On exit laws for semigroups in weak duality

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Abstract. Let $\mathbb{P} := (P_t)_{t>0}$ be a measurable semigroup and m a σ-finite positive measure on a Lusin space X. An m-exit law for $\mathbb{P}$ is a family $(f_t)_{t>0}$ of nonnegative measurable functions on X which are finite m-a.e. and satisfy for each $s, t > 0$ $P_s f_t = f_{s+t}$ m-a.e. An excessive function u is said to be in $\mathcal{R}$ if there exits an m-exit law $(f_t)_{t>0}$ for $\mathbb{P}$ such that $u = \int_0^\infty f_t\, dt$, m-a.e.

Let $\mathcal{P}$ be the cone of m-purely excessive functions with respect to $\mathbb{P}$ and $\mathcal{I}mV$ be the cone of m-potential functions. It is clear that $\mathcal{I}mV \subseteq \mathcal{R} \subseteq \mathcal{P}$. In this paper we are interested in the converse inclusion. We extend some results already obtained under the assumption of the existence of a reference measure. Also, we give an integral representation of the mutual energy function.

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1. Introduction

Let X be a Lusin metrizable topological space with its Borel tribe $\mathcal{B}$. We denote by $\mathcal{B}^+$ the cone of nonnegative Borel functions on X and by $\mathcal{M}$ the class of σ-finite positive measures on $(X, \mathcal{B})$.

In the sequel, let $\mathbb{P} := (P_t)_{t>0}$ and $\hat{\mathbb{P}} := (\hat{P}_t)_{t>0}$ be two submarkovian measurable semigroups on (X, B), in weak duality with respect to a fixed measure $m \in \mathcal{M}$, namely,

$$(1) \qquad \int P_t f(x) g(x) m(dx) = \int f(x) \hat{P}_t g(x) m(dx),\ \forall\, t > 0,\ \forall\, f, g \in \mathcal{B}^+.$$

The potential kernels $V := \int_0^\infty P_t\, dt$ and $\hat{V} := \int_0^\infty \hat{P}_t\, dt$ are assumed to be proper and satisfy the unicity of charges, that is, for each $\mu, \nu \in \mathcal{M}$

$$(2) \qquad \text{if } \ \mu\hat{V} = \nu\hat{V} \in \mathcal{M} \ \text{ then } \ \mu = \nu.$$

Throughout this paper we denote by $\mathcal{F}$ the set of Borel nonnegative functions which are finite m-a.e. and by $\mathcal{E}$ the cone of functions $u \in \mathcal{F}$ which are excessive with respect to $\mathbb{P}$.

Let $\mathcal{R}$ be the cone of functions $u \in \mathcal{E}$ such that there exists an m-exit law $(f_t)_{t>0}$ for $\mathbb{P}$ satisfying

$$u = \int_0^\infty f_t\, dt, \ m\text{-a.e.}$$

Denote by $\mathcal{P} := \{u \in \mathcal{E} : \inf_{t\to\infty} P_t u = o \ m\text{-a.e.}\}$, the cone of purely excessive functions with respect to $\mathbb{P}$ and $\mathcal{I}mV := \{u \in \mathcal{E} : \ u = V\varphi \ m\text{-a.e. for some } \varphi \in \mathcal{B}^+\}$.

Each object related to $\hat{\mathbb{P}}$ is equipped with "ˆ", so we define as above $\hat{\mathcal{E}}$ and $\hat{\mathcal{R}}$. Finally, we denote by $\prec$ the strong m-domination order defined by $\mathcal{E}$.

Obviously, we have the natural inclusion $\mathcal{R} \subseteq \mathcal{P}$, while in [10] Hmissi proved that $\mathcal{R} = \mathcal{P}$ provided that $\varepsilon_x P_t \ll m$ for each $x \in X$ and $t > 0$ (where ε_x denotes the Dirac mass at x) and in [12] he showed that if $\mathbb{P}$ is a lattice semigroup then $\mathcal{R} = \mathcal{I}mV$. Furthermore, Hmissi gave an example of a nonlattice semigroup for which $\mathcal{R} = \mathcal{I}mV$.

The first purpose of this paper is to give necessary and sufficient conditions on $\hat{\mathbb{P}}$ such that $\mathcal{R} = \mathcal{P}$. More precisely, we prove the following

Theorem 1. *The following statements are equivalent:*

(i) $\mathcal{R} = \mathcal{P}$*;*
(ii) *for every* $\mu \in \mathcal{M}$*, if* $\mu\hat{V} \ll m$ *then* $\mu\hat{P}_t \ll m$ *for all* $t > 0$.

As an application we obtain an integral representation of the mutual energy.

The second purpose of this paper is to give necessary and sufficient conditions on $\hat{\mathbb{P}}$ for $\mathcal{R} = \mathcal{I}mV \cap V(\mathbb{M})$ (where $V(\mathbb{M})$ will be defined later). More precisely, we have the following result:

Theorem 2. *The following statements are equivalent:*

(i) $\mathcal{I}mV = \mathcal{R} \cap V(\mathbb{M})$*;*
(ii) *for every* $\mu \in \mathcal{M}$*, if* $\mu\hat{P}_t \ll m$ *for all* $t > 0$ *then* $\mu \ll m$.

In Section 3 we suppose that the semigroup $\mathbb{P}$ is defined by a vaguely continuous convolution semigroup on $X = \mathbb{R}^n$ $(n \geq 1)$. Then we prove the existence of the largest element of $\mathcal{R}$ in the strong m-domination order which is strongly m-dominated by a given function $u \in \mathcal{P}$. Namely,

Theorem 3. *Let* $u \in \mathcal{P}$*. Then there exist a function* $r(u) \in \mathcal{R}$ *such that* $r(u) \prec u$ *and for each function* $v \in \mathcal{R}$ *satisfying* $v \prec u$*, we have* $v \prec r(u)$.

Note that the above result has been established by Hmissi ([11], Theorem 2.4) under the assumption of existence of a reference measure.

2. Characterization of $\mathcal{R}$

2.1 Exit laws in weak duality.

For the coming definitions we refer to [1], [3], [8] and [9].

Definition 1. (i) A family $(f_t)_{t>0} \subset \mathcal{F}$ is called an m-exit law for $\mathbb{P}$ if

$$P_s f_t = f_{s+t} \ m\text{-a.e.} \ \text{ for each } \ s,t>0. \tag{3}$$

(ii) Two m-exit laws $(f_t)_{t>0}$ and $(g_t)_{t>0}$ are equivalent if $f_t = g_t$ m-a.e. for each $t>0$.

Remark 1. Let $(f_t)_{t>0}$be an m-exit law for $\mathbb{P}$ such that $u := \int_0^\infty f_t\,dt \in \mathcal{F}$. Then $P_t u = V f_t$ m-a.e. for each $t>0$ and by [9, (6.19)] there exists $v \in \mathcal{E}$ such that $u = v$ m-a.e.

Definition 2. A function $u \in \mathcal{E}$ will be called the potential of a measure $\mu \in \mathcal{M}$ if $u \cdot m = \mu \hat{V}$, and we write $u = V(\mu)$ (see [8, (3.5)]).

In the sequel, we denote

$$\mathbb{M} := \{\mu \in \mathcal{M} : u \cdot m = \mu \hat{V} \ \text{ with } \ u \in \mathcal{E}\}$$

and

$$V(\mathbb{M}) := \{V(\mu) : \mu \in \mathbb{M}\}.$$

Remark 2. (i) $\mathcal{I}mV \subseteq \mathcal{R} \cap V(\mathbb{M})$ and $\mathcal{R} \subseteq \mathcal{P}$.

(ii) Let $\mathbb{P}$ be the heat semigroup on $\mathbb{R}^{n+1}$. Then we have $\mathcal{I}mV \neq \mathcal{R} \cap V(\mathbb{M})$ and $\mathcal{R} \neq \mathcal{P}$.

We recall that an entrance law for $\hat{\mathbb{P}}$ is a family $(\mu_t)_{t>0} \subset \mathcal{M}$ such that

$$\mu_t \hat{P}_s = \mu_{s+t} \ \text{ for each } \ s,t>0. \tag{4}$$

To prove Theorem 1, we need the following representation theorem (see K. Janssen [13]).

Theorem 4. *Any purely excessive measure μ can be uniquely decomposed as the integral $\mu = \int_0^\infty \mu_t\,dt$ of an entrance law $(\mu_t)_{t>0}$ with respect to $\mathbb{P}$.*

Proof of Theorem 1: (ii) $\Rightarrow$ (i). Let $u \in \mathcal{P}$, then the measure $u \cdot m$ is purely excessive with respect to $\hat{\mathbb{P}}$. So by Theorem 4 there exists a unique entrance law $(\nu_t)_{t>0}$ for $\hat{\mathbb{P}}$ such that

$$u \cdot m = \int_0^\infty \nu_t\,dt. \tag{5}$$

Using (1) and (4) it follows that $\nu_t \hat{V} = (P_t u) \cdot m \leq u \cdot m$ for each $t > 0$. Hence, $\nu_t = \nu_{\frac{t}{2}} \hat{P}_{\frac{t}{2}} \ll m$ for each $t > 0$.

Let $(f_t)_{t>0} \subset \mathcal{F}$ be such that $\nu_t = f_t \cdot m$ for each $t > 0$. Using again (1) and (4) we check that $(f_t)_{t>0}$ is an m-exit law with respect to $\mathbb{P}$. We deduce by (5) that $u \in \mathcal{R}$.

(i) $\Rightarrow$ (ii). Let $\mu \in \mathcal{M}$ and $V(\mu) \in \mathcal{P}$ be such that

$$\mu \hat{V} = V(\mu) \cdot m = \int_0^\infty (\mu \hat{P}_t)\, dt. \tag{6}$$

Let $(f_t)_{t>0}$ be an m-exit law with respect to $\mathbb{P}$ such that

$$V(\mu) \cdot m = \int_0^\infty (f_t \cdot m)\, dt. \tag{7}$$

From (3) and (1) we check that $(f_t \cdot m)_{t>0}$ is an entrance law with respect to $\hat{\mathbb{P}}$. Therefore by (2), (6) and (7), we have $\mu \hat{P}_t = f_t \cdot m$ for each $t > 0$.
Hence, $\mu \hat{P}_t \ll m$ for each $t > 0$. □

Proof of Theorem 2: (i) $\Rightarrow$ (ii). Let $\mu \in \mathcal{M}$ be such that $\mu \hat{P}_t \ll m$ for each $t > 0$. Then there exists $V(\mu) \in \mathcal{E}$ which satisfies

$$\mu \hat{V} = V(\mu) \cdot m. \tag{8}$$

Let $(f_t)_{t>0} \subset \mathcal{F}$ such that $\mu \hat{P}_t = f_t \cdot m$ for each $t > 0$. By (1) and (4) it is easy to check that $(f_t)_{t>0}$ is an m-exit law with respect to $\mathbb{P}$, and from (8) it follows that $V(\mu) \in \mathcal{R} \cap V(\mathbb{M})$. Now by (i), there exists a function $\varphi \in \mathcal{B}^+$ such that $\mu \hat{V} = V(\mu) \cdot m = V(\varphi) \cdot m$. Hence, using (1) and (2) we deduce that $\mu \ll m$.

(ii) $\Rightarrow$ (i). Let $u \in \mathcal{R} \cap V(\mathbb{M})$. There exists an m-exist law for $\mathbb{P}$, $(f_t)_{t>0} \subset \mathcal{F}$, and $\mu \in \mathcal{M}$ such that

$$u \cdot m = \int_0^\infty (f_t \cdot m)\, dt = \mu \hat{V} = \int_0^\infty (\mu \hat{P}_t)\, dt. \tag{9}$$

Using again (2) we get $\mu \hat{P}_t = f_t \cdot m$ for each $t > 0$.
Let $\varphi \in \mathcal{B}^+$ satisfy $\mu = \varphi \cdot m$. Then we get $u \in \mathcal{I}m V$ by (1). □

2.2 Mutual energy formula.

Definition 3 (see [5, XII, 39]). For $(u, v) \in \mathcal{E} \times \hat{\mathcal{E}}$, the mutual energy $E(u, v)$ is defined by

$$E(u,v) := \sup\{m(\varphi \cdot v), \quad \varphi \in \mathcal{B}^+, \quad V\varphi \leq u\} \tag{10}$$

$$= \sup\{m(u \cdot \psi), \quad \psi \in \mathcal{B}^+, \quad \hat{V}\psi \leq v\}. \tag{11}$$

Remark 3. Let $u, u_1 \in \mathcal{E}$ and $v, v_1 \in \hat{\mathcal{E}}$ be such that $u = u_1$ m-a.e. and $v = v_1$ m-a.e. Then, using (10) and (11) we deduce that $E(u, v) = E(u_1, v_1)$.

We recall briefly the following properties of E (cf. [5]).

Proposition 1.

(a) $E(Vf, v) = \int f(x)v(x)m(dx)$ *for each* $f \in \mathcal{B}^+$ *such that* $Vf \in \mathcal{E}$ *and* $v \in \hat{\mathcal{E}}$.
(b) *If* $(u_n)_n \subset \mathcal{E} \nearrow u \in \mathcal{E}$ *then* $(E(u_n, v))_n \nearrow E(u, v)$ *for each* $v \in \hat{\mathcal{E}}$.
(c) $E(\alpha u_1 + \beta u_2, v) = \alpha E(u_1, v) + \beta E(u_2, v)$ *for each* $u_1, u_2 \in \mathcal{E}$, $v \in \hat{\mathcal{E}}$ *and* α, $\beta \geq 0$.

Proposition 2. *For each* $(u, v) \in \mathcal{R} \times \hat{\mathcal{R}}$ *we have the following integral representation of the mutual energy:*

$$E(u, v) = 2 \int_0^\infty \int_X f_t(x) g_t(x) m(dx)\, dt,$$

where $(f_t)_{t>0}$ $((g_t)_{t>0})$ *is an* m*-exit law for* $\mathbb{P}$ $(\hat{\mathbb{P}})$ *such that* $u = \int_0^\infty f_t\, dt$ m*-a.e.* $(v = \int_0^\infty g_t\, dt$ m*-a.e., respectively).*

Proof: Let $t > 0$. Since $P_t u = V f_t$ m-a.e. and $\hat{P}_t v = \hat{V} g_t$ m-a.e., we have by Remark 3 and Proposition 1,

$$\begin{aligned} E(P_t u, \hat{P}_t v) &= \int_X f_t(x) \hat{V} g_t(x) m(dx) \\ &= \int_0^\infty \int_X f_t(x) \hat{P}_s g_t(x) m(dx)\, ds \\ &= \int_0^\infty \int_X P_{\frac{s}{2}} f_t(x) \hat{P}_{\frac{s}{2}} g_t(x) m(dx)\, ds \\ &= \int_0^\infty \int_X P_s f_t(x) \hat{P}_s g_t(x) m(dx)\, ds \\ &= 2 \int_t^\infty \int_X f_s(x) g_s(x) m(dx)\, ds. \end{aligned}$$

Therefore by the monotone convergence theorem and Proposition 1, we have

$$E(u, v) = \sup_{t \to 0} E(P_t u, \hat{P}_t v) = 2 \int_0^\infty \int_X f_s(x) g_s(x) m(dx)\, ds.$$

□

Remark 4. Suppose that $\varepsilon_x P_t \ll m$ and $\varepsilon_x \hat{P}_t \ll m$ for each $t > 0$, $x \in X$, and let G be the unique density of V and $\hat{V}$ with respect to m (see [3, XII,72]). Let $\mu, \nu \in \mathcal{M}$ be such that

$$G\mu := \int G(.,y)\mu(dy) \in \mathcal{F} \quad \text{and} \quad \nu G := \int G(x,.)\nu(dx) \in \mathcal{F}.$$

Then it is well known that $E(G\mu, \nu G) = \iint G(x,y)\mu(dy)\nu(dx)$. But since in general the cone $\mathcal{R}$ contains strictly the cone $\{G\mu : \mu \in \mathcal{M}, G\mu \in \mathcal{F}\}$ (see [10, 3.2(2)]) by Proposition 2 we have obtained an integral representation of the mutual energy for a wider class in $\mathcal{E} \times \hat{\mathcal{E}}$.

2.3 Examples.

2.3.1 Compound Poisson.

Let $c > 0$ and M, N be two submarkovian kernels on $(X, \mathcal{B})$ in weak duality with respect to a fixed σ-finite measure m. Let $\mathbb{P}^N := (P_t^N)_{t>0}$ be the submarkovian semigroup defined for each $t > 0$ by

$$P_t{}^N := e^{ct(N-I)} = e^{-ct} \sum_{k=0}^{\infty} \frac{(ct)^k}{k!} N^k.$$

Note that the triplet $(\mathbb{P}^N, \mathbb{P}^M, m)$ satisfies (1) and (2).
In this case, conditions (ii) of Theorem 2 and Theorem 3 are obviously satisfied. Hence $\mathcal{I}mV = \mathcal{R} \cap V(\mathbb{M})$ and $\mathcal{P} = \mathcal{R}$.

2.3.2 Semigroups in strong duality.

Assume that $\varepsilon_x V \ll m$ and $\varepsilon_x \hat{V} \ll m$ for each $x \in X$. In this case, condition (ii) of Theorem 2 is equivalent to $\varepsilon_x \hat{P}_t \ll m$ for each $x \in X$ and $t > 0$.
Therefore Theorem 2 extends Theorem 3.3 in [10].

2.3.3 Lattice semigroups.

Suppose that $\mathbb{P}$ is a measurable lattice semigroup of kernels on $(X, \mathcal{B})$, i.e.

$$P_t|f| = |P_t f|, \quad \text{for} \quad f \in \mathcal{B} \quad \text{and} \quad t > 0.$$

Then, condition (ii) of Theorem (3) is an obvious consequence of [12, Proposition 2.2]. Hence $\mathcal{I}mV = \mathcal{R} \cap V(\mathbb{M})$, and so we find again the result given in [12, Corollary 2.5].

2.3.4 Nearly symmetric semigroups.

Assume that $\mathbb{P}$ and $\hat{\mathbb{P}}$ satisfy the sector condition (see [4]). Then using [8, Proposition 3.8] and [4, Theorem 5.1] we deduce that condition (ii) of Theorem 2 is valid and, therefore, $\mathcal{R} = \mathcal{P}$.

2.3.5 Uniform motion on $\mathbb{R}$.

Let $X = \mathbb{R}$ and $\mathbb{P} := (P_t)_{t>0}$ the uniform motion on X defined by

$$P_t f(x) = f(x+t), \quad t > 0, \ x \in \mathbb{R} \ \text{ and } \ f \in \mathcal{B}^+(\mathbb{R}).$$

Then $\mathcal{R} \neq \mathcal{P}$ by Theorem 1.

3. Decomposition of elements in $\mathcal{P}$

In this section, we suppose that $X = \mathbb{R}^n (n \geq 1)$ and we denote by m the Lebesgue measure on $\mathbb{R}^n$. Let $(\mu_t)_{t>0}$ be a convolution semigroup on $\mathbb{R}^n$ (see [1]) and $\mathbb{P} := (P_t)_{t>0}$ be the associated semigroup of submarkovian kernels defined by

$$P_t f(x) := \int f(x+y)\mu_t(dy), \quad t > 0, \ x \in \mathbb{R}^n, \ f \in \mathcal{B}^+. \tag{12}$$

In the sequel we suppose that $\int_0^\infty \mu_t(f)\, dt < \infty$ for each $f \in C_c^+$.
If we denote by $\hat{\mu}_t$ the centrally symmetric image of μ_t and by $\hat{P}_t$ the induced kernel given by formula (12), then $\mathbb{P} := (P_t)_{t>0}$ and $\hat{\mathbb{P}} := (\hat{P}_t)_{t>0}$ are in weak duality with respect to m and $V, \ \hat{V}$ satisfy the unicity of charges.
In general $\mathcal{R} \neq P$. So we shall investigate the existence of the largest element of $\mathcal{R}$ in the strong m-domination order which is strongly m-dominated by a given function $u \in \mathcal{P}$.

Definition 4. A function $v \in \mathcal{E}$ strongly m-dominates a function $v \in \mathcal{E}$ and we write $u \prec v$ if there exists $w \in \mathcal{E}$ with $v = u + w$ m-a.e.

To prove Theorem 3 we need the following

Lemma 1. *Let $\mu \in \mathcal{M}$ and $A_\mu := \{\nu \in \mathcal{M} : \nu \leq \mu$ and $\nu\hat{P}_t \ll m \ \forall\, t > 0\}$. Put $\nu_0 := \sup A_\mu$. Then we have $\nu_0 \in A_\mu$.*

Proof: Let $\mu \in \mathcal{M}$.
(i) If $\nu, \nu' \in A_\mu$ then since $\sup(\nu,\nu') \leq \nu + \nu'$, we get $\sup(\nu,\nu') \in A_\mu$.
(ii) If $B \in \mathcal{B}$ is such that $\mu(B) < \infty$, then $\sup_{\nu \in A_\mu} \nu(B) \leq \mu(B) < \infty$.
Hence there exists a sequences $(\nu_k) \subset A_\mu$ such that

$$\lim_{k\to\infty} \nu_k(B) = \sup_{\nu \in A_\mu} \nu(B). \tag{13}$$

Using (i) we can assume that (ν_k) is nondecreasing in (13). Let $\nu_\infty := \lim\limits_{k\to\infty} \nu_k$. Then we have $\nu_\infty \hat{P}_t \ll m$ for any $t > 0$ and for each $\nu \in A_\mu, \ 1_B\nu \leq 1_B\nu_\infty$.

(iii) Write $\mathbb{R}^n = \bigcup_{p=0}^{\infty} B_p$ with $(B_p)_p \nearrow$ in $\mathcal{B}$, $p \to \infty$, and $\mu(B_p) < \infty$ for each $p \in \mathbb{N}$. Then using (ii), for each $p \in \mathbb{N}$ there exists a sequence $(\nu_{p,k})$ in A_μ such that $(\nu_{p,k})_k \nearrow (k \to \infty)$ and

$$\lim_{k\to\infty} \nu_{p,k}(B_p) = \sup_{\nu \in A_\mu} \nu(B_p).$$

Therefore $\nu_0 = \lim_{k\to\infty} \nu_{k,k}$ and $\nu_0 \in A_\mu$ by (ii). □

Proof of Theorem 3: Let $u \in \mathcal{P}$ and $\mu \in \mathcal{M}$ be the unique measure (see [1, Theorem 16.7]) such that

$$u \cdot m = \mu \hat{V}. \tag{14}$$

Let $A_\mu := \{\nu \in \mathcal{M} : \nu \leq \mu \text{ and } \nu \hat{P}_t \ll m \ \forall\, t > 0\}$ and $\nu_0 := \sup A_\mu$. Then it follows from Lemma 1 that $\nu_0 \in A_\mu$. Now consider $(f_t)_{t>0} \subset \mathcal{F}$ such that

$$\nu_0 \hat{P}_t = f_t \cdot m \quad \text{for each} \quad t > 0. \tag{15}$$

By (1) and (4) one can check that $(f_t)_{t>0}$ is an m-exit law with respect to $\mathbb{P}$. Therefore, by integrating (15) and using Remark 1 there exists a function $r(u) \in \mathcal{R}$ such that $r(u) \cdot m = \nu_0 \hat{V}$. Now since $\nu_0 \leq \mu$, it follows from (14) that $r(u) \prec u$. Finally, let $v \in \mathcal{R}$ satisfy $v \prec u$. Then (see [1, Theorem 16.7]) there exist $\sigma \in \mathcal{M}$ such that $v \cdot m = \sigma \hat{V}$. Using (2) we get $\sigma \leq \nu_0$ and, hence, $v \prec r(u)$. □

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