Σ-products and selections of set-valued mappings

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Abstract. Every lower semi-continuous closed-and-convex valued mapping $\Phi : X \to 2^Y$, where X is a Σ-product of metrizable spaces and Y is a Hilbert space, has a single-valued continuous selection. This improves an earlier result of the author.

Keywords: set-valued mapping, l.s.c. mapping, Σ-product, selection

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1. Introduction

In [11] it is proved that every lower semi-continuous closed-and-convex valued mapping $\Phi : X \to 2^Y$ where X is collectionwise normal, countably paracompact and pseudoparacompact (i.e. the Dieudonné completion of X is paracompact) and Y is a reflexive Banach space, has a single-valued continuous selection. It is easy to see that, in the above statement, the requirement on X to be collectionwise normal and countably paracompact is necessary. The pseudoparacompactness of X is a sufficient but not a necessary condition for such a selection to exist. Namely, as it is shown in [12], if X is a Σ-product of separable metric spaces (in particular-real lines) and Y is a Hilbert space, then every lower semi-continuous closed-and-convex valued mapping $\Phi : X \to 2^Y$ admits a single-valued continuous selection. In the present paper we prove that the last result remains true in case X is a Σ-product of arbitrary metric spaces as well. Note that the Σ-product of uncountably many real lines is known to be collectionwise normal and countably paracompact but not pseudoparacompact ([7]).

Theorem 1.1. *Let X be a Σ-product of metric spaces, Y be a Hilbert space and $\Phi : X \to 2^Y$ be an l.s.c. closed-and-convex valued mapping. Then Φ has a single-valued continuous selection.*

Note that Theorem 1.1 can be regarded as a new argument in support of the following

Conjecture 1.2 (M. Choban, V. Gutev, S. Nedev [2]). *Every l.s.c. closed-and-convex valued mapping $\Phi : X \to 2^Y$, where X is collectionwise normal and countably paracompact and Y is a Hilbert space, has a single-valued continuous selection.*

2. Notations and terminology

Let Y be a Banach space. We put $B_r = \{y \in Y : \|y\| < r\}$, $D_r = \{y \in Y : \|y\| \leq r\}$ for every $r \geq 0$, $B_\epsilon(y) = \{z \in Y : \|z - y\| < \epsilon\}$ and $D_\epsilon(y) = \{z \in Y : \|z - y\| \leq \epsilon\}$ for every $y \in Y$ and $\epsilon > 0$.

If A is a set, then 2^A denotes the set of all nonempty subsets of A. If X and Y are topological spaces, a set-valued mapping $\Phi : X \to 2^Y$ is called *lower semi-continuous* (*l.s.c.*) if $\Phi^{-1}(U) = \{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$.

A mapping $\psi : X \to 2^Y$ is called a *selection* for Φ if $\psi(x) \subset \Phi(x)$ for every $x \in X$. A T_2-space X is *paracompact* ([4]) (resp. *countably paracompact*) if every open cover (resp. every countable open cover) of X has an open locally finite refinement. A T_1-space is *collectionwise normal* ([1]) if every discrete family of its closed subsets can be separated by a disjoint family of open subsets. The Σ-*product* (see [3]) of a family of topological spaces $\{X_s\}_{s\in S}$ with the base point $x = \{x_s\} \in \prod_{s\in S} X_s$ is the subspace

$$\Sigma(x) = \{y = \{y_s\} : |\{s \in S : x_s \neq y_s\}| \leq \aleph_0\}$$

of the Tykhonov product $\prod_{s\in S} X_s$

3. Proof of Theorem 1.1

Let $X = \Sigma(a)$ be a Σ-product of metric spaces $\{M_s\}_{s\in S}$ with a base point $a = \{a_s\}_{s\in S}$. Since $\Sigma(a)$ is countably paracompact ([3, Corollary 1]), by [12, Lemma 4.1], we may suppose, without loss of generality that there exists $r > 0$ such that $\Phi(x) \subset B_r$ for every $x \in \Sigma(a)$. By E. Michael's technique [9, the proof of Theorem 3.2$''$], in order to construct a single-valued continuous selection for Φ it is sufficient to find for every $\epsilon > 0$ a locally finite open cover of $\Sigma(a)$ that refines $\mathcal{B} = \{\Phi^{-1}(B_\epsilon(y)) : y \in Y\}$. To this end, we shall construct by induction a σ-locally finite open cover $\mathcal{O}$ of $\Sigma(a)$ that refines $\mathcal{B}$ ($\Sigma(a)$ is countably paracompact and hence every σ-locally finite open cover of $\Sigma(a)$ has a locally finite open refinement).

Fix $\epsilon > 0$. For every $x \in \Sigma(a)$, denote by $f(x)$ the only point of $\Phi(x)$ whose norm is equal to $\inf\{\|y\| : y \in \Phi(x)\}$.

Following the proof of [12, Theorem 1.1] we put $X_0 = \Sigma(a)$ and $\xi_0 = \sup\{\|f(x)\| : x \in X_0\}$. For every $i \in \mathbb{N}$ define an l.s.c. $\Phi_0^i : X_0 \to 2^Y$ by

$$\Phi_0^i(x) = cl(\Phi(x) \cap B_{\xi_0 + 1/i}), \quad x \in X_0.$$

Since Y is a Hilbert space, by [12, Lemma 3.1] there exists $i(0) \in \mathbb{N}$ such that

$$(*(0)) : \qquad \operatorname{diam}(\Phi_0^{i(0)}(x)) < \epsilon/6 \ \text{ for each } \ x \in A_0 = X_0 \backslash \Phi^{-1}(B_{\xi_0 - 1/i(0)}).$$

Now we apply the construction used in [10]. For every intersection $V = \Sigma(a) \cap \prod_{s\in S} U_s$, where $\prod_{s\in S} U_s$ is an element of the canonical base $\mathcal{V}$ of the product

$\prod_{s\in S} M_s$ (that is U_s is open in M_s for every $s \in S$ and $U_s \neq M_s$ for no more than finitely many $s \in S$), put $S(V) = \{s \in S : U_s \neq M_s\}$. For each $x = \{x_s\} \in \Sigma(a)$ let $\{s \in S : x_s \neq a_s\} = \{s_{x,1}, s_{x,2}, \dots\}$. For every $s \in S$ fix a sequence $\mathcal{U}_{s,1}, \mathcal{U}_{s,2}, \dots$ of locally finite open covers of M_s such that any element of $\mathcal{U}_{s,i}$ is an union of elements of $\mathcal{U}_{s,i+1}$ and its diameter is less than $1/i$. Put $\mathcal{V}_i = \{V = \Sigma(a) \cap \prod_{s\in S} U_s : \emptyset \neq \prod_{s\in S} U_s \in \mathcal{V}$ and $U_s \in \mathcal{U}_{s,i}$ for $s \in S(V)\}$ for $i = 1, 2, \dots$.

Fix a mapping φ_0 assigning to every couple $(V, [x(1), x(2), \dots, x(n)])$, where V is an open subset of A_0 which is not covered by finitely many elements of $\mathcal{B}$ and $x(1), x(2), \dots, x(n)$ is a finite (or empty) sequence of points of A_0, a point

$$\varphi_0(V, [x(1), x(2), \dots, x(n)]) \in V \setminus \bigcup_{i=1}^{n} (\Phi_0^{i(0)})^{-1}(B_\epsilon(f(x(i)))),$$

(or $\varphi_0(V; \emptyset) \in V$).

We denote by $\mathcal{L}_0$ the family of all finite sequences $V_0, V_1, \dots, V_n$ of open subsets of $\Sigma(a)$ satisfying the following conditions:

(1) $\Sigma(a) = V_0 \supset V_1 \supset, \dots, \supset V_n$ and $V_i \in \mathcal{V}_i$ for $i = 1, 2, \dots, n$;

(2) $V_n \cap A_0 \neq \emptyset$ and $V_i \cap A_0$ is not covered by finitely many elements of $\mathcal{B}$ for $i = 1, 2, \dots, n-1$;

(3) $S(V_i) = \{s_{x(k),j} : k \leq i-1, j \leq i\}$ for $i = 1, 2, \dots, n$ where $x(0) = \varphi_0(A_0; \emptyset)$ and $x(k) = \varphi_0(V_k \cap A_0, [x(0), x(1), \dots, x(k-1)])$ for $k = 1, 2, \dots, n-1$.

Let $\mathcal{U}_0$ be the family of the last elements V_n of those elements of $\mathcal{L}_0$ for which $V_n \cap A_0$ is covered by finitely many elements of $\mathcal{B}$.

Let us verify that $\mathcal{U}_0$ covers A_0. Suppose that there exists $x = \{x_s\} \in A_0 \setminus \bigcup \mathcal{U}_0$ and construct a sequence $\{V_i\}_{i=0}^{\infty}$ of open subsets of $\Sigma(a)$ and a discrete subset $\{x(i)\}_{i=0}^{\infty}$ of A_0 in the following manner:

Put $V_0 = \Sigma(a)$ and $x(0) = \varphi_0(A_0; \emptyset)$. Take V_1 such that $x \in V_1$, $V_1 \in \mathcal{V}_1$ and $S(V_1) = \{s_{x(0),1}\}$. Obviously V_0, V_1 is a sequence of $\mathcal{L}_0$. Suppose $n \in \mathbb{N}$ and we have constructed $V_0, V_1, \dots, V_n$ and $x(0), x(1), \dots, x(n-1)$ satisfying (1), (2) and (3) with $x \in V_n$. By assumption $x \notin \bigcup \mathcal{U}_0$ and hence $V_n \cap A_0$ is not covered by finitely many elements of $\mathcal{B}$. So we put $x(n) = \varphi_0(V_n \cap A_0, [x(0), x(1), \dots, x(n-1)])$ and pick V_{n+1} such that $x \in V_{n+1} \subset V_n$, $V_{n+1} \in \mathcal{V}_{n+1}$ and $S(V_{n+1}) = \{s_{x(k),j} : k \leq n, j \leq n+1\}$.

It follows, from $(*(0))$ and the definition of φ_0 that, for every $\tilde{x} \in A_0$, the set $A_0 \cap (\Phi_0^{i(0)})^{-1}(B_{\epsilon/6}(f(\tilde{x})))$ is a neighborhood of $\tilde{x}$ in A_0 which meets no more than one element of $\{x(i)\}_{i=0}^{\infty}$, i.e. $\{x(i)\}_{i=0}^{\infty}$ is discrete in A_0 and hence in $\Sigma(a)$. Observe that if $s \in S(V_i)$ for some $i \in \mathbb{N}$, then $s \in S(V_j)$ for every $j \geq i$. Therefore, since $x_i \in V_i \in \mathcal{V}_i$ for every $i \in \mathbb{N}$, then $\lim_{i\to\infty} x(i)_s = x_s$ for every $s \in \bigcup_{i=1}^{\infty} S(V_i)$. In other words $\{x(i)\}_{i=0}^{\infty}$ converges to the point $x' = \{x'_s\}$ where

$$x'_s = \begin{cases} x_s, & s \in \bigcup_{i=1}^{\infty} S(V_i) \\ a_s, & s \notin \bigcup_{i=1}^{\infty} S(V_i). \end{cases}$$

This is a contradiction and hence $\mathcal{U}_0$ covers A_0.

Now let us verify that $\mathcal{U}_0^n = \{V_n : V_0, V_1, \ldots, V_n$ is an element of $\mathcal{L}_0\}$ is locally finite in $\Sigma(a)$ for every $n \in \mathbb{N}$. Obviously $\mathcal{U}_0^0 = \{\Sigma(a)\}$ is locally finite. Suppose $n \in \mathbb{N}$ and $\mathcal{U}_0^k$ is locally finite in $\Sigma(a)$ for every $k \leq n$. Let $x = \{x_s\} \in \Sigma(a)$ be arbitrary. Take a neighborhood T of x in $\Sigma(a)$ which meets only finitely many elements of $\bigcup\{\mathcal{U}_0^k : k = 1, 2, \ldots n\}$ and let $S' = \bigcup\{S(V_{n+1}) : V_{n+1} \in \mathcal{U}_0^{n+1}, V_{n+1} \cap T \neq \emptyset\}$. It is clear from (3), that $S(V_{n+1})$ is uniquely determined by $V_0, V_1, \ldots, V_n$. Hence, since there are only finitely many elements $V_0, V_1, \ldots, V_n$ of $\mathcal{L}_0$ with $T \cap V_n \neq \emptyset$, S' is finite. Define a neighborhood O of x in $\Sigma(a)$ in the following way:

$$O = T \cap (\prod_{s \in S} O_s) \cap \Sigma(a),$$

where $O_s = M_s$ for every $s \in S \backslash S'$ and O_s is a neighborhood of x_s in M_s which escapes all but finitely many elements of $\mathcal{U}_{s,n+1}$ for $s \in S'$. Clearly O intersects only finitely many elements of $\mathcal{U}_0^{n+1}$. Thus we have actually shown, by induction, that $\mathcal{U}_0^n$ is locally finite in $\Sigma(a)$ for every $n \geq 0$. Since $\mathcal{U}_0 \subset \bigcup_{n=0}^\infty \mathcal{U}_0^n$, $\mathcal{U}_0$ is a σ-locally finite family in $\Sigma(a)$ which covers A_0. By definition, for every $V \in \mathcal{U}_0$ there exists a finite family $\mathcal{B}(V) \subset \mathcal{B}$ that covers $V \cap A_0$. Hence the family $\mathcal{P}_0 = \bigcup\{\{V \cap A_0 \cap B : B \in \mathcal{B}(V)\} : V \in \mathcal{U}_0\}$ is a σ-locally finite open (in A_0) covering of A_0 whose elements are contained in the elements of $\mathcal{B}$. Since $\Sigma(a)$ is collectionwise normal ([6]) and countably paracompact, by [8], there exists a σ-locally finite and open (in $\Sigma(a)$) family $\mathcal{O}_0$ such that $\mathcal{P}_0 = \{O \cap A_0 : O \in \mathcal{O}_0\}$. Without loss of generality, we may assume that every element of $\mathcal{O}_0$ is contained in some element of $\mathcal{B}$.

Now, suppose, for every $\gamma < \alpha < \omega_1$, X_γ, A_γ and $\mathcal{O}_\gamma$ have been constructed with: X_γ and A_γ are nonvoid closed subsets of $\Sigma(a)$, $X_\gamma \supset A_\gamma$ and $\mathcal{O}_\gamma$ is a σ-locally finite open (in $\Sigma(a)$) cover of A_γ whose elements are contained in elements of $\mathcal{B}$.

If $\bigcup(\bigcup_{\gamma<\alpha} \mathcal{O}_\gamma) = \Sigma(a)$, then we merely take $\mathcal{O} = \bigcup_{\gamma<\alpha} \mathcal{O}_\gamma$. Otherwise put $X_\alpha = \Sigma(a) \backslash \bigcup(\bigcup_{\gamma<\alpha} \mathcal{O}_\gamma)$. As before, let $\xi_\alpha = \sup\{\|f(x)\| : x \in X_\alpha\}$ and for every $i \in \mathbb{N}$ define $\Phi_\alpha^i : X_\alpha \to 2^Y$ by $\Phi_\alpha^i(x) = cl(\Phi(x) \cap B_{\xi_\alpha+1/i})$, $\quad x \in X_\alpha$. We take $i(\alpha) \in \mathbb{N}$ such that

$$(*(\alpha)) \qquad \operatorname{diam}(\Phi_\alpha^{i(\alpha)}(x)) < \epsilon/6 \text{ for each } x \in A_\alpha = X_\alpha \backslash \Phi^{-1}(B_{\xi_\alpha - 1/i(\alpha)}).$$

In the same way as above we find a σ-locally finite open (in $\Sigma(a)$) cover $\mathcal{O}_\alpha$ of A_α such that every element of $\mathcal{O}_\alpha$ is contained in some element of $\mathcal{B}$, which completes the proof.

Note that $\alpha < \gamma < \omega_1$ implies $\xi_\gamma < \xi_\alpha$. Then we have $\bigcup(\bigcup_{\gamma<\alpha'} \mathcal{O}_\gamma) = \Sigma(a)$ for some $\alpha' < \omega_1$, otherwise we get a strictly decreasing transfinite sequence of real numbers $\{\xi_\gamma\}_{\gamma<\omega_1}$, which is impossible. Therefore $\mathcal{O} = \bigcup_{\gamma<\alpha'} \mathcal{O}_\gamma$ is a σ-locally finite open refinement of $\mathcal{B}$.

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