

## Selections on $\Psi$ -spaces

M. HRUŠÁK, P.J. SZEPTYCKI, A.H. TOMITA

*Abstract.* We show that if  $\mathcal{A}$  is an uncountable AD (almost disjoint) family of subsets of  $\omega$  then the space  $\Psi(\mathcal{A})$  does not admit a continuous selection; moreover, if  $\mathcal{A}$  is maximal then  $\Psi(\mathcal{A})$  does not even admit a continuous selection on pairs, answering thus questions of T. Nogura.

*Keywords:* MAD family, Vietoris topology, continuous selection

*Classification:* 54C65, 54B20, 03E05

The program of studying continuous selections on topological spaces was initiated by E. Michael in an influential series of papers in the 1950's (see [Mi]). Since then a number of both positive and negative results have been established and research in the area is blooming.

The concept of a  $\Psi$ -space, introduced independently by S. Mrówka and J. Isbell, provides an important class of examples in the theory of Fréchet spaces. Let us mention Mrówka's construction of a  $\Psi$ -space with a unique compactification ([Mr]) and P. Simon's example ([Si]) of two compact Fréchet spaces whose product is not Fréchet. The set-theoretic notation used here is standard and follows [Ku].

Recall that an infinite family  $\mathcal{A} \subseteq [\omega]^\omega$  is *almost disjoint (AD)* if every two distinct elements of  $\mathcal{A}$  have only finite intersection. A family  $\mathcal{A}$  is *MAD* if it is almost disjoint and maximal with this property. Given an almost disjoint family  $\mathcal{A}$ ,  $\mathcal{I}(\mathcal{A})$  denotes the ideal of those subsets of  $\omega$  which can be almost covered by finitely many elements of  $\mathcal{A}$ ,  $\mathcal{I}^*(\mathcal{A})$  denotes the dual filter and  $\mathcal{I}^+(\mathcal{A}) = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$  the coideal of large sets.

**Definition 0.1.** Let  $\mathcal{A}$  be an AD family. Define the space  $\Psi(\mathcal{A})$  as follows: The underlying set is  $\omega \cup \mathcal{A}$ , all elements of  $\omega$  are isolated and basic neighborhoods of  $A \in \mathcal{A}$  are of the form  $\{A\} \cup (A \setminus F)$  for some finite set  $F$ .

It follows immediately from the definition that  $\Psi(\mathcal{A})$  is a first countable, locally compact space. It is hardly surprising that there is a close relationship between topological properties of the space  $\Psi(\mathcal{A})$  and combinatorial properties of the almost disjoint family  $\mathcal{A}$ . If  $\mathcal{A}$  is infinite then  $\Psi(\mathcal{A})$  is not countably compact and

---

The first author's research was partially supported by a grant GAČR 201/00/1466.

$\Psi(\mathcal{A})$  is pseudocompact (contains no infinite discrete family of open subsets) if and only if  $\mathcal{A}$  is a MAD family.

The hyperspace of a space  $X$  (denoted by  $\exp(X)$ ) consists of all closed non-empty subsets of  $X$ . There are many ways to define a topology on  $\exp(X)$  the standard (and most useful) being the *Vietoris topology* generated by sets of the form:

$$\langle U_0, \dots, U_{n-1} \rangle = \{F \in \exp(X) : F \subseteq \bigcup_{i < n} U_i \text{ and } F \cap U_i \neq \emptyset \text{ for every } i < n\}$$

where  $U_0, \dots, U_{n-1}$  are nonempty open subsets of  $X$ . Let  $[X]^2$  denote the set of (unordered) pairs of elements of  $X$ . If  $X$  is a  $T_1$ -space then we consider  $[X]^2$  as a subspace of  $\exp(X)$  equipped with the Vietoris topology.

**Definition 0.2.** A space  $X$  admits a selection if there exists a continuous  $\phi : \exp(X) \rightarrow X$  such that  $\phi(F) \in F$  for every  $F \in \exp(X)$ . Similarly,  $X$  has a weak selection if there exists a continuous  $\phi : [X]^2 \rightarrow X$  such that  $\phi(\{x, y\}) \in \{x, y\}$  for every pair  $\{x, y\}$  of elements of  $X$ .

Note that the existence of a weak selection is equivalent to the existence of a continuous function  $\varphi : X^2 \rightarrow X$  such that  $\varphi((x, y)) = \varphi((y, x)) \in \{x, y\}$ , where  $X^2$  is given the product topology.

T. Nogura has asked the natural question whether  $\Psi(\mathcal{A})$  admits a selection for some (any) MAD family  $\mathcal{A}$ . We answer this question in the negative by proving:

**Theorem 0.3.** *The space  $\Psi(\mathcal{A})$  does not have a weak selection for any maximal almost disjoint family  $\mathcal{A}$ .*

It should be mentioned here that this theorem was proved independently by G. Artico, U. Marconi, J. Pelant, L. Rotter and M. Tkachenko in [A&al]. In fact, it follows directly from a much stronger theorem proved in [A&al].

Here we also show that

**Theorem 0.4.** *If  $X$  is regular, separable and contains an uncountable closed discrete set, then  $X$  does not admit a continuous selection.*

from which it directly follows that  $\Psi(\mathcal{A})$  does not admit a continuous selection for any uncountable almost disjoint family  $\mathcal{A}$ .

We offer our thanks to Salvador Garcia-Ferreira for communicating the question to us and to Jan Pelant for detecting and filling a gap in a preliminary draft of this note.

## I. Proofs of the main theorems

Our proof of Theorem 0.3 is based on a Ramsey theoretic property of the coideal  $\mathcal{I}^+(\mathcal{A})$ . Recall that if  $f : [\omega]^2 \rightarrow 2$  is a coloring of pairs into two colors,

then a set  $A \subseteq \omega$  is *f-homogeneous* if  $|f([A]^2)| = 1$ , in other words, if all pairs of elements of  $A$  are colored by the same color. The famous Ramsey Theorem states that for any coloring  $f$  there is an infinite  $f$ -homogeneous set. The following crucial lemma is well known in set-theoretic circles (see also [BDS]):

**Lemma I.1** ([Ma]). *For every MAD family  $\mathcal{A}$  and every decreasing sequence  $\{X_i : i \in \omega\} \subseteq \mathcal{I}^+(\mathcal{A})$  there is an  $X \in \mathcal{I}^+(\mathcal{A})$  such that  $X \setminus i \subseteq \bigcap_{j < i} X_j$  for every  $i \in X$ .*

**Lemma I.2.** *Let  $\mathcal{A}$  be a MAD family and let  $f : [\omega]^2 \rightarrow 2$ . Then there exists an  $f$ -homogeneous set  $B$  such that  $B \in \mathcal{I}^+(\mathcal{A})$ .*

PROOF: Extend the filter  $\mathcal{I}^*(\mathcal{A}) = \langle \{\omega \setminus A : A \in \mathcal{A}\} \rangle$  to an ultrafilter  $\mathcal{U}$ . We will construct an  $f$ -homogeneous set using this ultrafilter. Let  $g : \omega \rightarrow 2$  be such that  $X_n = \{m \in \omega : f(\{n, m\}) = g(n)\} \in \mathcal{U}$ . Note that  $X_n \in \mathcal{I}^+(\mathcal{A})$ . By previous lemma, there is an  $X \in \mathcal{I}^+(\mathcal{A})$  such that  $X \setminus n \subseteq \bigcap_{i < n} X_i$ , for every  $n \in X$ . Let  $B(i) = \{n \in X : g(n) = i\}$  for  $i \in 2$ . As  $X = B(0) \cup B(1)$ , there exists  $i \in 2$  such that  $B(i) \in \mathcal{I}^+(\mathcal{A})$ . The set  $B = B(i)$  is the desired  $f$ -homogeneous subset.  $\square$

PROOF OF THEOREM 0.3: The proof proceeds by contradiction. Assume that  $\phi : [\Psi(\mathcal{A})]^2 \rightarrow \Psi(\mathcal{A})$  is a weak selection. Consider  $\phi \upharpoonright [\omega]^2$  and define  $f : [\omega]^2 \rightarrow 2$  by:

$$f(\{n, m\}) = 0 \text{ if and only if } \phi(\{n, m\}) = \min\{n, m\}.$$

By Lemma I.2 there is a  $B \in \mathcal{I}^+(\mathcal{A})$  which is  $f$ -homogeneous. Let  $A_0, A_1$  be distinct elements of  $\mathcal{A}$  such that  $B \cap A_i$  is infinite for both  $i < 2$ . We will show that  $\phi$  is not continuous at  $\{A_0, A_1\}$ . Assume that  $\phi(\{A_0, A_1\}) = A_0$ . It suffices to show that the image of any open neighborhood of  $\{A_0, A_1\}$  is not contained in  $\{A_0\} \cup A_0$ , a neighborhood of  $A_0$ .

Suppose  $U$  is a neighborhood of  $\{A_0, A_1\}$ . Then  $U$  contains  $V = \langle \{A_0\} \cup (A_0 \setminus k), \{A_1\} \cup (A_1 \setminus k) \rangle$  for some  $k \in \omega$ .

Suppose that  $f([B]^2) = 0$ . Let  $n > k$  be such that  $n \in (A_1 \cap B) \setminus A_0$  and  $m > n$  such that  $m \in (A_0 \cap B) \setminus A_1$ . Then  $\{n, m\} \in V$  and  $\phi(\{n, m\}) = n \notin A_0$ . On the other hand, if  $f([B]^2) = 1$ , let  $n > k$  be such that  $n \in (A_0 \cap B) \setminus A_1$  and  $m > n$  such that  $m \in (A_1 \cap B) \setminus A_0$ . Then  $\{n, m\} \in V$  and  $\phi(\{n, m\}) = m \notin A_0$ .

Therefore,  $\phi''U \not\subseteq \{A_0\} \cup A_0$ .  $\square$

PROOF OF THEOREM 0.4: Let  $X$  be a separable regular space and let  $A$  be an uncountable closed discrete subset of  $X$ , without loss of generality without isolated points. By way of contradiction assume that  $\phi : \exp(X) \rightarrow X$  is a continuous selection. Define an enumeration

$$A = \{a_\alpha : \alpha < \lambda\}$$

by letting  $a_0 = \phi(A)$  and  $a_\alpha = \phi(A_\alpha)$  where

$$A_\alpha = A \setminus \{a_\beta : \beta < \alpha\}.$$

Fix open neighborhoods  $O_\alpha$  of each  $a_\alpha$  such that

$$\overline{O_\alpha} \cap A = \{a_\alpha\}.$$

By continuity, for each  $\alpha$ ,  $\phi^{-1}(O_\alpha)$  is an open set in  $\exp(X)$  containing  $A_\alpha$ . So, by definition of the Vietoris topology on  $\exp(X)$ , there are  $m_\alpha \in \omega$  and open sets  $U_\alpha^n$ ,  $n < m_\alpha$ , such that

$$A_\alpha \in \langle U_\alpha^n : n < m_\alpha \rangle \subseteq \phi^{-1}(O_\alpha).$$

Therefore,  $A_\alpha \subseteq \bigcup_{n < m_\alpha} U_\alpha^n$  and  $A_\alpha \cap U_\alpha^n \neq \emptyset$  for each  $n < m_\alpha$ .

By shrinking the  $U_\alpha^n$ 's we may assume that

- (a)  $U_\alpha^0 \subseteq O_\alpha$  for each  $\alpha < \lambda$ .
- (b)  $\overline{O_\alpha} \cap \bigcup_{0 < n < m_\alpha} U_\alpha^n = \emptyset$ .

Therefore, as  $\langle U_\alpha^n : n < m_\alpha \rangle \subseteq \phi^{-1}(O_\alpha)$ , we have

- (c) For each  $F \in [X]^{<\aleph_0}$  if  $F \in \langle U_\alpha^n : n < m_\alpha \rangle$  then  $\phi(F) \in F \cap U_\alpha^0$ .

Using that  $X$  is separable, fix  $D$  to be a countable dense subset of  $X$ .

**Claim.** *There is  $F \in [D]^{<\aleph_0}$ , and  $\alpha < \beta < \lambda$  such that*

- (d)  $F \cap U_\alpha^n \neq \emptyset$  for each  $n < m_\alpha$ ;
- (e)  $F \cap U_\beta^n \neq \emptyset$  for each  $n < m_\beta$ ;
- (f)  $F \subseteq (\bigcup_{n < m_\alpha} U_\alpha^n) \cap (\bigcup_{n < m_\beta} U_\beta^n)$ ;
- (g)  $(F \cap U_\alpha^0) \cap (F \cap U_\beta^0) = \emptyset$ .

First note that the Claim leads to a contradiction. Namely, by (b),  $\phi(F) \in U_\alpha^0 \cap U_\beta^0$  but by (g) this is impossible. Thus, proving the Claim will complete the proof of the theorem.

To this end let, for each  $\alpha$ ,

$$V_\alpha = \bigcup_{0 < n < m_\alpha} U_\alpha^n.$$

Then  $U_\alpha^0 \cap V_\alpha = \emptyset$  by (a) and (b). As  $D$  is countable, there is an uncountable set  $J \subset \omega_1$  and a finite set  $G \subset D$  such that

$$\forall \alpha \in J \forall n, 0 < n < m_\alpha : G \cap U_\alpha^n \neq \emptyset \ \& \ G \subset V_\alpha.$$

Let  $\{\delta_\alpha : \alpha \in \omega_1\}$  be an increasing enumeration of  $J$ .

For each  $\alpha \in J$  let

$$D_{\alpha+1} = D \cap U_{\delta_{\alpha+1}}^0 \cap V_{\delta_\alpha}.$$

Note that each  $D_{\alpha+1}$  is a nonempty subset of  $D$  ( $a_{\delta_{\alpha+1}} \in U_{\delta_{\alpha+1}}^0 \cap V_{\delta_\alpha}$  and  $a_{\delta_{\alpha+1}}$  is not isolated). Therefore  $\{D_{\alpha+1} : \alpha < \omega_1\}$  is not pairwise disjoint. So we may fix successor ordinals  $\alpha < \beta < \omega_1$  such that

$$U_{\delta_\alpha}^0 \cap V_{\delta_\beta} \neq \emptyset.$$

Let  $k_0 \in D \cap U_{\delta_\alpha}^0 \cap V_{\delta_\beta}$ . As  $D \cap U_{\delta_\beta}^0 \cap V_{\delta_\alpha} \neq \emptyset$  (recall that  $a_{\delta_\beta} \in V_{\delta_\alpha}$  as  $V_{\delta_\alpha}$  is an open set containing  $A_{\delta_{\alpha+1}}$  and  $a_{\delta_\beta} \in A_{\delta_{\alpha+1}}$ ), we may choose  $k_1 \in D \cap U_{\delta_\beta}^0 \cap V_{\delta_\alpha}$ . Now define  $F = G \cup \{k_0, k_1\}$ .

Notice that  $F \cap U_{\delta_\alpha}^0 = \{k_0\}$  and  $F \cap U_{\delta_\beta}^0 = \{k_1\}$ , thus  $F$  satisfies (g). It is clear that  $F$  satisfies the other conclusions of the Claim.  $\square$

## II. Concluding remarks

The proof of Theorem 0.3 is similar to the proof of the following proposition due to E. van Douwen ([vD1]).

**Proposition II.1** (van Douwen). *If  $X$  is a countably compact, not sequentially compact space, then  $X$  does not have a weak selection. In particular, it does not admit a continuous selection.*

A natural question arises as to for which almost disjoint families  $\Psi(\mathcal{A})$  admits a weak selection. Obviously, if  $\mathcal{A}$  is a countable almost disjoint family, then  $\Psi(\mathcal{A})$  is homeomorphic to an ordinal hence admits a continuous selection. For the proof of Theorem 0.3 we, in fact, only needed that  $\mathcal{A}$  is *somewhere MAD*, i.e. there is an  $X \in \mathcal{I}^+(\mathcal{A})$  such that for every infinite  $Y \subseteq X$  there is an  $A \in \mathcal{A}$  intersecting  $Y$  in an infinite set. If an AD family  $\mathcal{A}$  is not somewhere MAD we say that  $\mathcal{A}$  is *nowhere MAD*. Note that the one-point compactification of the locally compact space  $\Psi(\mathcal{A})$  is Fréchet if and only if  $\mathcal{A}$  is nowhere MAD (see e.g. [vD2]).

We will show that for some, but not all, uncountable nowhere MAD families  $\mathcal{A}$ ,  $\Psi(\mathcal{A})$  does admit a weak selection.

**Example II.2.** *There is an uncountable almost disjoint family  $\mathcal{A}$  such that  $\Psi(\mathcal{A})$  admits a weak selection.*

PROOF: Identify  $\omega$  with  $2^{<\omega}$  — the set of all finite sequences of 0's and 1's. For every  $f \in 2^\omega$  let  $A_f = \{f \upharpoonright n : n \in \omega\}$ . Let  $\mathcal{A} = \{A_f : f \in 2^\omega\}$ . For  $s, t \in 2^{<\omega} \cup 2^\omega$  let  $\Delta_{s,t} = \min\{n \in \omega : s(n) \neq t(n)\}$ . Of course,  $\Delta_{s,t}$  is not well-defined if  $s \subseteq t$

or  $t \subseteq s$ . Define an ordering on  $\Psi(\mathcal{A})$  by:

$$x \leq y \text{ if } \begin{cases} x, y \in 2^{<\omega} \text{ and } (x \subseteq y \text{ or } x(\Delta_{x,y}) < y(\Delta_{x,y})), \\ x \in 2^{<\omega}, y = A_f \text{ and } (x \subseteq f \text{ or } x(\Delta_{x,f}) < f(\Delta_{x,f})), \\ x = A_f, y \in 2^{<\omega} \text{ and } f(\Delta_{y,f}) < y(\Delta_{y,f}), \\ x = A_f, y = A_g \text{ and } (f = g \text{ or } f(\Delta_{f,g}) < g(\Delta_{f,g})). \end{cases}$$

The ordering  $\leq$  is a linear order on  $\Psi(\mathcal{A})$  and the usual topology on  $\Psi(\mathcal{A})$  is finer than the interval topology induced by  $\leq$ . It is easy to verify that putting

$$\phi(\{x, y\}) = x \text{ if and only if } x \leq y$$

defines a continuous weak selection for  $\Psi(\mathcal{A})$ . □

On the other hand:

**Proposition II.3.** *There are nowhere MAD families whose  $\Psi$ -spaces do not have a weak selection.*

PROOF: Let  $\mathcal{A}$  be the almost disjoint family  $\mathcal{A}$  from Example II.2. Note that  $\mathcal{A}$  is a nowhere MAD family of size  $\mathfrak{c}$ .

Enumerate all  $f : [\omega]^2 \rightarrow 2$  as  $\{f_\alpha : \alpha < \mathfrak{c}\}$  and enumerate  $\mathcal{A}$  as  $\{A_\alpha : \alpha \in \mathfrak{c}\}$ .

For every  $\alpha < \mathfrak{c}$ , find an infinite  $f_\alpha$ -homogeneous subset  $C_\alpha$  of  $A_\alpha$  and split it into two infinite pieces  $C_\alpha^0$  and  $C_\alpha^1$ . Let  $A_\alpha^0 = C_\alpha^0$  and  $A_\alpha^1 = A_\alpha \setminus C_\alpha^0$ . Let  $\mathcal{B} = \{A_\alpha^0, A_\alpha^1 : \alpha < \mathfrak{c}\}$ . Now, the proof of Theorem 0.3 goes through, so  $\Psi(\mathcal{B})$  does not have a weak selection, and  $\mathcal{I}(\mathcal{B}) = \mathcal{I}(\mathcal{A})$ , so  $\mathcal{B}$  is nowhere MAD. □

**Corollary II.4.** *There is a separable scattered compact Fréchet space without a weak selection.*

PROOF: Let  $X$  be a one-point compactification of  $\Psi(\mathcal{A})$  without a weak selection, where  $\mathcal{A}$  is nowhere MAD. Then  $X$  is compact, Fréchet and scattered, and does not have a weak selection since  $\Psi(\mathcal{A})$  does not admit one. □

As pointed out by the referee this follows directly from a result of J. van Mill and E. Wattel (see [vMW]) where they proved that *a compact space admits a weak selection if and only if it is orderable.*

REFERENCES

[A&al] G. Artico, U. Marconi, J. Pelant, L. Rotter and M. Tkachenko, *Selections and suborderability*, preprint.  
 [BDS] Balcar B., Dočkálková J., Simon P., *Almost disjoint families of countable sets*, Colloq. Math. Soc. János Bolyai, Finite and Infinite Sets **37** (1984), 59–88.  
 [vD1] van Douwen E.K., *Mappings from hyperspaces and convergent sequences*, Topology Appl. **34** (1990), 35–45.

- [vD2] van Douwen E., *The integers and topology*, in K. Kunen, J. Vaughn, editors, Handbook of Set Theoretic Topology (1984), North-Holland, 111–167.
- [Ku] Kunen K., *Set Theory: An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1980.
- [Ma] Mathias A.R.D., *Happy families*, Ann. Math. Logic **12** (1977), 59–111.
- [vMW] van Mill J., Wattel E., *Selections and orderability*, Proc. Amer. Math. Soc. **83** (1981), 601–605.
- [Mi] Michael E., *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182.
- [Mr] Mrówka S., *Some set-theoretic constructions in topology*, Fund. Math. **94** (1977), 83–92.
- [Si] Simon P., *A compact Fréchet space whose square is not Fréchet*, Comment. Math. Univ. Carolinae **21** (1980), 749–753.

DEPARTMENT OF MATH. AND COMPUTER SCIENCE, VRIJE UNIVERSITEIT, DE BOELELAAN 1081A, 1081 HV AMSTERDAM, THE NETHERLANDS

*E-mail:* michael@cs.vu.nl

DEPARTMENT OF MATHEMATICS, ATKINSON, YORK UNIVERSITY, 4700 KEELE STREET, M3J 1P3 TORONTO, CANADA

*E-mail:* szeptyck@yorku.ca

DEPARTAMENTO DE MATEMÁTICA, INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, CAIXA POSTAL 66281, CEP 05389-970, SÃO PAULO, BRASIL

*E-mail:* tomita@ime.usp.br

(Received October 23, 2000, revised August 28, 2001)