One of the degeneracy operators is however a little bit difficult, since it "folds" a newly created face (the faces of Figure 1.1 involving dotted lines) onto an old one.

Let $\mathcal{A}$ be an arbitrary category with all pullbacks, $T : \mathcal{A} \longrightarrow \mathcal{A}$ a covariant functor and $\varphi, \psi : T \longrightarrow Id_{\mathcal{A}}$, $\eta : Id_{\mathcal{A}} \longrightarrow T$ and $\varkappa : T \longrightarrow T^2$ natural transformations. Let us assume that $\langle T, \varphi, \psi, \eta, \varkappa \rangle$ is a simplicial construction in sense of the following definition.

**Definition 1.1.** We say that $\langle T, \varphi, \psi, \eta, \varkappa \rangle$ is a *simplicial construction* for $\mathcal{A}$ if $T$ preserves pullbacks and the five diagrams below are commutative.

$$
\begin{tikzcd}
& T^2 \arrow[dr, "T\varkappa"] & \\
T \arrow[ur, "\varkappa"] \arrow[dr, "\varkappa"'] & & T^3 \\
& T^2 \arrow[ur, "\varkappa T"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& T \arrow[dr, "\varphi"] & \\
I \arrow[ur, "\eta"] \arrow[dr, "\eta"'] \arrow[rr, equal] & & I \\
& T \arrow[ur, "\psi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& T \arrow[dr, "\eta T"] & \\
I \arrow[ur, "\eta"] \arrow[dr, "\eta"'] & & T^2 \\
& T \arrow[ur, "\varkappa"'] &
\end{tikzcd}
$$

$(a)$ $(b)$ $(c)$

$$
\begin{tikzcd}
& T^2 \arrow[dr, "T\varphi"] & \\
T \arrow[ur, "\varkappa"] \arrow[dr, "\varkappa"'] \arrow[r, "\varphi"] & I \arrow[r, "\eta"] & T \\
& T^2 \arrow[ur, "\varphi T"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& T^2 \arrow[dr, "T\psi"] & \\
T \arrow[ur, "\varkappa"] \arrow[dr, "\varkappa"'] \arrow[rr, equal] & & T \\
& T^2 \arrow[ur, "\psi T"'] &
\end{tikzcd}
$$

$(d)$ $(e)$

Figure 1.2. Diagrams for Simplicial Construction

We will use the simplicial construction $\langle T, \varphi, \psi, \eta, \varkappa \rangle$ to assign to a given object $A$ of $\mathcal{A}$ a simplicial $\mathcal{A}$-object $A_*$, with face operators $d_i^n$ and degeneracy operators $s_i^n$, such that $A_0 = A$. Nevertheless, we will not know until Proposition 1.1 that $A_*$ is indeed a simplicial $\mathcal{A}$-object, i.e., that the operators $d_i^n$ and $s_i^n$ satisfy the usual relations ([1, p. 175]). Thus, to be precise, we will proceed by recursion on $n \geq 0$ and in the induction step we will define:

A. principal items

(1) $\mathcal{A}$-object $A_{n+1}$;
(2) face operators $d_i^{n+1} \in \mathcal{A}\,(A_{n+1}, A_n)$;
(3) degeneracy operators $s_i^n \in \mathcal{A}\,(A_n, A_{n+1})$;

B. auxiliary morphisms

(1) $\beta_{n+1} \in \mathcal{A}\,(A_{n+1}, T(A_n))$, which corresponds to the collapse mapping as explained in the introduction to the present section;

(2) $\kappa_{n+1} \in \mathcal{A}\,(A_{n+1}, T(A_{n+1}))$, which is utilized to define the difficult degeneracy operator and to prove a contractibility of $A_{n+1}$ to $A_n$;
(3) $\alpha_{n+1} \in \mathcal{A}\,(A_{n+1}, A_{n-1})$ (only for $n \geq 1$), which is of no particular significance.

These items will be constructed in such a way that the following relations and property, which will be verified during the construction, hold true:

$$\left.\begin{array}{lrl}
(D1) & \beta_m d_i^{m+1} = T(d_i^m)\beta_{m+1} & \text{for } 0 \leq i \leq m-1 \\
(D2) & d_m^{m+1} = \psi_{A_m}\beta_{m+1} & \text{for } m \geq 0 \\
(D3) & d_{m+1}^{m+1} = \varphi_{A_m}\beta_{m+1} & \text{for } m \geq 0 \\
 & & \\
(S1) & \beta_{m+1} s_i^m = T(s_i^{m-1})\beta_m & \text{for } 0 \leq i \leq m-2 \\
(S2) & \beta_{m+1} s_{m-1}^m = \kappa_m & \text{for } m \geq 1 \\
(S3) & \beta_{m+1} s_m^m = \eta_{A_m} & \text{for } m \geq 0 \\
 & & \\
(K) & T(d_{m+1}^{m+1})\kappa_{m+1} = \eta_{A_m} d_{m+1}^{m+1} & \text{for } m \geq 0 \\
 & & \\
(P) & T^k(\beta_m) \text{ } \textit{is a monomorphism} & \text{for } m \geq 1, k \geq 0
\end{array}\right\} \quad (DSKP)$$

First of all, we initialize the recursive construction by setting:

A. principal items

(1) $A_0 = A$ and $A_1 = T(A_0)$;
(2) $d_0^1 = \psi_{A_0}$ and $d_1^1 = \varphi_{A_0}$;
(3) $s_0^0 = \eta_{A_0}$;

B. auxiliary morphisms

(1) $\beta_1 = A_1$;
(2) $\kappa_1 = \varkappa_{A_0}$.

One sees easily that the properties $(DSKP)$ (for $m = 0$) are satisfied.

Next, let us suppose that $A_m$, $d_i^m$, $s_i^{m-1}$, $\alpha_m$, $\beta_m$ and $\kappa_m$ have been constructed for all $m \leq n$, $n \geq 1$, and that they satisfy the corresponding properties of $(DSKP)$. The induction step reads as follows:

As far as $A_{n+1}$, $\alpha_{n+1}$ and $\beta_{n+1}$ are concerned, they are defined by the requirement that the diagram of Figure 1.3 be a pullback. Observe, that for $k \geq 0$ the morphism $T^k(\beta_{n+1})$ is a monomorphism, since the functor $T^k$ preserves pullbacks and $T^k(\beta_{n+1})$ lies opposite $T^k(\eta_{A_{n-1}})$, which is a monomorphism by diagram $(b)$

of Figure 1.2, in the corresponding $T^k$-image of the diagram of Figure 1.3.

$$\begin{tikzcd}
& T(A_n) \arrow[rr, "T(d^n_n)"] & & T(A_{n-1}) \\
A_{n+1} \arrow[ur, dashed, "\beta_{n+1}"] \arrow[rr, dashed, "\alpha_{n+1}"] & & A_{n-1} \arrow[ur, "\eta A_{n-1}"] &
\end{tikzcd}$$

Figure 1.3. Definition of $A_{n+1}$

In order to define $\kappa_{n+1}$, let us give our attention to the cube of Figure 1.4. Employ the naturality of $\varkappa$ and $\eta$ and the diagram $(c)$ of Figure 1.2 to see that the four complete faces of the cube are commutative. Moreover, since the functor $T$ preserves pullbacks, the top face is a pullback and we may (and do) use its universal property to define $\kappa_{n+1}$ by the requirement the two remaining faces of Figure 1.4 also be commutative.

$$\begin{tikzcd}
& T^2(A_n) \arrow[rr, "T^2(d^n_n)"] & & T^2(A_{n-1}) \\
T(A_{n+1}) \arrow[ur, "T(\beta_{n+1})"] \arrow[rr, "T(\alpha_{n+1})"] & & T(A_{n-1}) \arrow[ur, "T(\eta A_{n-1})"] & \\
& T(A_n) \arrow[uu, "\varkappa A_n"] \arrow[rr, "T(d^n_n)"] & & T(A_{n-1}) \arrow[uu, "\varkappa A_{n-1}"'] \\
A_{n+1} \arrow[uu, dashed, "\kappa_{n+1}"] \arrow[ur, "\beta_{n+1}"] \arrow[rr, "\alpha_{n+1}"] & & A_{n-1} \arrow[uu, "\eta A_{n-1}"'] \arrow[ur, "\eta A_{n-1}"] &
\end{tikzcd}$$

Figure 1.4. Definition of $\kappa_{n+1}$

Now we are going to define the face operators $d_i^{n+1}$. The last two of them $(i = n, n+1)$ are defined by the relations $(D2)$ and $(D3)$, where we set $m = n$. For $n = 1$, the face operator $d_0^2$ is given by

$$d_0^2 = T(d_0^1)\beta_2.$$

For $n \geq 2$, cf. the cube of Figure 1.5 to construct the operators $d_i^{n+1}$, $i \leq n-1$. Its top and bottom faces are commutative by definition (Figure 1.3) and the right-hand face is commutative by the naturality of $\eta$. Let us show that also the back face is commutative (for $i \leq n-1$): for $i < n-1$ we have (by the induction hypothesis and the naturality of $\varphi$)

$$d_i^{n-1}d_n^n = d_i^{n-1}\varphi_{A_{n-1}}\beta_n = \varphi_{A_{n-2}}T(d_i^{n-1})\beta_n = \varphi_{A_{n-2}}\beta_{n-1}d_i^n = d_{n-1}^{n-1}d_i^n,$$

whereas for $i = n-1$

$$d^{n-1}_{n-1}d^n_n = d^{n-1}_{n-1}\varphi_{A_{n-1}}\beta_n = \varphi_{A_{n-2}}T(d^{n-1}_{n-1})\beta_n = \varphi_{A_{n-2}}\eta_{A_{n-2}}\alpha_n = \alpha_n$$

and similarly

$$d^{n-1}_{n-1}d^n_{n-1} = d^{n-1}_{n-1}\psi_{A_{n-1}}\beta_n = \psi_{A_{n-2}}T(d^{n-1}_{n-1})\beta_n = \psi_{A_{n-2}}\eta_{A_{n-2}}\alpha_n = \alpha_n.$$

Now the morphisms $d^{n+1}_i$ ($i \le n-1$) are defined by the universal property of the top face, which is a pullback. The relations ($D1$) and ($K$) are at this moment readily verified.

$$\begin{tikzcd}
 & T(A_{n-1}) \arrow[rr, "T(d^{n-1}_{n-1})"] & & T(A_{n-2}) \\
A_n \arrow[ur, "\beta_n"] \arrow[rr, "\alpha_n"] & & A_{n-2} \arrow[ur, "\eta_{A_{n-2}}"] & \\
 & T(A_n) \arrow[uu, "T(d^n_i)"] \arrow[rr, "T(d^n_n)"] & & T(A_{n-1}) \arrow[uu, "T(d^{n-1}_i)"'] \\
A_{n+1} \arrow[uu, dashed, "d^{n+1}_i"] \arrow[ur, "\beta_{n+1}"] \arrow[rr, "\alpha_{n+1}"] & & A_{n-1} \arrow[uu, "d^{n-1}_i"'] \arrow[ur, "\eta_{A_{n-1}}"] &
\end{tikzcd}$$

Figure 1.5. Definition of $d^{n+1}_i$ for $i \le n-1$

Finally, let us define the degeneracy operators $s^n_i$. For $i \le n-2$ the operator $s^n_i$ is defined as indicated in the diagram of Figure 1.6, in which all the four complete faces are commutative — this is obvious except for the back one. However, by the induction hypothesis, for $i < n-2$ we have

$$d^n_n s^{n-1}_i = \varphi_{A_{n-1}}\beta_n s^{n-1}_i = \varphi_{A_{n-1}}T(s^{n-2}_i)\beta_{n-1} = s^{n-2}_i\varphi_{A_{n-2}}\beta_{n-1} = s^{n-2}_i d^{n-1}_{n-1},$$

and for $i = n-2$ we have

$$\begin{aligned}\beta_{n-1}d^n_n s^{n-1}_i &= \beta_n - 1\varphi_{A_{n-1}}\beta_n s^{n-1}_i = \beta_{n-1}\varphi_{A_{n-1}}\kappa_{n-1} = \varphi_{T(A_{n-2})}T(\beta_{n-1})\kappa_{n-1}\\ &= \varphi_{T(A_{n-2})}\varkappa_{A_{n-2}}\beta_{n-1} = \eta_{A_{n-2}}\varphi_{A_{n-2}}\beta_{n-1} = \beta_{n-1}s^{n-2}_{n-2}d^{n-1}_{n-1},\end{aligned}$$

therefore, in both cases we have

$$d^n_n s^{n-1}_i = s^{n-2}_i d^{n-1}_{n-1},$$

since $\beta_{n-1}$ is a monomorphism by ($P$). (The relations above make sense only for $n > 1$, but we do not have to bother about them for $n = 1$ as they are not needed in the latter case.) Thus, the commutativity of the back face of Figure 1.6 follows and, once again, we use the fact that the top face of Figure 1.6 is a pullback to define $s_i^n$.

$$\begin{tikzcd}
& T(A_n) \arrow[rr, "T(d_n^n)"] & & T(A_{n-1}) \\
A_{n+1} \arrow[ur, "\beta_{n+1}"] \arrow[rr, "\alpha_{n+1}"] & & A_{n-1} \arrow[ur, "\eta_{A_{n-1}}"] & \\
& T(A_{n-1}) \arrow[uu, "T(s_i^{n-1})"'] \arrow[rr, "T(d_{n-1}^{n-1})"] & & T(A_{n-2}) \arrow[uu, "T(s_i^{n-2})"'] \\
A_n \arrow[uu, dashed, "s_i^n"] \arrow[ur, "\beta_n"] \arrow[rr, "\alpha_n"] & & A_{n-2} \arrow[uu, "s_i^{n-2}"'] \arrow[ur, "\eta_{A_{n-2}}"] &
\end{tikzcd}$$

Figure 1.6. Definition of $s_i^n$ for $i \leq n-2$

The operators $s_{n-1}^n$ and $s_n^n$ are defined as in the diagram of Figure 1.7. As far as $s_{n-1}^n$ is concerned, the morphism $\kappa_n$ is used and it follows by the induction hypothesis, namely the relation $(K)$, that

$$T(d_n^n)\kappa_n = \eta_{A_{n-1}} d_n^n.$$

Once again, utilize the universal property of the pullback to define $s_{n-1}^n$.

Similarly the operator $s_n^n$ is defined — this time the naturality of $\eta$ is used to verify that the universal property can be employed.

$$\begin{tikzcd}
& T(A_n) \arrow[rr, "T(d_n^n)"] & & T(A_{n-1}) \\
A_{n+1} \arrow[ur, "\beta_{n+1}"] \arrow[rr, "\alpha_{n+1}"] & & A_{n-1} \arrow[ur, "\eta_{A_{n-1}}"] & \\
A_n \arrow[u, dashed, "{s_{n-1}^n, s_n^n}"] \arrow[uur, "{\kappa_n, \eta_{A_n}}"'] \arrow[urr, "d_n^n"'] & & &
\end{tikzcd}$$

Figure 1.7. Definition of $s_{n-1}^n$ and $s_n^n$

The recursion is complete.

**Proposition 1.1.** *The sequence $\{A_n\}_{n<\aleph_0}$ together with the morphisms $d_i^n$ and $s_i^n$ as constructed above yield a simplicial $\mathcal{A}$-object $A_*$.*

Proof: The proof depends in verifying the usual relations, which the morphisms $d_i^n$ and $s_i^n$ must satisfy. This is done easily, one only applies the elementary categorical calculus on the properties $(DSKP)$ and the diagrams of the Figure 1.2. □

Let $\rho : A \longrightarrow B$ be a morphism of $\mathcal{A}$. We leave it to the reader that $\rho$ can be (recursively) extended to a simplicial $\mathcal{A}$-morphism $\rho_* : A_* \longrightarrow B_*$ (i.e. $\rho_0 = \rho$).

**Proposition 1.2.** *The assignment indicated above yields a covariant functor from the category $\mathcal{A}$ to the category of $\mathcal{A}$-objects.*

In what follows we aim at showing that the simplicial $\mathcal{A}$-object $A_*$ has suitable contractibility properties. Though the definitions of homotopy, composed homotopy and contractibility could be carried out in a more general fashion, we still assume that $\langle T, \varphi, \psi, \eta, \varkappa \rangle$ is a simplicial construction for $\mathcal{A}$; the reason is that we use fragments of the properties of the data to conclude some useful properties, which will be needed in the sequel.

**Definition 1.2.** Let $\mu, \nu \in \mathcal{A}\,(A, B)$.

(1) We say that $\vartheta \in \mathcal{A}\,(A, T(B))$ is a *homotopy* from $\mu$ to $\nu$, if $\mu = \varphi_B \circ \vartheta$ and $\nu = \psi_B \circ \vartheta$. We write $\mu \rightharpoonup \nu$ if there is a homotopy from $\mu$ to $\nu$.

$$\begin{tikzcd} A \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\nu"'] \arrow[dr, dashed, "\vartheta"'] & B \\ & T(B) \arrow[u, shift left, "\varphi_B"] \arrow[u, shift right, "\psi_B"'] \end{tikzcd}$$

Figure 1.8. Homotopy of Morphisms

(2) The morphisms $\mu$ and $\nu$ are *composed-homotopic* if there exists a sequence of morphisms $\mu_0, \ldots, \mu_n \in \mathcal{A}\,(A, B)$ such that

$$\mu = \mu_0 \rightharpoonup \mu_1 \leftharpoondown \mu_2 \rightharpoonup \cdots \leftharpoondown \mu_n = \nu.$$

Then we write $\mu \rightleftharpoons \nu$.

**Proposition 1.3.** *The relation $\rightharpoonup$ is*

(1) *reflexive on every set $\mathcal{A}\,(A, B)$;*
(2) *compatible with the composition of morphisms in $\mathcal{A}$, i.e., for all $\mu, \nu \in \mathcal{A}\,(A, B)$ such that $\mu \rightharpoonup \nu$ we have*
   (a) $\tau \in \mathcal{A}\,(C, A) \implies \mu\tau \rightharpoonup \nu\tau$ *and*
   (b) $\tau \in \mathcal{A}\,(B, C) \implies \tau\mu \rightharpoonup \tau\nu$.

**Corollary 1.1.** *The relation $\rightleftharpoons$ is a congruence on $\mathcal{A}$, i.e.,*

(1) *it is an equivalence on all sets $\mathcal{A}(A, B)$;*
(2) *it is compatible with the composition of morphisms in $\mathcal{A}$.*

The relations discussed may, of course, happen to coincide. Since the category $\mathcal{A}$ has all pullbacks, there are induced a covariant functor $T^+ : \mathcal{A} \longrightarrow \mathcal{A}$ and natural transformations $\varphi^+, \psi^+ : T^+ \longrightarrow T$, determined uniquely up to natural equivalence by the requirement the diagram of Figure 1.9 be a pullback.

$$\begin{tikzcd}
& T(A) \arrow[dr, "\psi_A"] & \\
T^+(A) \arrow[ur, "\varphi^+_A"] \arrow[dr, "\psi^+_A"'] & & A \\
& T(A) \arrow[ur, "\varphi_A"'] &
\end{tikzcd}$$

Figure 1.9. Definition of $T^+$, $\varphi^+$ and $\psi^+$

Then we have:

**Proposition 1.4.** *The relation $\rightharpoonup$ is*

(1) *symmetric iff for all $A \in \operatorname{obj} \mathcal{A}$ $\psi_A \rightharpoonup \varphi_A$;*
(2) *transitive iff for all $A \in \operatorname{obj} \mathcal{A}$ $\varphi_A \varphi^+_A \rightharpoonup \psi_A \psi^+_A$.*

**Definition 1.3.** Let $A, B \in \operatorname{obj} \mathcal{A}$. Then $B$ is contractible to $A$ if there exist $\alpha \in \mathcal{A}(B, A)$ and $\beta \in \mathcal{A}(A, B)$ such that $B \rightleftharpoons \beta \circ \alpha$.

**Proposition 1.5.** *For all objects $A$, $B$ and $C$ of $\mathcal{A}$ we have:*

(1) *$A$ is contractible to $A$.*
(2) *If $C$ is contractible to $B$ and $B$ is contractible to $A$ then $C$ is contractible to $A$.*
(3) *$T(A)$ is contractible to $A$.*

**Proposition 1.6.** *For an arbitrary object $A \in \operatorname{obj} \mathcal{A}$, the $n$-th $\mathcal{A}$-object $A_n$ of $A_*$ is contractible to $A$.*

Proof: It suffices to show that $A_{n+1}$ is contractible to $A_n$. Consider the diagram of Figure 1.10.

$$\begin{tikzcd}
& A_n \arrow[dr, "s^n_n"] & \\
A_{n+1} \arrow[ur, "d^{n+1}_{n+1}"] \arrow[rr, "A_{n+1}"] \arrow[drr, "\kappa_{n+1}"'] & & A_{n+1} \\
& & T(A_{n+1}) \arrow[u, shift right, "\varphi_{A_{n+1}}"] \arrow[u, shift left, "\psi_{A_{n+1}}"']
\end{tikzcd}$$

Figure 1.10.

From the naturality of $\varphi$, definitions of $\kappa_{n+1}$, $d^{n+1}_{n+1}$, $s^n_n$ and diagram $(d)$ of Figure 1.2 it follows that

$$\beta_{n+1}\varphi_{A_{n+1}}\kappa_{n+1} = \beta_{n+1}s^n_n d^{n+1}_{n+1}$$

thus

$$\varphi_{A_{n+1}}\kappa_{n+1} = s^n_n d^{n+1}_{n+1}$$

since $\beta_{n+1}$ is a monomorphism.

Similarly we conclude that

$$\psi_{A_{n+1}}\kappa_{n+1} = A_{n+1}.$$

Therefore $\kappa_{n+1}$ is a homotopy from $s^n_n d^{n+1}_{n+1}$ to $A_{n+1}$. □

**Remark 1.1.** For all $n$ there is a canonical monomorphism

$$\gamma_n : A_n \longrightarrow T^n(A_0),$$

namely the composition

$$A_n \xrightarrow{\beta_n} T(A_{n-1}) \xrightarrow{T(\beta_{n-1})} T^2(A_{n-2}) \xrightarrow{T^2(\beta_{n-2})} \cdots \xrightarrow{T^{n-1}(\beta_1)} T^n(A_0).$$

**Example 1.1.** Let $\mathcal{A}$ be the category of unital algebras over a commutative unital ring $\Bbbk$. The functor $T$ assigns to every algebra $A$ the polynomial algebra $A[t]$ and is defined on homomorphisms in the obvious way. The homomorphisms

$$\varphi_A, \psi_A : A[t] \longrightarrow A$$

are given by

$$\varphi_A(f) = f(0)$$
$$\psi_A(f) = f(1)$$

and

$$\eta_A : A \longrightarrow A[t]$$

is the canonical inclusion. Finally,

$$\varkappa_A : A[t] \longrightarrow A[t_1, t_2]$$

is the unique homomorphism identical on $A$ and satisfying

$$\varkappa_A(t) = t_1 t_2.$$

It is readily checked that $\langle T, \varphi, \psi, \eta, \varkappa\rangle$ just defined is a simplicial construction for $\mathcal{A}$.

The simplicial $\Bbbk$-algebra $A_*$ assigned to $\Bbbk$ can be described as follows: let us for natural numbers $r_1, \cdots, r_n$ denote by $[r_1, \cdots, r_n]$ the element $x_1^{r_1} \cdots x_n^{r_n}$ of the polynomial algebra $\Bbbk[x_1, \cdots, x_n]$. Then $A_n$ is the subalgebra of $\Bbbk[x_1, \cdots, x_n]$ generated (as a $\Bbbk$-module) by the elements $[r_1, \cdots, r_n]$ satisfying

$$r_i = 0 \implies r_{i+1} = 0 \qquad i = 1, \cdots, n-1.$$

Note that $A_n$ is not (as a $\Bbbk$-algebra) finitely generated.

The face and degeneracy operators are given by

$$\begin{aligned}
d_i[r_1, \cdots, r_n] &= [r_1, \cdots, r_i, \widehat{r_{i+1}}, r_{i+2}, \cdots, r_n] \qquad & i < n \\
d_n[r_1, \cdots, r_n] &= \begin{cases} [r_1, \cdots, r_{n-1}] & r_n = 0 \\ 0 & r_n > 0 \end{cases} \\
s_i[r_1, \cdots, r_n] &= [r_1, \cdots, r_{i+1}, r_{i+1}, \cdots, r_n] \qquad & i < n \\
s_n[r_1, \cdots, r_n] &= [r_1, \cdots, r_n, 0],
\end{aligned}$$

where $\hat{}$ denotes omittion.

As far as homotopy is concerned, it is symmetric but, in general, not transitive — f.g. whenever $\Bbbk$ is an integral domain.

The next example uses the dual of the simplicial construction, the cosimplicial construction. We keep the same notation, thus

$$T : \mathcal{A} \longrightarrow \mathcal{A}$$

remains a *covariant* functor and the natural transformations have the opposite directions:

$$\varphi, \psi : Id_{\mathcal{A}} \longrightarrow T$$
$$\eta : T \longrightarrow Id_{\mathcal{A}}$$
$$\varkappa : T^2 \longrightarrow T.$$

Only the face and degeneracy operators are denoted, as usually, by $\delta_i^n$ and $\sigma_i^n$ respectively.

**Example 1.2.** Let $\mathcal{A} = \mathrm{T{\small O}P}$ be the category of topological spaces. The functor $T = (- \times \mathbb{I})$, where $\mathbb{I}$ denotes the closed unit interval, preserves all pushouts — in fact, it is a left adjoint ($\mathbb{I}$ is Hausdorff (locally) compact!).

The natural transformations $\varphi$, $\psi$, $\eta$ and $\varkappa$ are given by (here $P$ is a topological space)

$$\begin{aligned} \varphi_P(p) &= \langle p, 0\rangle \\ \psi_P(p) &= \langle p, 1\rangle \\ \eta_P(p,\tau) &= p \\ \varkappa_P(p,\tau_1,\tau_2) &= \langle p, \tau_1\tau_2\rangle \end{aligned}$$

for all $p \in P$ and $\tau, \tau_1, \tau_2 \in \mathbb{I}$.

**Proposition 1.7.** *The cosimplicial topological space assigned by the cosimplicial construction described above to a singleton is isomorphic to the standard cosimplicial topological space $\Delta^*$.*

Proof: We proceed by induction. It is immediate that $\Delta^0$ is a singleton and $\Delta^1 \simeq \mathbb{I}$, which is how the recursive cosimplicial construction is initialized. Also the operators $\delta^1_0$, $\delta^1_1$ and $\sigma^0_0$ are defined correctly. Let us identify $\Delta^1$ with $\mathbb{I}$. In order to treat the induction step we assume that $\Delta^m$ together with $\delta^m_i$, $\sigma^{m-1}_i$ and the maps $\alpha_m$, $\beta_m$, $\kappa_m$ has been constructed for all $m \le n$, where $n \ge 1$.

$$\begin{tikzcd} \Delta^{n-1}\times\mathbb{I} \arrow[rr, "\delta^n_n\times\mathbb{I}"] \arrow[dr, "\eta_{\Delta^{n-1}}"'] & & \Delta^n\times\mathbb{I} \arrow[dr, "\beta_{n+1}"] & \\ & \Delta^{n-1} \arrow[rr, "\alpha_{n+1}"'] & & \Delta^{n+1} \end{tikzcd}$$

Figure 1.11.

First of all observe, that the diagram above is a pushout — the maps $\alpha_{n+1}$ and $\beta_{n+1}$ are *defined* by the formulae

$$\begin{aligned} \alpha_{n+1}(\xi_0,\cdots,\xi_{n-1}) &= (\xi_0,\cdots,\xi_{n-1},0,0), \\ \beta_{n+1}(\xi_0,\cdots,\xi_n,\tau) &= (\xi_0,\cdots,\xi_{n-1},\xi_n(1-\tau),\xi_n\tau). \end{aligned}$$

From this it is obvious that the face operators $\delta^{n+1}_i$ constructed by the cosimplicial construction coincide with the usual ones.

Next, the map $\kappa_{n+1}$ (see Figure 1.4 for the definition) satisfies

$$\kappa_{n+1}(\xi_0,\cdots,\xi_{n+1},\tau) = (\xi_0,\cdots,\xi_{n-1},\xi_n+\xi_{n+1}(1-\tau),\xi_{n+1}\tau).$$

It remains to verify that the usual degeneracy operators $\sigma^n_i$ coincide with those defined by the cosimplicial construction, which is, at this moment, an easy task. $\square$

**Remark 1.2.** The explicit definition of $\gamma_n : \mathbb{I}^n \longrightarrow \Delta^n$ reads as follows:

$$\gamma_n(\tau_1, \cdots, \tau_n) = (1-\tau_1, \tau_1(1-\tau_2), \tau_1\tau_2(1-\tau_3), \cdots, \tau_1 \cdots \tau_{n-1}(1-\tau_n), \tau_1 \cdots \tau_n).$$

**Remark 1.3.** Observe that an arbitrary locally compact monoid with an annihilating element can be used in the example above instead of the closed unit interval.

## 2. Standard simplicial locally convex algebra

In this section we will apply the simplicial construction developed in Section 1 in order to define the *standard simplicial locally convex algebra* and we will investigate its basic properties.

Once for all, *locally convex algebra* will always mean *real Hausdorff locally convex topological unital commutative algebra with* (*jointly*) *continuous multiplication.* The category of all locally convex algebras (with morphisms the unital continuous algebra homomorphisms) is denoted by $\mathrm{L^CA}$. Note, that $\mathrm{L^CA}$ is both complete and cocomplete.

Let us by $C^\infty(\mathbb{I}, -)$ denote the functor assigning to a locally convex algebra the topological algebra of all infinitely differentiable mappings of the closed unit interval $\mathbb{I}$ into $A$; the topology of $C^\infty(\mathbb{I}, A)$ is the topology of uniform convergence on $\mathbb{I}$ of the mappings together with their derivations of all orders. We write $C^\infty(\mathbb{I})$ provided $A = \mathbb{R}$.

Thus, in the sequel, $\mathcal{A}$ will be the category of locally convex algebras $\mathrm{L^CA}$, which is complete; in particular, it has all pullbacks. Further, we set $T = C^\infty(\mathbb{I}, -)$. The functor $C^\infty(\mathbb{I}, -)$ is easily verified to preserve all pullbacks of $\mathrm{L^CA}$.

The natural transformations $\varphi$, $\psi$, $\eta$ and $\varkappa$ are given by the following formulae ($A \in \mathrm{obj}\ \mathrm{L^CA}$):

$$\begin{aligned}
\eta_A(x)(\xi) &= x && \forall\, \xi \in \mathbb{I}, x \in A \\
\varphi_A(f) &= f(0) && \forall\, f \in C^\infty(\mathbb{I}, A) \\
\psi_A(f) &= f(1) && \forall\, f \in C^\infty(\mathbb{I}, A) \\
\varkappa_A(f)(\xi_1, \xi_2) &= f(\xi_1\xi_2) && \forall\, f \in C^\infty(\mathbb{I}, A), \xi_1, \xi_2 \in \mathbb{I}.
\end{aligned}$$

Let us first of all note that the homotopy relation (see Definition 1.2) associated with the simplicial construction $\langle T, \varphi, \psi, \eta, \varkappa \rangle$ defined above is both symmetric and transitive, hence it coincides with the composed homotopy. To prove the transitivity of $\rightharpoonup$, we proceed according to Proposition 1.4:

Let $\flat : \left[0, \frac{1}{2}\right] \longrightarrow \mathbb{I}$ be a $C^\infty$-function flat at $\frac{1}{2}$, $\flat(0) = 0$ and $\flat(\frac{1}{2}) = 1$.

For an arbitrary locally convex algebra $A$, the algebra $A^+$ in the pullback

$$\begin{tikzcd}
 & C^{\infty}(\mathbb{I}, A) \arrow[dr, "\psi_A"] & \\
A^{+} \arrow[ur, "\varphi_A^{+}"] \arrow[dr, "\psi_A^{+}"'] & & A \\
 & C^{\infty}(\mathbb{I}, A) \arrow[ur, "\varphi_A"'] &
\end{tikzcd}$$

can be identified with the (locally convex) subalgebra of $C^{\infty}(\mathbb{I}, A) \times C^{\infty}(\mathbb{I}, A)$ consisting of all the pairs $\langle f, g\rangle$ such that $f(1) = g(0)$. The homomorphism

$$\vartheta : A^{+} \longrightarrow C^{\infty}(\mathbb{I}, A)$$

given by

$$\vartheta(\langle f, g\rangle)(\xi) = \begin{cases} f(\flat(\xi)) & \text{for } \ \xi \in \left[0, \frac{1}{2}\right] \\ g(\flat(1-\xi)) & \text{for } \ \xi \in \left[\frac{1}{2}, 1\right] \end{cases}$$

yields a homotopy from $\varphi_A \varphi_A^{+}$ to $\psi_A \psi_A^{+}$.

**Definition 2.1.** The simplicial locally convex algebra assigned to $\mathbb{R}$ by the simplicial construction $\langle T, \varphi, \psi, \eta, \varkappa\rangle$ described above will be denoted by $\Delta_*$ and called the *standard simplicial locally complex algebra*.

Let us establish the relationship between the simplicial object $\Delta_*$ and the cosimplicial one $\Delta^*$.

Let

$$H : \mathrm{LCA} \longrightarrow \mathrm{Top}$$

be the hom-functor $\mathrm{LCA}(-, \mathbb{R})$, where, for an algebra $A \in \operatorname{obj} \mathrm{LCA}$, the set $\mathrm{LCA}(A, \mathbb{R})$ is endowed with the weak topology. Note that $H(A)$ is always Hausdorff.

**Definition 2.2.** Let $\beta : A \longrightarrow B$ be a morphism of locally convex algebras.

(1) We say that $B$ has the $(INV)$-property if

$$(\forall\, b \in B)(\forall\, \zeta \in H(B))(\zeta(b) \neq 0 \implies b \text{ is invertible}).$$

(2) We say that $\beta$ has the $(NIP)$-property if

$$(\forall\, a \in A)(\beta(a) \text{ is invertible} \implies a \text{ is invertible}).$$

**Lemma 2.1.** *Let $\beta : A \longrightarrow B$ be a morphism of locally convex algebras and suppose that $B$ has the $(INV)$-property and $\beta$ has the $(NIP)$-property. Then*

(1) *$C^{\infty}(\mathbb{I}, B)$ has the $(INV)$-property;*
(2) *$A$ has the $(INV)$-property;*
(3) *in the pullback below, $\beta_1$ has the $(NIP)$-property.*

$$\begin{tikzcd} & B_1 \arrow[rr] & & B \\ A_1 \arrow[ur, "\beta_1"] \arrow[rr] & & A \arrow[ur, "\beta"'] & \end{tikzcd}$$

(4) *if $H(B)$ is compact, $H(\beta) : H(B) \longrightarrow H(A)$ is surjective.*

Proof of (4): We use a standard argument. Let $\xi \in H(A)$. For all $x \in \ker(\xi)$ we set

$$F_x = \{\zeta \in H(B) : \beta(x) \in \ker(\zeta)\}.$$

Then $F_x$ is a closed subset of $H(B)$ since the topology of $H(B)$ is the weak one, $F_x$ is non-empty, since $F_x = \emptyset$ implies that $\beta(x)$ be invertible by the $(INV)$-property of $B$, consequently $x$ be invertible by the $(NIP)$-property of $\beta$, which would be a contradiction with the assumption $x \in \ker(\xi)$. Finally, $F_{x_1^2+x_2^2} \subseteq F_{x_1} \cap F_{x_2}$. Thus the family

$$\langle F_x : x \in \ker(\xi)\rangle$$

is a centered system in $H(B)$ and by the compactness of $H(B)$ the intersection

$$\bigcap\{F_x : x \in \ker(\xi)\}$$

is non-empty. It is a routine to show that for an arbitrary $\zeta$ belonging to the intersection in question $H(\beta)(\zeta) = \xi$. □

**Consequence 2.1.** *For all $n$, the locally convex algebras $\Delta_n$ and $T(\Delta_n)$ have the $(INV)$-property and the corresponding morphisms $\beta_n$ have the $(NIP)$-property.*

Proof: Realize that $\eta_A$ has the $(NIP)$-property for all $A \in \operatorname{obj} \mathrm{L^{C}A}$, $\Delta_0 = \mathbb{R}$ has the $(INV)$-property and apply items 1–3 of the preceding lemma. □

**Definition 2.3.** We say that a locally convex algebra $A$ has the $(UC)$-property if

$$(\forall\, \varepsilon > 0)(\exists U,\ \text{a neighborhood of zero in } A)(\forall\, u \in U)(\forall\, \alpha \in H(A))(|\alpha(u)| < \varepsilon).$$

**Lemma 2.2.** *Let $A$ have the $(UC)$-property. Then*

(1) *$C^\infty(\mathbb{I}, A)$ has the $(UC)$-property;*
(2) *for all morphisms $\pi \in \mathrm{L^CA}(B, A)$, if $H(\pi)$ is surjective, then $B$ has the $(UC)$-property;*
(3) *the canonical map*

$$f : H(A) \times \mathbb{I} \longrightarrow H(C^\infty(\mathbb{I}, A))$$

*given by*

$$f(\alpha, \xi)(x) = \alpha(x(\xi))$$

*is a homeomorphism.*

**Lemma 2.3.** *In $\mathrm{L^CA}$, consider a pullback (Figure 2.1) with $\alpha_1$ a surjective homomorphism and $p, q \in \mathrm{L^CA}(A_1, \mathbb{R})$ distinct homomorphisms such that $p\pi_1 = q\pi_1$. Suppose the topology of $A_0$ is the terminal topology w.r.t. $\alpha_1$. There exist homomorphisms $p_0, q_0 \in \mathrm{L^CA}(A_0, \mathbb{R})$ such that $p = p_0\alpha_1$, $q = q_0\alpha_1$ and $p_0\alpha_2 = q_0\alpha_2$.*

$$\begin{tikzcd}
& A_1 \arrow[rr, "\alpha_1"] & & A_0 \\
A \arrow[ur, "\pi_1"] \arrow[rr, "\pi_2"'] & & A_2 \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

Figure 2.1.

Proof: We will prove the existence of $p_0$. For simplicity we suppose $A \subseteq A_1 \times A_2$ and $\pi_1, \pi_2$ are the projections. Let $z \in A_1$ be such that $p(z) = 1$ and $q(z) = 0$; such an element exists since $p \neq q$. Now for an arbitrary $x \in \ker(\alpha_1)$ we have $\langle xz, 0\rangle \in A$, hence $p(xz) = q(xz)$ by the hypothesis. Thus

$$p(x) = p(xz) = q(xz) = 0.$$

We have proved that

$$\ker(\alpha_1) \subseteq \ker(p),$$

therefore the existence of a homomorphism $p_0 : A_0 \longrightarrow \mathbb{R}$ (a priori not continuous) such that $p = p_0 \circ \alpha_1$ follows by a standard argument. The continuity of $p_0$ is a consequence of the assumption the topology of $A_0$ be the terminal one w.r.t. $\alpha_1$. The morphism $q_0$ is found in the same way.

Next we have

$$p_0\alpha_2\pi_2 = p_0\alpha_1\pi_1 = p\pi_1 = q\pi_1 = q_0\alpha_1\pi_1 = q_0\alpha_2\pi_2,$$

hence $p_0\alpha_2 = q_0\alpha_2$ since $\pi_2$ is surjective as so is $\alpha_1$ (this holds true in $\mathrm{L^CA}$). $\square$

**Proposition 2.1.** *The image $H(\Delta_*)$ of the standard simplicial locally convex algebra $\Delta_*$ is homeomorphic to the standard cosimplicial topological space $\Delta^*$.*

Proof: We have to prove that for all $n \geq 0$ there are homeomorphisms

$$\iota_n : H(\Delta_n) \longrightarrow \Delta^n$$

such that the diagrams

$$\begin{tikzcd} H(\Delta_{n-1}) \arrow[r, "\iota_{n-1}"] \arrow[d, "H(d_i^n)"'] & \Delta^{n-1} \arrow[d, "\delta_i^n"] \\ H(\Delta_n) \arrow[r, "\iota_n"'] & \Delta^n \end{tikzcd} \qquad \begin{tikzcd} H(\Delta_{n+1}) \arrow[r, "\iota_{n+1}"] \arrow[d, "H(s_i^n)"'] & \Delta^{n+1} \arrow[d, "\sigma_i^n"] \\ H(\Delta_n) \arrow[r, "\iota_n"'] & \Delta^n \end{tikzcd}$$

Figure 2.2.

are commutative.

In addition to the statement of the proposition we will also prove that all the locally convex algebras $\Delta_n$ have the $(UC)$-property; this is necessary for the induction step.

We proceed by induction on $n$. The assertions are obvious for $n = 0, 1$. Let us suppose that they hold true up to some $n \geq 1$ and that the isomorphisms $\iota_m$ $(m \leq n)$ are defined. From Lemma 2.2 and the induction hypothesis it follows that

$$(1) \qquad H(C^\infty(\mathbb{I}, \Delta_n)) \cong H(\Delta_n) \times \mathbb{I} \simeq \Delta^n \times \mathbb{I},$$

where the canonical isomorphism $\cong$ is that of Lemma 2.2 and the isomorphism $\simeq$ is $\iota_n \times \mathbb{I}$. We apply the functor $H$ to the pullback defining the algebra $\Delta_{n+1}$ and after the obvious identifications we obtain

$$(D) \qquad \begin{tikzcd} & \Delta^n \times \mathbb{I} \arrow[dl, "H(\beta_{n+1})"'] & & \Delta^{n-1} \times \mathbb{I} \arrow[ll, "\delta_n^n \times \mathbb{I}"'] \arrow[dl, "\eta_{\Delta^{n-1}}"] \\ H(\Delta_{n+1}) & & \Delta^{n-1} \arrow[ll, "H(\alpha_{n+1})"'] & \end{tikzcd} .$$

Let us show that the diagram $(D)$ is a pushout. Assume $P$ is a topological space and $f \in \mathrm{TOP}(\Delta^n \times \mathbb{I}, P)$ and $g \in \mathrm{TOP}(\Delta^{n-1}, P)$ satisfy

$$(2) \qquad f \circ (\delta_n^n \times \mathbb{I}) = g \circ \eta_{\Delta^{n-1}}.$$

From (1) it results that $H(C^\infty(\mathbb{I}, \Delta_n))$ is compact and we use Lemma 2.1(4) and Consequence 2.1 to conclude that $H(\beta_{n+1})$ is surjective. Hence the domain of the relation

$$h = f \circ H(\beta_{n+1})^{-1}$$

is the whole of $H(\Delta_{n+1})$ and it is an immediate consequence of Lemma 2.3 and equality (2) that $h$ is a mapping. One sees easily that

$$f = h \circ H(\beta_{n+1}),$$
$$g = h \circ H(\alpha_{n+1}).$$

To prove continuity of $h$, observe first that $H(\beta_{n+1})$ is closed, for it is a continuous mapping of a compact space into a Hausdorff space. Now continuity of $h$ follows from the relation

$$h^{-1}(F) = H(\beta n + 1)(f^{-1}(F)),$$

valid for all (a fortiori closed) subsets of $P$, and from continuity of $f$.

It therefore follows that the diagram $(D)$ is a pushout. In particular, the universal property of the pushout yields a uniquely determined isomorphism $\iota_{n+1} : H(\Delta_{n+1}) \longrightarrow \Delta^{n+1}$.

Since the definitions of the face and degeneracy operators involve only universal properties of the appropriate pullbacks (for $\Delta_*$) and pushouts (for $\Delta^*$ — cf. Proposition 1.7) and $H$ transforms the fragment of the simplicial construction for LCA involved in constructing $\Delta_*$ to that of cosimplicial construction for TOP involved in constructing $\Delta^*$, which is now readily proved, the commutativity of the two squares of Figure 2.2, in which every occurrence of $n$ is once for now replaced with $n+1$, follows. $\square$

Let us describe the locally convex algebras $\Delta_n$ in a more transparent way. From Remark 1.1 it follows that the homomorphisms $\gamma_n : \Delta_n \longrightarrow C^\infty(\mathbb{I}^n)$ are injective. We will prove more:

**Proposition 2.2.** *The homomorphisms $\gamma_n$ are embeddings.*

PROOF: It suffices to prove that the homomorphisms $T^k(\beta_n)$ are embeddings for all $k \geq 0$, $n \geq 1$. This is obvious for $n = 1$, while for $n \geq 2$ we use the fact that $T^k(\beta_n)$ lies opposite the homomorphism $T^k(\eta_{\Delta_{n-2}})$ in the pullback

$$\begin{tikzcd}
 & T^{k+1}(\Delta_{n-1}) \arrow[rr, "T^{k+1}(d^{n-1}_{n-1})"] & & T^{k+1}(\Delta_{n-2}) \\
T^k(\Delta_n) \arrow[ur, "T^k(\beta_n)"] \arrow[rr, "\alpha_n"'] & & T^k(\Delta_{n-2}) \arrow[ur, "T^k(\eta_{\Delta_{n-2}})"'] &
\end{tikzcd}$$

and $T^k(\eta_{n-2})$ is an embedding, since it has a left (continuous) inverse, f.g. the homomorphism $\varphi_{\Delta_{n-2}}$. $\square$

Therefore the locally convex algebra $\Delta_n$ can be identified with a subalgebra of $C^\infty(\mathbb{I}^n)$. Then the algebra $\Delta_n$ consists of all those mappings of $C^\infty(\mathbb{I}^n)$ which factor through $\gamma_n : \mathbb{I}^n \longrightarrow \Delta^n$. Here a more detailed description of the elements of $\Delta_n$ follows.

**Proposition 2.3.** *For all $n \geq 0$ the following assertions hold true*[1]*:*

(1) *for all $x \in \Delta_n$ there exists a unique mapping $x' : \Delta^n \longrightarrow \mathbb{R}$ such that $\gamma_n(x) = x' \circ \gamma_n$; the mapping $x'$ is continuous;*
(2) *for all mappings $x' : \Delta^n \longrightarrow \mathbb{R}$ such that $x' \circ \gamma_n \in C^\infty(\mathbb{I}^n)$ there exists a unique $x \in \Delta_n$ such that $\gamma_n(x) = x' \circ \gamma_n$;*
(3) *for all $z \in C^\infty(\mathbb{I}^n)$ which factor through $\gamma_n : \mathbb{I}^n \longrightarrow \Delta^n$, there exists a unique $x \in \Delta_n$ such that $\gamma_n(x) = z$.*

PROOF: From the proof of Proposition 2.1 it results that the diagram

$$\begin{tikzcd} H\left(C^\infty\left(\mathbb{I}^n\right)\right) \arrow[r, "H(\gamma_n)"] \arrow[d, "i_n"'] & H\left(\Delta_n\right) \arrow[d, "\iota_n"] \\ \mathbb{I}^n \arrow[r, "\gamma_n"'] & \Delta^n, \end{tikzcd}$$

where $\iota_n$ is the homeomorphism of (the proof of) Proposition 2.1 and $i_n$ is determined by

$$x(i_n(p)) = p(x) \qquad \forall\, x \in C^\infty\left(\mathbb{I}^n\right), p \in H\left(C^\infty\left(\mathbb{I}^n\right)\right),$$

is commutative. Therefore, since $i_n$ is a homeomorphism *onto* $\mathbb{I}^n$, the implication

$$\gamma_n(\xi_1) = \gamma_n(\xi_2) \implies \gamma_n(x)(\xi_1) = \gamma_n(x)(\xi_2)$$

holds for all $\xi_1, \xi_2 \in C^\infty\left(\mathbb{I}^n\right)$ and $x \in \Delta_n$. From the implication the first part of the item (1) follows.

A similar proof as the one of Proposition 2.2 shows that the topology of $\Delta^n$ is the terminal topology w.r.t. the mapping $\gamma_n : \mathbb{I}^n \longrightarrow \Delta^n$. Hence the mapping $x'$ is continuous, since $\gamma_n(x)$ is.

The items (2) and (3) are apparently equivalent. We prove the item (2), by induction on $n$. We deal with the existence part only, since the uniqueness is straightforward. The assertion is obvious for $n = 0, 1$. Let us suppose, the assertion is true for all $n \leq m$, $m \geq 1$; we shall prove it for $n = m + 1$. Let $x' : \Delta^{m+1} \longrightarrow \mathbb{R}$ be such that $x' \circ \gamma_{m+1} \in C^\infty\left(\mathbb{I}^{m+1}\right)$. For all $\tau \in \mathbb{I}$ we define a mapping $y'_\tau : \Delta^m \longrightarrow \mathbb{R}$ by the formula

$$y'_\tau(\xi) = x'(\beta_{m+1}(\xi, \tau)) \qquad \forall\, \xi \in \Delta^m.$$

Therefore we have

$$(y'_\tau \circ \gamma_m)(\xi) = (x' \circ \gamma_{m+1})(\xi, \tau).$$

[1]The symbol $\gamma_n$ is overloaded in what follows — it denotes not only the monomorphism $\gamma_n : \Delta_n \longrightarrow C^\infty\left(\mathbb{I}^n\right)$ but also the map $\gamma_n : \mathbb{I}^n \longrightarrow \Delta^n$. The meaning is always clear from the context.

Hence $y'_\tau \circ \gamma_m \in C^\infty(\mathbb{I}^m)$ and, by the induction hypothesis, there is a mapping $y : \mathbb{I} \longrightarrow \Delta_m$ such that

$$\gamma_n(y(\tau)) = y'_\tau \circ \gamma_n.$$

From the definition of $y$ it follows that $y \in C^\infty(\mathbb{I}, \Delta_m)$.

Use the formulae of the proof of Proposition 1.7, for $\beta_{m+1}$ and $\beta_m$, to show that the map

$$\varphi_{C^\infty(\mathbb{I}^{m-1})} \circ C^\infty(\mathbb{I}, \gamma_{m-1}) \circ \beta_m \circ y$$

of the unit interval $\mathbb{I}$ into $C^\infty\left(\mathbb{I}^{m-1}\right)$ is constant. Then one derives from the commutative diagram

$$\begin{tikzcd}
C^\infty(\mathbb{I}, \Delta_{m-1}) \arrow[r, "\varphi_{\Delta_{m-1}}"] \arrow[d, "C^\infty(\mathbb{I}, \gamma_{m-1})"'] & \Delta_{m-1} \arrow[d, "\gamma_{m-1}"] \\
C^\infty\left(\mathbb{I}, C^\infty\left(\mathbb{I}^{m-1}\right)\right) \arrow[r, "\varphi_{C^\infty(\mathbb{I}^{m-1})}"'] & C^\infty\left(\mathbb{I}^{m-1}\right).
\end{tikzcd}$$

that also the map

$$\gamma_{m-1} \circ \varphi_{\Delta_{m-1}} \circ \beta_m \circ y$$

is constant. Hence, since $\gamma_{m-1}$ is injective and $d^m_m = \varphi_{\Delta_{m-1}} \circ \beta_m$ (see $(D3)$, p. 4), the mapping $C^\infty(\mathbb{I}, d^m_m)(y) \in C^\infty(\mathbb{I}, \Delta_{m-1})$ is constant and, by the definition of $\Delta_{m+1}$, there is an element $x \in \Delta_m$ such that $\beta_{m+1}(x) = y$. For this $x$ we have

$$\gamma_{m+1}(x) = x' \circ \gamma_{m+1},$$

as required. □

**Corollary 2.1.** *The locally convex algebra $\Delta_n$ can be identified with the locally convex subalgebra of $C^\infty(\mathbb{I}^n)$ consisting of the elements $z \in C^\infty(\mathbb{I}^n)$ satisfying*

$$z(\tau_1, \cdots, \tau_{k-1}, 0, \tau_{k+1}, \cdots, \tau_n) = z(\tau_1, \cdots, \tau_{k-1}, 0, \cdots, 0), \qquad \forall\, \tau_1, \cdots, \tau_n \in \mathbb{I},$$

*where $k = 1, \cdots, n$.*

*The face and degeneracy operators are given as follows (cf. Example 1.1)*

$$(d_i z)(\tau_1, \cdots, \tau_n) = \begin{cases} z(\tau_1, \cdots, \tau_i, 1, \tau_{i+1}, \cdots, \tau_n), & i < n+1 \\ z(\tau_1, \cdots, \tau_n, 0), & i = n+1 \end{cases}$$

$$(s_i z)(\tau_1, \cdots, \tau_{n+1}) = \begin{cases} z(\tau_1, \cdots, \tau_i, \tau_{i+1}\tau_{i+2}, \cdots, \tau_{n+1}), & i < n \\ z(\tau_1, \cdots, \tau_{n+1}), & i = n. \end{cases}$$

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