

## On clopen sets in Cartesian products

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*Abstract.* The results concern clopen sets in products of topological spaces. It is shown that a clopen subset of the product of two separable metrizable (or locally compact) spaces is not always a union of clopen boxes. It is also proved that any clopen subset of the product of two spaces, one of which is compact, can always be represented as a union of clopen boxes.

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In notation and terminology we will follow [ENG]. In particular, a set  $A$  is *clopen* in a space  $X$  if it is closed and open in  $X$ . A *clopen box* in a space  $X \times Y$  is a clopen subset of the form  $U \times V$ , where  $U$  and  $V$  are clopen subsets of  $X$  and  $Y$ , respectively. It is known that a subset of the Cartesian product of two spaces is open if and only if it can be represented as a union of open boxes. When the same is true for clopen subsets?

### §1. Example 1

The following question was first formulated by Alexander Shostak at the beginning of 90's. He posed it in his talk at a seminar on General Topology at Moscow University. Independently, Andrej Bauer asked the same question, motivated by some problems in Computer Science on which he was working on.

**Question 1** (A. Bauer and A. Shostak). *Is it always true that any clopen subset of the Cartesian product of two spaces can be represented as a union of clopen boxes?*

Bauer also noticed the following interpretation of Question 1.

For a topological space  $X$  let  $z(X)$  be its zero-dimensional reflection, i. e., the same underlying set  $X$  but with topology generated by the collection of all clopen subsets of  $X$ . *Does zero-dimensional reflection commute with product operation?*

Motivation of Shostak for the question was purely topological. He considered the following property  $P$  of a space  $X$ : *every covering of  $X$  by clopen subsets contains a finite subcovering*. Shostak wanted to know if this property is preserved by finite products. Clearly, if the answer to Question 1 were affirmative then the answer to the question about productivity would be positive as well. Later, several

counterexamples to the latter question were found (see [STE], [SHO], and [SaS]). That provides counterexamples to Question 1. However, the spaces involved in those counterexamples are of large cardinality, non-metrizable, and non-locally compact. So, it would be interesting to restrict Question 1 to separable metrizable spaces or to spaces with strong compactness-type properties.

**Question 2** (A. Bauer and A. Shostak). *Is it always true that any clopen subset of the Cartesian product of two separable metrizable spaces can be represented as a union of clopen boxes?*

**Example 1.** *There exist two spaces,  $X$  (a locally compact subspace of  $\mathbb{R}^2$ ) and  $Z$  (a countable  $G_\delta$ -subspace of  $\mathbb{R}$ ), whose product contains a clopen set that cannot be represented as a union of clopen boxes.*

**Construction of  $X$ .**

Let  $X_1$  be a subset of  $\mathbb{R}^2$  consisting of the ray  $\{(x, y) \in \mathbb{R}^2 : y = 0, x \geq 1\}$ . For any  $n \in \mathbb{N}$ , take the sequence  $S_n = \{(n, 1/k) : k \in \mathbb{N} \setminus \{1\}\}$  that converges to  $(n, 0)$ . Put  $X = X_1 \cup (\bigcup\{S_n : n \in \mathbb{N}\})$ . The topology in  $X$  is inherited from  $\mathbb{R}^2$ . For further reference, let  $O_n^k$  be a fixed neighborhood of  $(n, 0)$  of radius  $1/k$ .

**Construction of  $Z$ .**

Let  $Z = \{0\} \cup \{K_n : n \in \mathbb{N} \setminus \{1\}\}$ , where  $K_n = \{a_k^n : a_k^n = 1/n + (1/(n - 1) - 1/n)/2^k, k \in \mathbb{N}\}$ . That is,  $K_n$  is a sequence of numbers converging to  $1/n$  and lying in between  $1/n$  and  $1/(n - 1)$ . So,  $Z$  is a countable metrizable non-compact space with only one non-isolated point 0.

The space  $X \times Z$  is the space we are looking for.

Indeed,  $X \times Z$  contains a clopen set  $A$  that cannot be represented as a union of clopen boxes.

**Construction of  $A$ .**

Let  $A_1$  be the union of all copies of the connected part  $X_1$  of  $X$  in  $X \times Z$ . That is,

$$A_1 = X_1 \times Z.$$

$A_1$  is closed in  $X \times Z$  but not open since points of the form  $(n, 0, z)$  are on the boundary. Let us first supply points  $(n, 0, 0)$  with their neighborhoods. Let  $U(n, 0, 0) = O_n^1 \times (Z \setminus \bigcup\{K_l : l < n\})$ . So, the sets  $U(n, 0, 0)$ 's form together a staircase.

Let  $A_2 = A_1 \cup (\bigcup\{U(n, 0, 0) : n \in \mathbb{N}\})$ . The set  $A_2$  is still not clopen since points  $(n, 0, z)$ , where  $z \in K_l$  for  $l < n$ , are on the boundary. Let us supply these points with neighborhoods.

Consider  $K_l = \{a_k^l : k \in \mathbb{N}\}$  (see the definition of  $Z$ ). For each  $a_k^l \in K_l$ , let  $U(n, 0, a_k^l) = O_n^k \times \{a_k^l\}$ . That is,  $U(n, 0, a_k^l)$  is the neighborhood of  $(n, 0)$  of radius  $1/k$  on level  $X \times \{a_k^l\}$ .

Put

$$A = A_2 \cup \left( \bigcup \{U(n, 0, a_k^l) : n, l, k \in \mathbb{N}\} \right).$$

$A$  is the set we need.

*The set  $A$  is open for the following reasons:*

1. Every point of form  $(x, y, z)$ , where  $x$  is not a natural number, is in  $A$  with an open neighborhood since  $(X_1 \setminus \{(n, 0) : n \in \mathbb{N}\}) \times Z$  is open, is a subset of  $A$ , and contains all points of this form. (That is, the ray  $X_1$  is in  $A$  on each level).
2. All other points in  $A$  are either in  $U(n, 0, 0)$  or in  $U(n, 0, a_k^l)$ .

*The set  $A$  is closed for the following reasons:*

1. The closure of  $A$  is the union of the closures of the traces of  $A$  on each level  $X \times \{z\}$ , where  $z$  is in  $Z$ , since  $X \times \{0\}$  is the only non-isolated level in our product and it is entirely in  $A$ .
2. On each discrete level  $X \times \{a_k^l\}$ , where  $a_k^l$  is in  $K_l$  (constructed for  $Z$ ), the complement of  $A$  is a collection of isolated points, and therefore, open. So, the trace of  $A$  on each level is closed.

Thus,  $A$  is clopen. Now, let us prove that  $A$  cannot be represented as a union of clopen sets of the form  $U \times V$ . Indeed, any clopen neighborhood of the point  $(1, 0, 0)$  of the form  $U \times V$  in  $X \times Z$  must contain the entire ray  $X_1$  on 0's level (that is, set  $X_1 \times \{0\}$ ) because of connectedness. And therefore, any clopen box of  $(1, 0, 0)$  in  $X \times Z$  must contain a set of the form

$$\bigcup \{O_n^k : O_n^k \text{ is one of fixed neighborhoods of } (n, 0), n \in \mathbb{N}\} \times (Z \setminus \bigcup \{K_l : l < m, \text{ for some fixed } m \in \mathbb{N}\}),$$

that is, a set obtained by multiplying the union of neighborhoods of  $(n, 0)$ 's by the space  $Z$  without a finite number of  $K_l$ 's.

(This is shown by an obvious connectedness argument).

But this is impossible, since for each  $n$ ,  $O_n^k \times (Z \setminus \bigcup \{K_l : l < m\})$ , where  $m$  is fixed, is contained in  $A$  only when  $n \leq m$ .

*Note.* There is nothing special about  $(1, 0, 0)$ . You can take any point that lies in the connected part of  $X \times \{0\}$ .

## §2. Arhangel'skii's observations

Example 1 gives us a negative answer to Question 2. However, it would be interesting to find some conditions under which a clopen set in a product is always a union of clopen boxes. Arhangel'skii noticed that the following theorem holds.

**Theorem 1.** *Let  $X$  and  $Y$  be topological spaces and let  $X$  be compact. Then any clopen subset of  $X \times Y$  can be represented as a union of clopen boxes.*

This fact is a direct consequence of Theorem 2 proved below.

The next statement is well known and easy to prove (see Lemma 3.1.15 in [ENG]):

**Lemma 1.** *Suppose  $F$  is a compact subspace of a space  $Y$ ,  $x$  is a point of a space  $X$ , and  $W$  is an open subset of the product space  $X \times Y$  such that  $\{x\} \times F \subset W$ . Then there exists an open subset  $V$  in  $X$  such that  $x \in V$  and  $V \times F \subset W$ .*

Now we apply Lemma 1 to prove the next statement:

**Lemma 2.** *Suppose  $F$  is a compact subspace of a space  $Y$ ,  $X$  a space, and  $W$  an open and closed subset of the product space  $X \times Y$ . Then the set  $U_F = \{x \in X : \{x\} \times F \subset W\}$  is an open and closed subset of  $X$ .*

PROOF: Indeed, it is immediate from Lemma 1 that  $U_F$  is open in  $X$ . Now take any  $x \in \overline{U_F}$ . By the definition of  $U_F$ , we have  $U_F \times F \subset W$ . Therefore,

$$\{x\} \times F \subset \overline{U_F} \times F \subset \overline{U_F \times F} \subset \overline{W} = W,$$

which implies that  $x \in U_F$ . Hence, the set  $U_F$  is closed.  $\square$

**Theorem 2.** *Suppose  $Y$  is a compact space,  $X$  is a space,  $W$  an open and closed subset of the product space  $X \times Y$ , and  $(a, b)$  a point in  $W$ . Then there exist an open and closed subset  $U$  in  $X$  and an open and closed subset  $V$  in  $Y$  such that  $(a, b) \in U \times V \subset W$ .*

PROOF: Put  $F = \{y \in Y : (a, y) \in W\}$ . Since  $W$  is closed, the set  $F$  is closed in  $Y$  and therefore, compact. Besides,  $F$  is open, since  $W$  is open. We also have  $b \in F$  and  $\{b\} \times F \subset W$ . From Lemma 2 it follows that the set  $U_F = \{x \in X : \{x\} \times F \subset W\}$  is an open and closed subset of  $X$ . Clearly,  $a \in U_F$  and  $(a, b) \in U_F \times F \subset W$ . Thus,  $U = U_F$  and  $V = F$  are the sets we are looking for.  $\square$

**Remark.** Theorem 2 obviously generalizes to the case when  $Y$  is any space satisfying the following condition:

(lc) For each  $y \in Y$  there exists an open and closed subset  $V$  of  $Y$  such that the subspace  $V$  is compact.

**§3. Example 2**

Since in Example 1  $X$  is locally compact, compactness of  $X$  is important in Theorem 1. However, a natural question arises.

**Question 3** (M. Reed). *Let  $X$  and  $Y$  be locally compact Hausdorff spaces. Is it true that any clopen subset of  $X \times Y$  can be represented as a union of clopen boxes?*

**Example 2.** *There exist two locally compact Hausdorff spaces  $X$  and  $Y$  whose product contains a clopen subset that cannot be represented as a union of clopen boxes.*

Let  $X$  and  $Z$  be spaces from Example 1.

Let  $Y_1 = \bigcup\{N_n : N_n \text{ is a copy of } \mathbb{N}, n \in \mathbb{N}\}$  with discrete topology. Consider  $Y_2 = \beta Y_1 \setminus [\bigcup\{\beta N_n \setminus N_n : n \in \mathbb{N}\}]_{\beta Y_1}$ . That is,  $Y_2$  is obtained from  $\beta Y_1$  by removing the closure of the union of the remainders of  $N_n$ 's.

The space  $Y_2$  is locally compact (since it is obtained from a compactum by removing a closed subset) and satisfies the following property.

(0) *Any clopen neighborhood of closed set  $Y_2 \setminus Y_1$  contains almost all  $N_n$ 's except maybe finite number of them.* (Proof: any sequence of elements  $\{a_n \in N_n : n \in \mathbb{N}\}$  and the set  $\bigcup\{\beta N_n \setminus N_n : n \in \mathbb{N}\}$  are disjoint closed subsets of  $\sigma$ -compact space  $\bigcup\{\beta N_n : n \in \mathbb{N}\}$ . Therefore, their closures in Stone-Ćech compactification (which is  $\beta Y_1$ ) do not intersect. So the remainder of any sequence  $\{a_n \in N_n : n \in \mathbb{N}\}$  lies entirely in  $Y_2 \setminus Y_1$ .)

The space  $Y_2$  is almost what we need, but we want  $Y_2 \setminus Y_1$  to live in a connected component. To achieve this, let us again consider  $\beta Y_1$ , take the cone over  $\beta Y_1 \setminus \bigcup\{\beta N_n : n \in \mathbb{N}\}$ , and denote the resulting space by  $Y_3$ . Now, the space

$$Y = Y_3 \setminus [\bigcup\{\beta N_n \setminus N_n : n \in \mathbb{N}\}]_{\beta Y_1}$$

is the space we need. The space  $Y$  has the following properties:

- (1)  *$Y$  consists of closed discrete sets  $N_n$ 's and a connected component  $Y \setminus Y_1$  (by construction);*
- (2)  *$Y$  is locally compact (since obtained from a compactum by removing a closed subset);*
- (3) *any neighborhood of  $Y \setminus Y_1$  contains almost all  $N_n$ 's except maybe finite number of them (see property (0)).*

Let  $Y^*$  be a quotient space of  $Y$  under a partition whose only non-trivial element is the connected component  $Y \setminus Y_1$ . And let  $p : Y \rightarrow Y^*$  be a quotient map. The map  $p$  and space  $Y^*$  have the following properties.

(4)  $Y^*$  is homeomorphic to  $Z$  ( $Z$  is the space in Example 1). This follows from properties (1), (3), and from construction of  $Z$ .

(5) For any clopen subset  $U$  of  $Y$ ,  $p(U)$  is clopen in  $Y^*$ . (It follows from the definition of  $p$  and the fact that any clopen set that intersect  $Y \setminus Y_1$  must contain  $Y \setminus Y_1$  (due to connectedness)).

Now,  $X$  and  $Y$  are both locally compact. Let us prove that  $X \times Y$  contains a clopen set that cannot be represented as a union of clopen boxes. Let  $f = i \times p : X \times Y \rightarrow X \times Z$  be the product of the identity map  $i : X \rightarrow X$  and the quotient map  $p : Y \rightarrow Z$  (see property (5)). The map  $f$  satisfies the following property.

(6) Image of any clopen box in  $X \times Y$  under  $f$  is a clopen box in  $X \times Z$  (follows from the definition of product maps, property (5), and the fact that  $i$  is identity).

Let  $A$  be a clopen subset of  $X \times Z$  that cannot be represented as a union of clopen boxes (we constructed such a set in Example 2). Consider  $f^{-1}(A)$  in  $X \times Y$ . By continuity of  $f$ ,  $f^{-1}(A)$  is clopen in  $X \times Y$  and, by property (6),  $f^{-1}(A)$  cannot be represented as a union of clopen boxes.

The space  $Y$  in our example is not metrizable. Can we make it metrizable? The following theorem shows that it is impossible.

**Theorem 3** (Kenneth Kunen). *Suppose  $X$  and  $Y$  are both locally compact Hausdorff and paracompact. Then any clopen subset of  $X \times Y$  is a union of clopen boxes.*

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