Non-autonomous vector integral equations with discontinuous right-hand side

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Abstract. We deal with the integral equation $u(t) = f(t, \int_I g(t,z)u(z)\,dz)$, with $t \in I := [0,1]$, $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : I \times I \to [0,+\infty[$. We prove an existence theorem for solutions $u \in L^s(I,\mathbb{R}^n)$, $s \in]1,+\infty]$, where f is not assumed to be continuous in the second variable. Our result extends a result recently obtained for the special case where f does not depend explicitly on the first variable $t \in I$.

Keywords: vector integral equations, discontinuity, multifunctions, operator inclusions

Classification: 45P05, 47H15

1. Introduction

Let $I := [0,1]$, and consider the integral equation

$$u(t) = f\Big(\int_I g(t,z)\,u(z)\,dz\Big) \quad \text{for a.a.} \quad t \in I, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : I \times I \to [0,+\infty[$ are given functions. Recently [3], an existence theorem for solutions $u \in L^\infty(I,\mathbb{R})$ to equation (1) was established, where, unlike other recent results in the field, the continuity of the function f was not assumed. More precisely, f was required to be a.e. equal in a suitable interval $[0,\sigma]$ to a function $f^* : [0,\sigma] \to \mathbb{R}$ such that the set $\{x \in [0,\sigma] : f^*$ is discontinuous at $x\}$ has null 1-dimensional Lebesgue measure. Later [4], such result was extended to the case where $f : \mathbb{R}^n \to \mathbb{R}^n$, establishing an existence theorem for solutions $u \in L^\infty(I,\mathbb{R}^n)$ (Theorem 1 of [4]). In the latter result, the above assumption (which specifies what kind of discontinuity is allowed for f) has the following form: there exist a function $f^* : \prod_{i=1}^n [0,\sigma_i] \to \mathbb{R}^n$ (with suitable positive σ_i) and n subsets $E_1, \dots, E_n$ of $\prod_{i=1}^n [0,\sigma_i]$ such that the projection of each set E_i over the i-th axis has null 1-dimensional Lebesgue measure and

$$\{x \in \prod_{i=1}^n [0,\sigma_i] : f^* \text{ is discontinuous at } x\} \cup \\ \cup \{x \in \prod_{i=1}^n [0,\sigma_i] : f^*(x) \neq f(x)\} \subseteq \bigcup_{i=1}^n E_i. \tag{2}$$

Moreover, it was proved that such result is no longer true if the set $\bigcup_{i=1}^n E_i$ is replaced by an arbitrary set $E \subseteq \prod_{i=1}^n [0, \sigma_i]$ with null n-dimensional Lebesgue measure.

Our aim in this note is to prove a further extension of Theorem 1 of [4] to the more general case where the function f can depend explicitly on the variable $t \in I$. That is, we are interested in the study of the vector integral equation

$$u(t) = f\big(t\,, \int_I g(t,z)\, u(z)\, dz\big) \quad \text{for a.a. } t \in I, \tag{3}$$

where $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : I \times I \to [0, +\infty[$. We establish an existence result for solutions $u \in L^s(I, \mathbb{R}^n)$ (with $s \in]1, +\infty]$) which contains Theorem 1 of [4] as a special case. In particular, the function f will not be assumed to be continuous in the second variable, but only to satisfy, for a.a. $t \in I$, a condition analogous to (2) with respect to a function $f^* : I \times \prod_{i=1}^n]0, \sigma_i[\to \mathbb{R}^n$ (with suitable positive σ_i). The function $f^*(\,\cdot\,, x)$ will be assumed to be measurable for each fixed x in a countable dense subset of $\prod_{i=1}^n]0, \sigma_i[$. Consequently, as regards regularity of f, our assumptions are weaker than the usual Carathéodory condition assumed in the literature (f measurable with respect to $t \in I$ for all $x \in \mathbb{R}^n$ and continuous in $x \in \mathbb{R}^n$ for a.a. $t \in I$). In this direction, the reader can see for instance [2], [5], [6] (where the same equation is studied in the scalar case $n = 1$ to obtain existence of integrable solutions) and also [7], and references therein. In particular, we refer to [2], [7] for motivations for studying equation (3).

Before concluding this section, we point out that our result is obtained as an application of an existence result for inclusions of the type $\Psi(u)(t) \in F(\,t\,, \Phi(u)(t))$ established by O. Naselli Ricceri and B. Ricceri ([13]).

2. Notations

Essentially, we follow the same notations as in [4]. Let $n \in \mathbb{N}$ be fixed. We denote by m_n the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $i \in \{1, \dots, n\}$, we denote by $\pi_i : \mathbb{R}^n \to \mathbb{R}$ the projection over the i-th axis. If $x \in \mathbb{R}^n$, we put $x_i := \pi_i(x)$ (namely, we use subscripts to denote component of vectors). If $x, y \in \mathbb{R}^n$, we write $x < y$ (resp., $x \le y$) to indicate that $x_i < y_i$ (resp., $x_i \le y_i$) for all $i = 1, \dots, n$. If $x, y \in \mathbb{R}^n$, with $x < y$ (resp., $x \le y$), we put $]x, y[:= \prod_{i=1}^n \,]x_i, y_i[$ (resp., $[x, y] := \prod_{i=1}^n [x_i, y_i]$).

The space $\mathbb{R}^n$ (whose origin is denoted by 0_n) is considered with its Euclidean norm $\|\cdot\|_n$. If $x \in \mathbb{R}^n$, $\varepsilon > 0$, $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$, we put

$$\begin{aligned} B(x,\varepsilon) &:= \big\{y \in \mathbb{R}^n : \|x - y\|_n < \varepsilon\big\}, \\ \overline{B}(x,\varepsilon) &:= \big\{y \in \mathbb{R}^n : \|x - y\|_n \le \varepsilon\big\}, \\ d(x, A) &:= \inf_{v \in A} \|x - v\|_n. \end{aligned}$$

Moreover, we denote by $\overline{A}$ and $\overline{\mathrm{co}}\, A$ the closure and the closed convex hull of A, respectively.

If $p \in [1, +\infty]$, we denote by p' the conjugate exponent of p. Moreover, we denote by $L^p(I, \mathbb{R}^n)$ the space of all (equivalence classes of) measurable functions $u : I \to \mathbb{R}^n$ such that

$$\begin{aligned} \int_I \|u(t)\|_n^p \, dt < +\infty \qquad & \text{if} \quad p < +\infty, \\ \operatorname{ess\,sup}_{t \in I} \|u(t)\|_n < +\infty \qquad & \text{if} \quad p = +\infty, \end{aligned}$$

with the usual norm

$$\begin{aligned} \|u\|_{L^p(I,\mathbb{R}^n)} &:= \left(\int_I \|u(t)\|_n^p \, dt \right)^{\frac{1}{p}} \qquad & \text{if} \quad p < +\infty, \\ \|u\|_{L^\infty(I,\mathbb{R}^n)} &:= \operatorname{ess\,sup}_{t \in I} \|u(t)\|_n \qquad & \text{if} \quad p = +\infty. \end{aligned}$$

We put $L^p(I) := L^p(I, \mathbb{R})$. As usual, we denote by $C^0(I, \mathbb{R}^n)$ the space of all continuous functions $v : I \to \mathbb{R}^n$. Finally, we put $I_0 :=]0, 1[$.

We refer the reader to [1], [11] for the definitions and the basic facts about multifunctions.

3. The result

We now state our main result.

Theorem 1. *Let $\sigma \in \mathbb{R}^n$, with $0_n < \sigma$, $s \in]1, +\infty]$, and let $f : I \times]0_n, \sigma[\to \mathbb{R}^n$, $g : I \times I \to [0, +\infty[$, $\alpha : I \to \mathbb{R}^n$ measurable, $\beta \in L^s(I, \mathbb{R}^n)$, $\phi_0 \in L^j(I)$, with $j \geq s'$ and $j > 1$, $\phi_1 \in L^{s'}(I)$, and P a countable dense subset of $]0_n, \sigma[$. Assume that:*

(i) *for a.a. $t \in I$, one has*

$$0 < \alpha_i(t) < \operatorname{ess\,inf}_{x \in]0_n, \sigma[} \, f_i(t, x) \leq \operatorname{ess\,sup}_{x \in]0_n, \sigma[} \, f_i(t, x) < \beta_i(t) \quad \textit{for all} \quad i = 1, \ldots, n; \tag{4}$$

(ii) *one has*

$$0 < \|\phi_0\|_{L^{s'}(I)} \leq \min_{1 \leq i \leq n} \frac{\sigma_i}{\|\beta_i\|_{L^s(I)}} \; ;$$

(iii) *there exist sets $E_1, \ldots, E_n \subseteq]0_n, \sigma[$, with $m_1(\pi_i(E_i)) = 0$ for all $i = 1, \ldots, n$, and a function $f^* : I \times]0_n, \sigma[\to \mathbb{R}^n$ such that for each $x \in P$ the function $f^*(\cdot, x)$ is measurable and for a.a. $t \in I$ one has*

$$\Big(\big\{ x \in]0_n, \sigma[: f^*(t, x) \neq f(t, x) \big\} \cup \cup \big\{ x \in]0_n, \sigma[: f^*(t, \cdot) \ \textit{is discontinuous at} \ x \big\} \Big) \subseteq \bigcup_{i=1}^n E_i; \tag{5}$$

(iv) *for each* $t \in I$, *the function* $g(t,\cdot)$ *is measurable;*

(v) *for a.a.* $z \in I$, *the function* $g(\cdot\,,z)$ *is continuous in* I, *differentiable in* I_0 *and*

$$g(t,z) \leq \phi_0(z), \quad 0 < \frac{\partial g}{\partial t}(t,z) \leq \phi_1(z) \ \ \text{for all} \ \ t \in I_0.$$

Then there exists a solution $u \in L^s(I,\mathbb{R}^n)$ *to equation* (3).

Before proving Theorem 1, we need the two following propositions.

Proposition 1. *Let* $\sigma \in \mathbb{R}^n$, *with* $0_n < \sigma$, *let* $f : I\times\,]0_n,\sigma[\to \mathbb{R}^n$, $\alpha : I \to \mathbb{R}^n$ *and* $\beta : I \to \mathbb{R}^n$ *three given functions, with* α *and* β *measurable, and let* $K \subseteq I$ *measurable, with* $K \neq I$, *such that for each* $t \in I \setminus K$ *and each* $i = 1,\dots,n$ *one has*

$$\alpha_i(t) < \operatorname{ess\,inf}_{x\in\,]0_n,\sigma[}\, f_i(t,x) \leq \operatorname{ess\,sup}_{x\in]0_n,\sigma[}\, f_i(t,x) < \beta_i(t).$$

Moreover, assume that there exist a function $f^* : I\times\,]0_n,\sigma[\to \mathbb{R}^n$, *a set* $E \subseteq\,]0_n,\sigma[\,$, *with* $m_n(E) = 0$, *and a nonempty set* $P \subseteq\,]0_n,\sigma[$ *such that:*

(i) *for each* $t \in I \setminus K$, *one has*

$$\big\{x \in\,]0_n,\sigma[: f^*(t,x) \neq f(t,x)\big\} \cup \\ \cup \big\{x \in\,]0_n,\sigma[: f^*(t,\cdot\,) \ \text{ is discontinuous at } \ x\big\} \subseteq E;$$

(ii) *for each* $x \in P$, *the function* $f^*(\,\cdot\,,x)$ *is measurable.*

Then there exists a function $\hat{f} : I\times\,]0_n,\sigma[\to \mathbb{R}^n$ *satisfying:*

(a) *for all* $i = 1,\dots,n$ *one has*

$$\alpha_i(t) \leq \hat{f}_i(t,x) \leq \beta_i(t) \ \ \text{for all} \ \ t \in I \setminus K \ \ \text{and all} \ \ x \in\,]0_n,\sigma[\,;$$

(b) *for each* $t \in I \setminus K$, *one has*

$$\big\{x \in\,]0_n,\sigma[: \hat{f}(t,x) \neq f(t,x)\big\} \cup \big\{x \in\,]0_n,\sigma[: \hat{f}(t,\cdot\,) \text{ is discontinuous at } x\big\} \subseteq E;$$

(c) *for each* $x \in P$, *the function* $\hat{f}(\,\cdot\,,x)$ *is measurable.*

Proof: Let $t \in I \setminus K$ be fixed. For each $i = 1,\dots,n$, let

$$R_i(t) := \big\{x \in\,]0_n,\sigma[: f_i^*(t,x) \leq \alpha_i(t)\big\},$$
$$S_i(t) := \big\{x \in\,]0_n,\sigma[: f_i^*(t,x) \geq \beta_i(t)\big\},$$

and let

$$T(t) := \bigcup_{i=1}^{n} (R_i(t) \cup S_i(t)).$$

We claim that $T(t) \subseteq E$. Arguing by contradiction, assume that there exists $\hat{x} \in T(t) \setminus E$. Therefore, there is some $\hat{i} \in \{1, \dots, n\}$ such that $\hat{x} \in R_{\hat{i}}(t) \cup S_{\hat{i}}(t)$. Assume that $\hat{x} \in R_{\hat{i}}(t)$ (if $\hat{x} \in S_{\hat{i}}(t)$, we can argue in an analogous way). Hence we have

$$f^*_{\hat{i}}(t, \hat{x}) \leq \alpha_{\hat{i}}(t) < \operatorname{ess\,inf}_{x \in]0_n, \sigma[} \ f_{\hat{i}}(t, x).$$

Since $\hat{x} \notin E$, by assumption (i) the function $f^*(t, \cdot\,)$ is continuous at $\hat{x}$. Consequently, there exists $\lambda \in \mathbb{R}^n$, with $0_n < \lambda$, such that

$$f^*_{\hat{i}}(t, u) < \operatorname{ess\,inf}_{x \in]0_n, \sigma[} \ f_{\hat{i}}(t, x) \quad \text{for all} \quad u \in V :=]\hat{x} - \lambda, \hat{x} + \lambda[\subseteq]0_n, \sigma[\,,$$

which contradicts assumption (i) since $m_n(V) > 0$. Such a contradiction implies $T(t) \subseteq E$, as claimed. Therefore, we have proved that

$$T(t) \subseteq E \quad \text{for all} \quad t \in I \setminus K. \tag{6}$$

Now, let $\hat{f} : I \times]0_n, \sigma[\to \mathbb{R}^n$ be defined by setting

$$\hat{f}(t, x) = \begin{cases} f^*(t, x) & \text{if } \ t \in I \setminus K \text{ and } x \in]0_n, \sigma[\setminus T(t) \\ \beta(t) & \text{otherwise.} \end{cases}$$

Taking into account (6) and assumption (i), it follows easily from the construction that $\hat{f}$ satisfies conclusion (a) and also $\hat{f}(t, x) = f(t, x)$ for all $(t, x) \in (I \setminus K) \times (\,]0_n, \sigma[\setminus E)$. To conclude the proof of conclusion (b), let $\bar{t} \in I \setminus K$ and $\overline{x} \in]0_n, \sigma[\setminus E$ be fixed, and let us show that the function $\hat{f}(\bar{t}, \cdot\,)$ is continuous at $\overline{x}$. By (6) we have $\overline{x} \notin T(\bar{t})$, hence

$$\alpha_i(\bar{t}) < f^*_i(\bar{t}, \overline{x}) < \beta_i(\bar{t}) \ \text{ for all } \ i = 1, \dots, n.$$

Since by assumption (i) the function $f^*(\bar{t}, \cdot\,)$ is continuous at $\overline{x}$, there exists a neighborhood U of $\overline{x}$, with $U \subseteq]0_n, \sigma[\,$, such that

$$\alpha_i(\bar{t}) < f^*_i(\bar{t}, z) < \beta_i(\bar{t}) \ \text{ for all } \ i = 1, \dots, n \ \text{ and all } \ z \in U.$$

Consequently, we have $U \cap T(\bar{t}) = \emptyset$, hence $\hat{f}(\bar{t}, z) = f^*(\bar{t}, z)$ for all $z \in U$. This implies that $\hat{f}(\bar{t}, \cdot\,)$ is continuous at $\overline{x}$, as claimed. Finally we prove conclusion (c). To this aim, fix $x \in P$. Let

$$S := \big\{t \in I \setminus K : x \notin T(t)\big\} = \bigcap_{i=1}^{n} \big\{t \in I \setminus K : \alpha_i(t) < f^*_i(t, x) < \beta_i(t)\big\}.$$

By our assumptions, the set S is measurable. Since we have

$$\hat{f}(t, x) = \begin{cases} f^*(t, x) & \text{if } \ t \in S \\ \beta(t) & \text{if } \ t \in I \setminus S, \end{cases}$$

it follows from assumption (ii) that $\hat{f}(\,\cdot\,, x)$ is measurable. □

The following proposition recollects some known facts about multifunctions. For the reader's convenience, we provide a short proof.

Proposition 2. *Let $\psi : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a given function, and let D be a countable dense subset of $\mathbb{R}^n$. Assume that:*

(i) *for each $t \in I$, the function $\psi(t, \cdot\,)$ is bounded;*
(ii) *for each $x \in D$, the function $\psi(\,\cdot\,, x)$ is measurable.*

Let $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be the multifunction defined by setting

$$F(t,x) := \bigcap_{m\in\mathbb{N}} \overline{\mathrm{co}}\Big(\bigcup_{\substack{y\in D \\ \|y-x\|_n \le \frac{1}{m}}} \{\psi(t,y)\} \Big). \tag{7}$$

Then one has:

(a) *$F(t,x) \neq \emptyset$ for all $(t,x) \in I \times \mathbb{R}^n$;*
(b) *for each $x \in \mathbb{R}^n$, the multifunction $F(\,\cdot\,, x)$ is measurable;*
(c) *for each $t \in I$, the multifunction $F(t, \cdot\,)$ has closed graph;*
(d) *if $t \in I$ and $\psi(t,\cdot\,)$ is continuous at $x \in \mathbb{R}^n$, then $F(t,x) = \{\psi(t,x)\}$.*

Proof: (a). Let $(t,x) \in I \times \mathbb{R}^n$ be fixed. For each $m \in \mathbb{N}$, put

$$A_m := \overline{\mathrm{co}}\Big(\bigcup_{\substack{y\in D \\ \|y-x\|_n \le \frac{1}{m}}} \{\psi(t,y)\} \Big).$$

Since the set D is dense in $\mathbb{R}^n$, it is immediate to see that $A_m \neq \emptyset$ for all $m \in \mathbb{N}$. Consequently, since $A_{m+1} \subseteq A_m$ for all $m \in \mathbb{N}$, the family $\{A_m\}_{m\in\mathbb{N}}$ has the finite intersection property. Since each A_m is closed, by assumption (i) it follows that $F(t,x) = \bigcap_{m\in\mathbb{N}} A_m \neq \emptyset$, as desired.

(b). Fix $x \in \mathbb{R}^n$. By assumption (ii) and Theorems 8.2.2 and 8.2.4 of [1], for each fixed $m \in \mathbb{N}$ the multifunction

$$t \in I \to \overline{\mathrm{co}}\Big(\bigcup_{\substack{y\in D \\ \|y-x\|_n \le \frac{1}{m}}} \{\psi(t,y)\} \Big)$$

is measurable. Again by Theorem 8.2.4 of [1], the multifunction $t \to F(t,x)$ is measurable.

(c). Fix $t \in I$. Let $\{\hat{x}^p\}$ and $\{\hat{y}^p\}$ be two sequences in $\mathbb{R}^n$, converging to $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^n$, respectively, such that

$$\hat{y}^p \in F(t, \hat{x}^p) \quad \text{for all} \quad p \in \mathbb{N}. \tag{8}$$

Let $m \in \mathbb{N}$ be chosen. Let $\nu \in \mathbb{N}$ be such that

$$\|\hat{x}^p - x^*\|_n \le \frac{1}{2m} \quad \text{for all} \quad p \ge \nu. \tag{9}$$

By (8) and (9), for each $p \geq \nu$ we have

$$\hat{y}^p \in \overline{\mathrm{co}}\Big(\bigcup_{\substack{y \in D \\ \|y - \hat{x}^p\|_n \leq \frac{1}{2m}}} \{\psi(t,y)\}\Big) \subseteq \overline{\mathrm{co}}\Big(\bigcup_{\substack{y \in D \\ \|y - x^*\|_n \leq \frac{1}{m}}} \{\psi(t,y)\}\Big).$$

Since the last set does not depend on p, we get

$$y^* \in \overline{\mathrm{co}}\Big(\bigcup_{\substack{y \in D \\ \|y - x^*\|_n \leq \frac{1}{m}}} \{\psi(t,y)\}\Big).$$

As $m \in \mathbb{N}$ was arbitrary, we get $y^* \in F(t, x^*)$, as desired.

(d). Let $t \in I$ be fixed, and let $x \in \mathbb{R}^n$ be such that $\psi(t, \cdot\,)$ is continuous at x. Let $\varepsilon > 0$ be fixed. Then, there exists $\delta > 0$ such that

$$\psi(t, \overline{B}(x,\delta)) \subseteq \overline{B}(\psi(t,x), \varepsilon).$$

Consequently, for each $m > \frac{1}{\delta}$ one has

$$\overline{\mathrm{co}}\Big(\bigcup_{\substack{y \in D \\ \|y - x\|_n \leq \frac{1}{m}}} \{\psi(t,y)\}\Big) \subseteq \overline{B}(\psi(t,x), \varepsilon),$$

hence $F(t,x) \subseteq \overline{B}(\psi(t,x), \varepsilon)$. Since ε was arbitrary and $F(t,x) \neq \emptyset$, we easily get $F(t,x) = \{\psi(t,x)\}$, as claimed. $\square$

Proof of Theorem 1: We can suppose $j < +\infty$. Put $E := \bigcup_{i=1}^n E_i$ (of course, $m_n(E) = 0$), and let $K \subseteq I$, with $m_1(K) = 0$, such that (4) and (5) hold for each $t \in I \setminus K$. Now, let $\hat{f} : I \times]0_n, \sigma[\to \mathbb{R}^n$ be a function satisfying the conclusion of Proposition 1 (the assumptions of Proposition 1 are satisfied), and let $\psi : I \times \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$\psi(t,x) = \begin{cases} \hat{f}(t,x) & \text{if } (t,x) \in (I \setminus K) \times]0_n, \sigma[\\ \beta(t) & \text{otherwise.} \end{cases} \tag{10}$$

In particular, observe that

$$\alpha(t) \leq \psi(t,x) \leq \beta(t) \quad \text{for all} \quad (t,x) \in (I \setminus K) \times \mathbb{R}^n. \tag{11}$$

Let Ω be a dense countable subset of $\mathbb{R}^n \setminus]0_n, \sigma[$. Hence, the set $D := P \cup \Omega$ is a dense countable subset of $\mathbb{R}^n$. It follows easily from the above construction that

ψ and D satisfy the assumptions of Proposition 2. Consequently, the multifunction $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ defined by (7) satisfies the conclusion of Proposition 2. Moreover, by (10) and (11) we get

$$\begin{cases} F(t,x) \subseteq [\alpha(t), \beta(t)] & \text{if } \ (t,x) \in (I \setminus K) \times \mathbb{R}^n \\ F(t,x) = \beta(t) & \text{if } \ (t,x) \in K \times \mathbb{R}^n. \end{cases} \tag{12}$$

Now we want to apply Theorem 1 of [13] taking $T = I$, $X = Y = \mathbb{R}^n$, $p = s$, $q = j'$, $V = L^s(I, \mathbb{R}^n)$, $\Psi(u) = u$, $r = \|\beta\|_{L^s(I,\mathbb{R}^n)}$, $\varphi(\lambda) \equiv +\infty$,

$$\Phi(u)(t) = \int_I g(t,z)\, u(z)\, dz,$$

and $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ as above. In particular, we observe the following facts.

(a) $\Phi(L^s(I,\mathbb{R}^n)) \subseteq C^0(I,\mathbb{R}^n)$. This follows easily from our assumptions (iv) and (v) and the Lebesgue's dominated convergence theorem.

(b) If $v \in L^s(I,\mathbb{R}^n)$ and $\{v^k\}$ is a sequence in $L^s(I,\mathbb{R}^n)$, weakly convergent to v in $L^{j'}(I,\mathbb{R}^n)$, then the sequence $\{\Phi(v^k)\}$ converges to $\Phi(v)$ strongly in $L^1(I,\mathbb{R}^n)$. This follows by Theorem 2 at p. 359 of [10], since g is j-th power summable in $I \times I$ (note that g is measurable on $I \times I$ by the classical Scorza-Dragoni's theorem; see [14] or also [9]).

(c) By (12) (taking into account that $0_n < \alpha(t)$ for all $t \in I \setminus K$), the function

$$h : t \in I \to \sup_{x \in \mathbb{R}^n} d(0_n, F(t,x))$$

belongs to $L^s(I)$ and $\|h\|_{L^s(I)} \le \|\beta\|_{L^s(I,\mathbb{R}^n)}$.

Therefore, taking into account the above construction, all the assumptions of Theorem 1 of [13] are satisfied. Consequently, there exist a function $\hat{u} \in L^s(I,\mathbb{R}^n)$ and a set $H \subseteq I$, with $m_1(H) = 0$, such that

$$\hat{u}(t) \in F(\,t\,, \Phi(\hat{u})(t)) \quad \text{for all} \quad t \in I \setminus H. \tag{13}$$

In particular, by (12) we have

$$\hat{u}(t) \in [\alpha(t), \beta(t)] \quad \text{for all} \quad t \in I \setminus (H \cup K). \tag{14}$$

For each fixed $i = 1, \ldots, n$, let $\gamma_i : I \to \mathbb{R}$ be defined by

$$\gamma_i(t) := \pi_i(\Phi(\hat{u})(t)) = \int_I g(t,z)\, \hat{u}_i(z)\, dz.$$

For each $t \in I$, by (ii), (v) and (14) we have

$$0 \le \gamma_i(t) \le \|\phi_0\|_{L^{s'}(I)} \cdot \|\hat{u}_i\|_{L^s(I)} \le \frac{\sigma_i}{\|\beta_i\|_{L^s(I)}} \cdot \|\beta_i\|_{L^s(I)} = \sigma_i \,,$$

hence

$$\gamma_i(I) \subseteq [0, \sigma_i]. \tag{15}$$

By (iv), (v) and (14), it is easy to see that γ_i is strictly increasing, and also by Lemma 2.2 at p. 226 of [12], we have

$$\frac{d}{dt}\,\gamma_i(t) = \int_I \frac{\partial g}{\partial t}(t,z)\,\hat{u}_i(z)\,dz > 0 \quad \text{for all} \quad t \in I_0.$$

By Theorem 2 of [15] (taking into account (a)), the function γ_i^{-1} is absolutely continuous. Put

$$S_i := \gamma_i^{-1}\big[(\pi_i(E_i) \cup \{0, \sigma_i\}) \cap \gamma_i(I)\big].$$

By assumption (iii) and Theorem 18.25 of [8], we get $m_1(S_i) = 0$. At this point, put

$$S := (\bigcup_{i=1}^{n} S_i) \cup K \cup H.$$

Choose any point $t^* \in I \setminus S$. We claim that

$$\Phi(\hat{u})(t^*) \in]0_n, \sigma[\, \setminus E. \tag{16}$$

To see this, observe that for each $i = 1, \dots, n$ we have $\gamma_i(t^*) \notin \pi_i(E_i) \cup \{0, \sigma_i\}$, hence by (15) we get $\gamma_i(t^*) \in]0, \sigma_i[$ and also $\Phi(\hat{u})(t^*) \notin E_i$. Therefore, (16) follows. Since $\psi(t^*, x) = \hat{f}(t^*, x)$ for all $x \in]0_n, \sigma[$, and by (16) the function $\hat{f}(t^*, \cdot\,)$ is continuous at $\Phi(\hat{u})(t^*)$, it follows that $\psi(t^*, \cdot\,)$ is continuous at $\Phi(\hat{u})(t^*)$, hence (taking into account conclusion (d) of Proposition 2) we have

$$F(\,t^*\,, \Phi(\hat{u})(t^*)) = \{\psi(\,t^*\,, \Phi(\hat{u})(t^*))\} = \{\hat{f}(\,t^*\,, \Phi(\hat{u})(t^*))\} = \{f(\,t^*\,, \Phi(\hat{u})(t^*))\}.$$

Consequently, (13) implies

$$\hat{u}(t^*) = f(\,t^*\,, \Phi(\hat{u})(t^*)).$$

As t^* was any point in $I \setminus S$ and $m_1(S) = 0$, the proof is complete. $\square$

Remark. The example at p. 245 of [3] shows that in assumption (v) of Theorem 1 one cannot assume $0 \leq \frac{\partial g}{\partial t}(t,z) \leq \phi_1(z)$. Moreover, as we pointed out in Section 1, the example provided at the end of [4] shows that in assumption (iii) of Theorem 1 the set $\bigcup_{i=1}^{n} E_i$ cannot be replaced by any set $E \subseteq]0_n, \sigma[$ with $m_n(E) = 0$.

The next example shows that the sets $E_1, \dots, E_n$ in assumption (iii) of Theorem 1 cannot be assumed to depend on $t \in I$.

Example. Let $n = 1$, $s = +\infty$, $\alpha(t) \equiv \frac{1}{2}$, $\beta(t) \equiv 3$, $\sigma = 4$, $g(t,z) = t$, $\phi_0(z) \equiv 1$, $\phi_1(z) \equiv 1$ and

$$f(t,x) = \begin{cases} 1 & \text{if } x \neq t \\ 2 & \text{if } x = t. \end{cases} \tag{17}$$

It is easy to check that all the assumptions of Theorem 1 are satisfied, with the exception of assumption (iii). Moreover, observe that if one puts $f^*(t,x) \equiv 1$, than for each $t \in]0,1]$ one has $\{x \in]0,4[: f^*(t,x) \neq f(t,x)\} = \{t\}$ (or also, one can take $f^* = f$ and observe that for each $t \in]0,1]$ one has $\{x \in]0,4[: f(t,\cdot\,)$ is discontinuous at $x\} = \{t\}$; in both cases, the function $f^*(\,\cdot\,,x)$ is measurable for all $x \in]0,4[\,$). Now we prove that there is no solution $u \in L^1(I)$ to problem (3). Arguing by contradiction, assume that such a solution exists. Consequently, by (17) we get $u(t) \in \{1,2\}$ for a.a. $t \in I$. Therefore, we have

$$u(t) = f(\,t\,, t\,\|u\|_{L^1(I)}) \quad \text{for a.a.} \quad t \in I. \tag{18}$$

Now, assume that $\|u\|_{L^1(I)} = 1$. By (17) and (18) we get $u(t) = 2$ a.e. in I, a contradiction. If, conversely, we assume that $\|u\|_{L^1(I)} > 1$, again by (17) and (18) we get $u(t) = 1$ a.e. in I, another contradiction. This proves our claim.

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