Comment.Math.Univ.Carolin. 42,2 (2001)379–384

On remote points, non-normality and π-weight ω_1

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Abstract. We show, in particular, that every remote point of X is a nonnormality point of βX if X is a locally compact Lindelöf separable space without isolated points and $\pi w(X) \leq \omega_1$.

Keywords: remote point, butterfly-point, nonnormality point

Classification: 54D35

1. Introduction

We investigate some types of points in remainders $X^* = \beta X \setminus X$ of Čech-Stone compactifications.

A point $p \in X^*$ is called a remote point of X if it is not in the closure of any nowhere dense subset of X. This kind of points became popular after the papers [3], [4] of van Douwen had been published. The existence of remote points in the remainders of ccc nonpseudocompact spaces with π-weight ω_1 was proved by Dow [2]. An inspection of the relevant results in the literature reveals that the remote points constructed so far satisfy our condition $(*)$ below. This leads us to the notion of a strong remote point. It is unknown to the author whether there is an example of a remote point, which is not a strong remote point.

If removing a point p from a compact Hausdorff space results in obtaining a nonnormal subspace, then p is called a nonnormality point of the space. There are several simple proofs that, under CH, any point of ω^* is a nonnormality point of ω^* ([8], [9]). "Naively", it is known only for special points of ω^*. If p is an accumulation point of some countable discrete subset of ω^*, or if p is a strong *R-point*, or if p is a Kunen's point, then p is a nonnormality point of ω^* (Blaszczyk and Szymanski [1], Gryzlov [5], van Douwen, respectively). If X is a normal second countable space without isolated points, which is either locally compact or zero-dimensional, then every point of its remainder is a nonnormality point of βX ([6], [7]).

In some cases the fact that $p \in X^*$ is a strong remote point of X permits to show that p is a *b-point* of βX, i.e. that there are sets F and $G \subset X^* \setminus \{p\}$ which are closed in $\beta X \setminus \{p\}$, disjoint and have p as a limit point [7], [10] (see below). It easily implies that p is a nonnormality point of βX, i.e. $\beta X \setminus \{p\}$ is not normal.

In our paper, the following results are obtained.

Theorem 1.1. *Let X be a locally compact Lindelöf separable space without isolated points and $\pi w(X) \leq \omega_1$. Then every remote point $p \in X^*$ of X is a b-point (and, consequently, a nonnormality point) of βX.*

Theorem 1.2. *Let $X = \bigcup_{i \in \omega} X_i$ be a normal separable space without isolated points and $\pi w(X) \leq \omega_1$. Then every strong remote point $p \in X^*$ of X is a b-point (and, consequently, a nonnormality point) of βX.*

2. Proofs

We will present a proof of Theorem 1.2 below, assuming its conditions hold. By Claims 1 and 2 it is clear that Theorem 1.1 is an easy corollary to Theorem 1.2.

The set of all functions from ω to ω is denoted by ω^ω. For a set $U \subset X$ let $U^\epsilon = \beta X \setminus Cl_{\beta X}(X \setminus U)$ if U is open and $U^* = Cl_{\beta X} U \setminus X$ if U is closed. A set $U \subset X^*$ is called *τ-bounded* for a cardinal τ iff for any $F \subset U$, $|F| < \tau$ implies $Cl_{\beta X} F \subset U$. A *π-base* $\mathcal{U}$ for X is a set of nonempty open subsets of X with the property that each nonempty open subset of X contains a member of $\mathcal{U}$. The *π-weight* of X, $\pi w(X)$, is the minimum cardinality of a π-base for X.

Let 2^X be set of all subsets of X. A subset π of 2^X is called *strong cellular* if the closures of its members in X form a pairwise disjoint family. One *refines* a subset σ of 2^X, $\pi > \sigma$, if $U \cap V \neq \emptyset$ implies $U \subset V$ for any $U \in \pi$ and $V \in \sigma$. If, in addition, $\{U \in \pi : U \subset V\}$ is finite for every $V \in \sigma$, then π *finitally refines* σ, $\pi >_{fin} \sigma$. And, finally, π *$*$-refines* σ, $\pi >_* \sigma$, iff there is a finite subset $\delta \subset \pi$ such that $\pi \setminus \delta$ refines σ.

If $\pi_0, \ldots, \pi_n$ are nonempty subsets of 2^X, then the collection

$$\prod_{k=0}^{n} \pi_k = \{\bigcap_{k=0}^{n} U_k : U_k \in \pi_k \text{ and } \bigcap_{k=0}^{n} U_k \neq \emptyset\}$$

is said to be their *product*.

From now on $X = \bigcup_{i \in \omega} X_i$ is a free topological sum and $\pi_0 = \{X_i : i \in \omega\}$.

Definition 2.1. A point $p \in X^*$ is called a *strong remote point* of X iff p is a remote point of X and

- ($*$) for any family of open sets $\mathcal{W} \subset 2^X$ the following holds: if $\mathcal{W} > \pi_0$ and $p \in \bigcup \mathcal{W}^\epsilon$, then there is a subfamily $\mathcal{W}' \subset \mathcal{W}$ such that $\mathcal{W}' >_{fin} \pi_0$ and $p \in (\bigcup \mathcal{W}')^\epsilon$.

From now on a strong remote point $p \in X^*$ is fixed. It is easy to see that $p \notin Cl_{\beta X} X_i$ for each $i \in \omega$ and that ($*$) is trivial if every X_i is compact.

A discrete in X countable family of nonempty open sets $\pi \subset 2^X$ is called a *p-chain* if $\pi >_{fin} \pi_0$ and $p \in \bigcup \pi^\epsilon$. Thus π_0 is a p-chain. Next we put

$$[\pi] = \bigcap \{Cl_{\beta X} \bigcup \sigma : \sigma \subset \pi \text{ is a } p\text{-chain}\}$$

for any p-chain π and $S = \{s \in [\pi_0] : s$ is a strong remote point of $X\}$. We fix $Y = \bigcup_{i\in\omega} Y_i$, where $Y_i = \{y_{ij} : j \in \omega\}$ is a countable everywhere dense subset of X_i, and put

$$T = \{t \in [\pi_0] : t \in Cl_{\beta X} D \text{ for some } D \subset Y, \text{ for which every } D \cap Y_i \text{ is finite } \}.$$

From now on

$$\xi(p) = \{A \subset \omega : p \in (\bigcup_{i\in A} X_i)^\epsilon\}$$

is an ultrafilter on ω. For any $f, g \in \omega^\omega$, $f <_p g$ iff $\{i \in \omega : f(i) < g(i)\} \in \xi(p)$. It is a folklore and easy to see that there are so called $\xi(p)$-*dominant* families $\{f_\alpha : \alpha < \tau\} \subset \omega^\omega$ having the following properties: $f_\alpha <_p f_\beta$ whenever $\alpha < \beta < \tau$ and for any $g \in \omega^\omega$, $g <_p f_\alpha$ for some $\alpha < \tau$. We fix one of them $\mathcal{F} = \{f_\alpha : \alpha < \lambda(p)\}$ of the smallest cardinality $\lambda(p)$. Then, obviously, $\lambda(p) \geq \omega_1$. For any $\mathcal{G} \subset \omega^\omega$, $|\mathcal{G}| < \lambda(p)$ implies $g <_p f$ for each $g \in \mathcal{G}$ and for some $f \in \omega^\omega$.

Now for every $i \in \omega$ we fix a π-base $\mathcal{U}_i = \{U_{i\alpha} : \alpha \in \omega_1\}$ for X_i. For any $\beta \in \omega_1$, for $\{U_{i\alpha} : \alpha < \beta\} \subset \mathcal{U}_i$ we fix a cellular refinement $\{\mathcal{V}_{ij}(\beta) : j \in \omega\}$ with the following properties:

1) every $\mathcal{V}_{ij}(\beta)$ is a maximal strong cellular family of nonempty open subsets of X_i;
2) $\mathcal{V}_{ij+1}(\beta) > \mathcal{V}_{ij}(\beta)$ for each $j \in \omega$;
3) for every $\alpha < \beta$, $\mathcal{V}_{ij(i,\alpha,\beta)}(\beta) > \{U_{i\alpha}\}$ for some $j(i, \alpha, \beta) \in \omega$.

We put, also, $\mathcal{V}_g(\beta) = \bigcup_{i\in\omega} \mathcal{V}_{ig(i)}(\beta)$ for each $g \in \omega^\omega$ and fix a p-chain $\pi_g(\beta)$ so that $\pi_g(\beta) \subset \mathcal{V}_g(\beta)$.

Claims 1 through 4 are easy and sometimes well-known and are left as exercises to the reader.

Claim 1. *If $p \in X^*$ is a b-point of βX, then $\beta X \setminus \{p\}$ is not normal.*

Claim 2. *Let $p \in X^*$, where X is a locally compact Lindelöf space. Then there exists a family $\{X_n : n \in \omega\}$ of compact regularly closed subsets of X such that $\{X_n : n \in \omega\}$ is a discrete in X family and $p \in Cl_{\beta X} \bigcup\{X_n : n \in \omega\}$.*

Claim 3. *For any p-chains π and σ, if $\pi >_* \sigma$, then $[\pi] \subset [\sigma]$.*

Claim 4. *For any finite family of p-chains $\{\pi_i\}_{i=0}^n$, $\prod_{i=0}^n \pi_i$ is a p-chain refining every π_i.*

Claim 5. *For any countable family of p-chains $\{\pi_i : i \in \omega\}$ there is a p-chain π *-refining every π_i.*

Proof: Let $\sigma = \bigcup_{n\in\omega} \sigma(n)$, where

$$\sigma(n) = \prod_{i=0}^{n} \{U \subset X_n : \text{ either } U \in \pi_i \text{ or } U = X_n \setminus Cl \bigcup \pi_i\}.$$

Then $Cl\bigcup\sigma = X$. So $Cl_{\beta X}Op \subset \bigcup\sigma^{\epsilon}$ for some neighborhood $Op \subset \beta X$. Any p-chain π such that $\pi \subset \{Op \cap U : U \in \sigma \text{ meets } Op\}$ is as required. $\square$

Claim 6. *T is $\lambda(p)$-bounded.*

Proof: Let $F \subset T$ and $|F| < \lambda(p)$. For every $x \in F$, $x \in Cl_{\beta X}\bigcup_{i\in\omega}\{y_{ij} \in Y : j \leq f_x(i)\}$ for some $f_x \in \omega^\omega$. For some $f \in \omega^\omega$, $f_x <_p f$ for each $x \in F$. But then

$$Cl_{\beta X}F \subset Cl_{\beta X}\bigcup_{i\in\omega}\{y_{ij} \in Y : j \leq f(i)\} \cap [\pi_0] \subset T.$$

$\square$

Claim 7. *S is $\lambda(p)$-bounded.*

Proof: Let $q \in [\pi_0] \setminus S$. Then there is a maximal strong cellular family of open sets $\mathcal{W} = \{V_{ij} \subset X_i : i, j \in \omega\}$ such that $q \notin Cl_{\beta X}\bigcup\sigma$ for any $\sigma \subset \mathcal{W}$, $\sigma >_{fin} \pi_0$. Let $F \subset S$ and $|F| < \lambda(p)$. Then for every $x \in F$, $x \in (\bigcup_{i\in\omega}\bigcup_{j\leq f_x(i)} V_{ij})^{\epsilon}$ for some $f_x \in \omega^\omega$. For some $f \in \omega^\omega$, $f_x <_p f$ for each $x \in F$. But then

$$Cl_{\beta X}F \subset Cl_{\beta X}\bigcup_{i\in\omega}\bigcup_{j\leq f(i)} V_{ij} \subset \beta X \setminus \{q\}.$$

$\square$

Claim 8. *For any family of p-chains $\{\pi_\alpha\}_{\alpha<\tau}$, if $\tau < \lambda(p)$ then $\bigcap_{\alpha<\tau}[\pi_\alpha] \cap T \neq \emptyset$.*

Proof: For any finite $\rho \subset \tau$ we can fix a point $t(\rho) \in T$ so that

$$t(\rho) \in [\prod_{\alpha\in\rho}\pi_\alpha] \subseteq \bigcap_{\alpha\in\rho}[\pi_\alpha].$$

But then the set $Cl_{\beta X}\{t(\rho) : \rho \subset \tau \text{ is finite}\}$, which is contained in T by Claim 6, meets $\bigcap_{\alpha<\tau}[\pi_\alpha]$. $\square$

Claim 9. *For any family of p-chains $\{\pi_\alpha\}_{\alpha<\tau}$, if $\tau < \lambda(p)$, then p is not isolated in $\bigcap_{\alpha<\tau}[\pi_\alpha]$.*

Proof: Let $\bigcap_{\alpha<\tau}[\pi_\alpha] \cap Cl_{\beta X}Op = \{p\}$ for some neighborhood $Op \subset \beta X$. Then for a p-chain $\pi = \{Op \cap X_i : i \in \omega\}$ we have $\bigcap_{\alpha<\tau}[\pi_\alpha] \cap [\pi] \cap T = \emptyset$ in a contradiction to Claim 8. $\square$

Claim 10. *Let $\bigcap_{\alpha<\tau}[\pi_\alpha] \cap S = \{p\}$ for some family of p-chains $\{\pi_\alpha\}_{\alpha<\tau}$ of cardinality $\tau < \lambda(p)$. Then p is a b-point of βX.*

Proof: For any finite $\rho \subset \tau$ we can fix a point $s(\rho) \in S \setminus \{p\}$ so that $s(\rho) \in [\prod_{\alpha\in\rho}\pi_\alpha]$ by [2]. But then the sets $Cl_{\beta X}\{s(\rho) : \rho \subset \tau \text{ is finite}\} \setminus \{p\}$ and $\bigcap_{\alpha<\tau}[\pi_\alpha] \setminus \{p\}$ are as required. $\square$

Below we have only to examine the case when the hypotheses of Claim 10 are wrong.

Claim 11. *For an arbitrary neighborhood $Op \subset \beta X$, $[\pi_{f_\alpha}(\beta)] \subset Op$ for some $f_\alpha \in \mathcal{F}$ and $\beta \in \omega_1$.*

PROOF: Let $Cl_{\beta X} O'p \subset Op$ for a neighborhood $O'p \subset \beta X$. As p is a strong remote point, $p \in (\bigcup \bigcup_{i\in\omega} \mathcal{U}_i^{'})^\epsilon \subset O'p$ for some finite $\mathcal{U}_i^{'} \subset \mathcal{U}_i$. For some $\beta < \omega_1$, $\mathcal{U}_i^{'} \subset \{U_{i\alpha} : \alpha < \beta\}$ for each $i \in \omega$. For every $U_{i\alpha} \in \mathcal{U}_i^{'}$ we can choose $j(i,\alpha,\beta) \in \omega$ so that $\mathcal{V}_{ij(i,\alpha,\beta)}(\beta) > \{U_{i\alpha}\}$ (see above). Let $g \in \omega^\omega$ be defined for any $i \in \omega$ as follows: $g(i) = \max\,\{j(i,\alpha,\beta) : U_{i\alpha} \in \mathcal{U}_i^{'}\}$ if $\mathcal{U}_i^{'} \neq \emptyset$ and $g(i) = 1$ otherwise. Then $\mathcal{V}_g(\beta) > \bigcup_{i\in\omega} \mathcal{U}_i^{'}$ by our construction. Let, finally, $f_\alpha \in \mathcal{F}$ be chosen so that $g <_p f_\alpha$. But then $[\pi_{f_\alpha}(\beta)] \subset [\pi_g(\beta)] \subset Cl_{\beta X} \bigcup \bigcup_{i\in\omega} \mathcal{U}_i^{'} \subset Op$. □

Claim 12. *If $|\mathcal{F}| > \omega_1$, then p is a b-point of βX.*

PROOF: For every $f_\alpha \in \mathcal{F}$ there are points $t_\alpha \in T$ and $s_\alpha \in S \setminus \{p\}$, belonging to $B_{f_\alpha} = \bigcap_{\beta<\omega_1} [\pi_{f_\alpha}(\beta)]$ by Claims 8 and 10. Then the sets $F = Cl_{\beta X}\{t_\alpha : \alpha < \lambda(p)\} \setminus \{p\}$ and $G = Cl_{\beta X}\{s_\alpha : \alpha < \lambda(p)\} \setminus \{p\}$ are as required. Indeed, they have p as a limit point by Claim 11. For every $\lambda < \gamma < \lambda(p)$, $f_\lambda <_p f_\gamma$ clearly implies $[\pi_{f_\gamma}(\beta)] \subset [\pi_{f_\lambda}(\beta)]$ for each $\beta < \omega_1$, so $B_{f_\gamma} \subset B_{f_\lambda}$. But then

$$F \cap G \setminus B_{f_\lambda} \subset Cl_{\beta X}\{t_\alpha : \alpha < \lambda\} \cap Cl_{\beta X}\{s_\alpha : \alpha < \lambda\} \subset T \cap S = \emptyset.$$

□

Claim 13. *If $|\mathcal{F}| = \omega_1$, then p is a b-point of βX.*

PROOF: Let $\{\pi_{f_\alpha}(\beta) : f_\alpha \in \mathcal{F}, \beta \in \omega_1\}$ be listing into the form $\{\pi_\gamma : \gamma \in \omega_1\}$. By Claim 5 we can construct p-chains σ_γ $(\gamma < \omega_1)$ so that $\sigma_\gamma >_* \pi_\gamma$ and $\sigma_\gamma >_* \sigma_\lambda$ if $\lambda < \gamma < \omega_1$. We can fix points $t_\gamma \in T$ and $s_\gamma \in S \setminus \{p\}$, belonging to $[\sigma_\gamma]$, and repeat the proof of Claim 12, using these points.

Our proof is complete. □

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(*Received* December 6, 1999, *revised* November 17, 2000)