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Change-point estimator in continuous quadratic regression

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Abstract. The paper deals with the asymptotic distribution of the least squares estimator of a change point in a regression model where the regression function has two phases — the first linear and the second quadratic. In the case when the linear coefficient after change is non-zero the limit distribution of the change point estimator is normal whereas it is non-normal if the linear coefficient is zero.

Keywords: change-point estimator, nonlinear regression, limit distribution

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1. Introduction

In applications we often observe a sequence of random variables $Y_1, \dots, Y_n$ that is related to a sequence of regression constants $x_1, \dots, x_n$ by a linear relationship

$$Y_i = a + b\, x_i + e_i, \quad i = 1, \dots, n,$$

where $\{e_i\}$ are i.i.d. errors. However, it can happen that at some unknown time point t^* the relationship between $\{Y_i\}$ and $\{x_i\}$ changes in the following way:

$$Y_i = a^* + b^*\, x_i + \beta^*\, (x_i - t^*)_+ + e_i, \quad i = 1, \dots, n. \tag{1}$$

The model (1) is called bi-linear and can be applied, for example, when the stress-strain relationship for certain types of material is studied. It describes an idealized behavior where a material is perfectly elastic until an elastic limit strain and after it becomes plastic. However, there exist also some other materials that behave differently. For some of them we suppose that the stress-strain relationship has three phases, i.e. elastic, inelastic and plastic. Here, the stress-strain relationship corresponding to the first two phases is often described by the model:

$$Y_i = a^* + b^*\, x_i + \beta^*(x_i - t^*)_+ + \gamma^*(x_i - t^*)_+^2 + e_i, \quad i = 1, \dots, n. \tag{2}$$

The basic statistical problem is to estimate the change-point t^* as well as the other unknown parameters. The problem of parameters estimation in the bi-linear model (1) was solved by several authors, e.g. Hinkley (1969), Feder (1975),

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Bhattacharya (1990), Hušková (1998). In our paper we deal with parameters estimation in the model (2). However, we simplify the model (2) and suppose that the parameters a^* and b^* are known so that without any loss of generality they may be set to zero. The generalization of our result for the case $a^* \neq 0$ and/or $b^* \neq 0$ includes no other mathematical difficulties. Moreover, we suppose that the explanatory variable attains equidistant values and therefore the regression constants $\{x_i\}$ may be set $x_i = i/n$ for $i = 1, \ldots, n$ and $t^* \in (0, 1)$. The variance of random errors σ^2 is supposed to be known so that it may be set to 1. By all simplifications above we arrive to the model:

$$Y_i = \beta^* \, (i/n - t^*)_+ + \gamma^* \, (i/n - t^*)_+^2 + e_i, \quad i = 1, \ldots, n, \tag{3}$$

where $\{e_i\}$ are i.i.d. with $E\, e_i = 0$ and $E\, e_i^2 = 1$. Finally, for mathematical simplicity we take time in reverse order and instead of (3) we consider the following model:

$$Y_i = \beta^* \, (t^* - i/n)_+ + \gamma^* \, (t^* - i/n)_+^2 + e_i, \quad i = 1, \ldots, n, \tag{4}$$

with coefficients $\beta^* \neq 0$ and/or $\gamma^* \neq 0$ and errors $\{e_i\}$ that are i.i.d. with $E\, e_i = 0$, $E\, e_i^2 = 1$ and $E\, |e_i|^{2+\Delta} < \infty$.

In our paper the least squares estimators of unknown parameters t^*, β^* and γ^* are considered. Obviously, for normally distributed random errors $\{e_i\}$ the maximum likelihood estimators coincide with the least squares estimators. We suppose that $t^* \in [\delta, 1 - \delta]$ for a certain known constant $\delta \in (0, 1/2)$ so that the least squares estimators $\widehat{t^*}$, $\widehat{\beta^*}$, $\widehat{\gamma^*}$ are solutions of the following minimization problem:

$$\min \Big\{ S_n(t, \beta, \gamma);\; t \in [\delta, 1 - \delta],\, \beta \in R^1, \gamma \in R^1 \Big\} \tag{5}$$

with

$$S_n(t, \beta, \gamma) = \sum_{i=1}^{n} \Big(Y_i - \beta (t - i/n)_+ - \gamma (t - i/n)_+^2 \Big)^2.$$

The model under the study (4) is a non-linear regression model. It can be proved that the least squares estimators are consistent as n tends to infinity. The aim of the present paper is to find their asymptotic distribution.

We would like to mention that the problem described above was in a completely different setting treated by Feder (1975). In his paper he found the right rate of convergence. Unfortunately, as the model (4) does not always fulfill the assumptions of his theorems, we were not able to apply them directly to find the asymptotic distribution. The method applied here is the method derived by Hušková (1998, 1999).

2. Main theorems

Supposing $\gamma^* \neq 0$, the limit behavior of the studied least squares estimates differ completely in two different cases, i.e. whether $\beta^* \neq 0$ or $\beta^* = 0$. For $\beta^* \neq 0$ the limit distribution of estimators $\widehat{t^*}$, $\widehat{\beta^*}$ and $\widehat{\gamma^*}$ is normal as expected, while for $\beta^* = 0$ the studied distribution is not normal.

Before we state the main theorems it is convenient to realize that the least squares estimator $\widehat{t^*}$ is the argument that minimizes $S_n(t, \widetilde{\beta}_t, \widetilde{\gamma}_t)$, over the interval $[\delta, 1-\delta]$, where

$$S_n(t, \widetilde{\beta}_t, \widetilde{\gamma}_t) = \min_{\beta \in R^1, \gamma \in R^1} \sum_{i=1}^{[nt]} \Big(Y_i - \beta \left(t - (i/n)\right)_+ - \gamma \left(t - (i/n)\right)_+^2 \Big)^2,$$

and it is the same value that maximizes (over the same interval)

$$\boldsymbol{Y}_n^T \boldsymbol{D}_n(t) \big(\boldsymbol{D}_n^T(t) \boldsymbol{D}_n(t)\big)^{-1} \boldsymbol{D}_n^T(t) \boldsymbol{Y}_n - \boldsymbol{Y}_n^T \boldsymbol{D}_n(t^*) \big(\boldsymbol{D}_n^T(t^*) \boldsymbol{D}_n(t^*)\big)^{-1} \boldsymbol{D}_n^T(t^*) \boldsymbol{Y}_n$$

with

$$\boldsymbol{D}_n(t) = \begin{pmatrix} t - \frac{1}{n} & \left(t - \frac{1}{n}\right)^2 \\ \vdots & \vdots \\ t - \frac{[nt]}{n} & \left(t - \frac{[nt]}{n}\right)^2 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{Y}_n = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}.$$

Theorem A. *Suppose that* $\beta^* \neq 0$. *Then* $n^{1/2} \left(\widehat{t^*} - t^*, \widehat{\beta^*} - \beta^*, \widehat{\gamma^*} - \gamma^* \right)$ *has asymptotically a zero-mean normal distribution with a variance-covariance matrix* $\boldsymbol{G}^{-1}$, *where*

$$\boldsymbol{G} = \begin{pmatrix} \beta^{*2} t^* + 4\beta^* \gamma^* t^{*2}/2 + 4\gamma^{*2} t^{*3}/3 & \dots & \dots \\ \beta^* t^{*2}/2 + 2\gamma^* t^{*3}/3 & t^{*3}/3 & \dots \\ \beta^* t^{*3}/3 + 2\gamma^* t^{*4}/4 & t^{*4}/4 & t^{*5}/5 \end{pmatrix}.$$

Especially, it means that $n^{1/2}(\widehat{t^*} - t^*)$ *has asymptotically a normal distribution* $N\Big(0, \frac{9}{\beta^{*2} t^*}\Big)$.

Proof: The proof of Theorem A follows the same pattern as in Hušková (1998). □

Theorem B. *Suppose that* $\beta^* = 0$ *and* $\gamma^* \neq 0$*. Let* (Z_1, Z_2, Z_3) *be a normal vector with zero mean and the following variance – covariance matrix:*

$$\begin{pmatrix} t^* & t^{*2}/2 & t^{*3}/3 \\ t^{*2}/2 & t^{*3}/3 & t^{*4}/4 \\ t^{*3}/3 & t^{*4}/4 & t^{*5}/5 \end{pmatrix}.$$

Let us introduce a random variable $X = -Z_1 + \frac{4}{t^*} Z_2 - \frac{10}{3\,t^{*2}} Z_3$ *having a normal distribution* $N(0, t^*/9)$ *and* $U_+ = \max(0,\ X/\gamma^*)$*. Then as* $n \to \infty$

$$\mathcal{L}\Big(n^{1/2}(\widehat{t^*} - t^*)^2\Big) \;\to\; \mathcal{L}\big(\frac{9}{t^*}\, U_+\big) \equiv \max\Big(0, N\big(0, \frac{9}{\gamma^{*2} t^*}\big)\Big), \tag{6}$$

$$\mathcal{L}\Big(n^{1/2}\widehat{\beta^*}^2\Big) \;\to\; \mathcal{L}\Big(\frac{36}{t^*}\gamma^{*2}\, U_+\Big) \equiv \max\Big(0, N\big(0, \frac{144\,\gamma^{*2}}{t^*}\big)\Big), \tag{7}$$

$$\mathcal{L}\Big(n^{1/2}\big(\widehat{\gamma^*} - \gamma^*\big)\Big) \;\to\; \mathcal{L}\Big(-\frac{30}{t^{*3}}\gamma^*\, U_+ - \frac{60}{t^{*4}}\, Z_2 + \frac{80}{t^{*5}}\, Z_3\Big), \tag{8}$$

$$\mathcal{L}\Big(S_n\big(t^*, \widetilde{\beta_{t^*}}, \widetilde{\gamma_{t^*}}\big) \;-\; S_n\big(\widehat{t^*}, \widehat{\beta^*}, \widehat{\gamma^*}\big)\Big) \;\to\; \mathcal{L}\Big(\frac{9}{t^*}\gamma^{*2} U_+^2\Big) \equiv \Big(\max\big(0, N(0,1)\big)\Big)^2. \tag{9}$$

Proof: Denoting $k = [nt]$ and $k^* = [nt^*]$ we introduce

$$\begin{aligned}
r_{jl}^n(t^*, t) &= \frac{1}{n} \sum_{i=1}^{\min(k^*,k)} (t^* - i/n)^j (t - i/n)^l \quad \text{for} \quad j, l = 1, 2, \\
\boldsymbol{R}^n(t^*, t) &= ||\, r_{jl}^n(t^*, t)\, ||_{j,l=1}^2, \\
\boldsymbol{r}_{2.}^n(t^*, t) &= \big(r_{21}^n(t^*, t), r_{22}^n(t^*, t)\big), \\
\boldsymbol{d}^n(t^*, t) &= \big(d_1^n(t^*, t), d_2^n(t^*, t)\big) = \boldsymbol{r}_{2.}^n(t^*, t)\big(\boldsymbol{R}^n(t, t)\big)^{-1}, \\
\boldsymbol{e}^n(t) &= \Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{k} (t - i/n)\, e_i,\ \frac{1}{\sqrt{n}} \sum_{i=1}^{k} (t - i/n)^2 e_i\Big), \\
\boldsymbol{e}^n(t^*) &= \Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{k^*} (t^* - i/n)\, e_i,\ \frac{1}{\sqrt{n}} \sum_{i=1}^{k^*} (t^* - i/n)^2 e_i\Big).
\end{aligned}$$

By introducing

$$C^n(t^*, t) = \boldsymbol{r}_{2.}^n(t^*, t)\big(\boldsymbol{R}^n(t, t)\big)^{-1}\big(\boldsymbol{r}_{2.}^n(t^*, t)\big)^T \;-\; \frac{1}{n} \sum_{i=1}^{k^*} (t^* - i/n)^4,$$

$$B^n(t^*, t) = \boldsymbol{r}_{2.}^n(t^*, t)\big(\boldsymbol{R}^n(t, t)\big)^{-1}\big(\boldsymbol{e}^n(t)\big)^T \;-\; \frac{1}{\sqrt{n}} \sum_{i=1}^{k^*} (t^* - i/n)^2 e_i,$$

$$A^n(t^*, t) = \boldsymbol{e}^n(t)\big(\boldsymbol{R}^n(t, t)\big)^{-1}\big(\boldsymbol{e}^n(t)\big)^T \;-\; \boldsymbol{e}^n(t^*)\big(\boldsymbol{R}^n(t^*, t^*)\big)^{-1}\big(\boldsymbol{e}^n(t^*)\big)^T,$$

we can express

$$S_n(t,\widetilde{\beta_t},\widetilde{\gamma_t}) \;-\; S_n(t^*,\widetilde{\beta_{t^*}},\widetilde{\gamma_{t^*}}) \;=\; n\,\gamma^{*2}\,C^n(t^*,t) \;+2\sqrt{n}\,\gamma^* B^n(t^*,t) \;+\; A^n(t^*,t).$$

Further we introduce

$$\begin{aligned}
r_{jl}(t^*,t) &= \int_0^1 (t^*-x)_+^j (t-x)_+^l \,dx \quad \text{for} \quad j,l=1,2,\\
\boldsymbol{R}(t,t) &= \|\, r_{jl}(t,t)\,\|_{j,l=1}^2,\\
\boldsymbol{r}_{2.}(t^*,t) &= \big(r_{21}(t^*,t), r_{22}(t^*,t)\big),\\
\boldsymbol{d}(t^*,t) &= \boldsymbol{r}_{2.}(t^*,t)\big(\boldsymbol{R}(t,t)\big)^{-1} \;=\; \big(d_1(t^*,t), d_2(t^*,t)\big),\\
C(t^*,t) &= \boldsymbol{r}_{2.}(t^*,t)\big(\boldsymbol{R}(t,t)\big)^{-1}\big(\boldsymbol{r}_{2.}(t^*,t)\big)^T - t^{*5}/5.
\end{aligned}$$

Clearly

$$\boldsymbol{R}(t,t) \;=\; \begin{pmatrix} t^3/3 & t^4/4 \\ t^4/4 & t^5/5 \end{pmatrix}, \quad \big(\boldsymbol{R}(t,t)\big)^{-1} \;=\; \begin{pmatrix} 48/t^3 & -60/t^4 \\ -60/t^4 & 80/t^5 \end{pmatrix},$$

$$\boldsymbol{d}(t^*,t^*) \;=\; (0,1).$$

Lemma 1. *It holds*

$$\begin{aligned}
&\Big|\Big|\Big(\big(\boldsymbol{R}^n(t,t)\big)^{-1} - \big(\boldsymbol{R}^n(t^*,t^*)\big)^{-1}\Big) - \Big(\big(\boldsymbol{R}(t,t)\big)^{-1} - \big(\boldsymbol{R}(t^*,t^*)\big)^{-1}\Big)\Big|\Big|\\
&\qquad = O\Big(\frac{|\,t-t^*|}{n}\Big),\\
&\Big|\Big|\,\boldsymbol{d}^n(t^*,t) - \boldsymbol{d}(t^*,t)\,\Big|\Big| \;=\; O\Big(\frac{|\,t-t^*|}{n}\Big),\\
&|C^n(t^*,t) - C(t^*,t)| \;=\; O\Big(\frac{|\,t-t^*|}{n}\Big),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
d_1(t^*,t) &= -2\,(t-t^*) \;+\; 4\frac{(t-t^*)^2}{t^*} \;+\; 4\frac{(t-t^*)^3}{t^{*2}} \;+\; o\big((t-t^*)^4\big),\\
d_2(t^*,t) &= 1 - \frac{10}{3}\frac{(t-t^*)^2}{t^*} \;+\; \frac{20}{3}\frac{(t-t^*)^3}{t^{*2}} \;-\; 5\frac{(t-t^*)^4}{t^{*3}} \;+\; o\big((t-t^*)^4\big),\\
C(t^*,t) &= -\frac{t^*}{9}(t-t^*)^4 \;+\; o\big((t-t^*)^4\big).
\end{aligned} \tag{11}$$

Proof: As for $j,l=1,2$

$$\begin{aligned}
r_{jl}^n(t^*,t) - r_{jl}^n(t^*,t^*) &= r_{jl}(t^*,t) - r_{jl}(t^*,t^*) \;+\; O\big(|\,t-t^*|/n\big),\\
r_{jl}^n(t,t) - r_{jl}^n(t^*,t^*) &= r_{jl}(t,t) - r_{jl}(t^*,t^*) \;+\; O\big(|\,t-t^*|/n\big),\\
r_{jl}^n(t^*,t) - r_{jl}(t^*,t) &= O\big(1/n\big),
\end{aligned}$$

the approximations (10) hold true. The assertions (11) are the Taylor expansions of the corresponding terms. □

To prove the following two lemmas we apply the relationships between $\boldsymbol{e}^n(t)$ and $\boldsymbol{e}^n(t^*)$ together with the expansion of $\boldsymbol{d}(t^*,t)\big(\boldsymbol{e}^n(t)\big)^T - e_2^n(t^*)$:

For $t^* < t$

$$\sum_{i=1}^{k}(t-i/n)\,e_i = \sum_{i=1}^{k^*}(t^*-i/n)\,e_i + (t-t^*)\sum_{i=1}^{k^*} e_i$$

$$+ \sum_{i=k^*+1}^{k}(t^*-i/n)\,e_i + (t-t^*)\sum_{i=k^*+1}^{k} e_i,$$

$$\sum_{i=1}^{k}(t-i/n)^2 e_i = \sum_{i=1}^{k^*}(t^*-i/n)^2\,e_i + 2\,(t-t^*)\sum_{i=1}^{k^*}(t^*-i/n)\,e_i + (t-t^*)^2\sum_{i=1}^{k^*} e_i$$

$$+ \sum_{i=k^*+1}^{k}(t^*-i/n)^2\,e_i + 2\,(t-t^*)\sum_{i=k^*+1}^{k}(t^*-i/n)\,e_i + (t-t^*)^2\sum_{i=k^*+1}^{k} e_i,$$

and

(12)

$$\sqrt{n}\,\boldsymbol{d}(t^*,t)\big(\boldsymbol{e}^n(t)\big)^T - \sum_{i=1}^{k^*}(t^*-i/n)^2 e_i =$$

$$\Big(-2(t-t^*) + \frac{4}{t^*}(t-t^*)^2 + K_1(t)\,(t-t^*)^3\Big)\Big(\sum_{i=1}^{k}(t-i/n)\,e_i\Big) +$$

$$\Big(-\frac{10}{3}\frac{1}{t^*}(t-t^*)^2 + K_2(t)(t-t^*)^3\Big)\Big(\sum_{i=1}^{k}(t-i/n)^2 e_i\Big) +$$

$$\Big(\sum_{i=1}^{k}(t-i/n)^2 e_i - \sum_{i=1}^{k^*}(t^*-i/n)^2 e_i\Big) =$$

$$\Big(-2(t-t^*) + \frac{4}{t^*}(t-t^*)^2 + K_1(t)\,(t-t^*)^3\Big)\Big(\sum_{i=1}^{k}(t-i/n)\,e_i - \sum_{i=1}^{k^*}(t^*-i/n)\,e_i\Big) +$$

$$\Big(\frac{4}{t^*}(t-t^*)^2 + K_1(t)\,(t-t^*)^3\Big)\Big(\sum_{i=1}^{k^*}(t^*-i/n)\,e_i\Big) +$$

$$\Big(-\frac{10}{3}\frac{1}{t^{*2}}(t-t^*)^2 + K_2(t)\,(t-t^*)^3\Big)\Big(\sum_{i=1}^{k}(t-i/n)^2 e_i\Big) + (t-t^*)^2\sum_{i=1}^{k^*} e_i +$$

$$\sum_{i=k^*+1}^{k} (t^* - i/n)^2 e_i + 2(t - t^*) \sum_{i=k^*+1}^{k} (t^* - i/n) e_i + (t - t^*)^2 \sum_{i=k^*+1}^{k} e_i,$$

where $K_1(t)$ and $K_2(t)$ are some continuous function on $[\delta, 1-\delta]$.

A similar relationship can be shown for $t < t^*$.

Lemma 2. *It holds*

$$\widehat{t^*} - t^* = O_P\big((1/n)^{1/4}\big) \quad \text{as} \quad n \to \infty. \tag{13}$$

Proof: It is known that

$$\widehat{t^*} - t^* = o_P(1) \quad \text{as} \quad n \to \infty,$$

see Seber and Wild (1989). Moreover, Feder (1975) showed that

$$\widehat{t^*} - t^* = O_P\big((\ln\ln n/n)^{1/4}\big) \quad \text{as} \quad n \to \infty.$$

As

$$C(t^*, t) = \frac{-(t-t^*)^4\, t^{*5}\, \big(9(t-t^*) + 5\, t^*\big)}{45\, t^5} \quad \text{for} \quad t - t^* > 0,$$

$$C(t^*, t) = \frac{(t-t^*)^4 \big(4\,(t-t^*) - 5\, t^*\big)}{45} \quad \text{for} \quad t - t^* < 0,$$

there exist a constant K and $n_0 \in N$ such that for $n \geq n_0$

$$C^n(t^*, t) \leq -K(t-t^*)^4 \quad \text{for} \quad t, t^* \in (0,1).$$

For the assertion (13) it is sufficient to prove that for any sequence $\{r_n\}$ such that $r_n \to \infty$

$$\max_{|t-t^*| \geq \frac{r_n}{n^{1/4}}} \frac{|2\,\gamma^* \sqrt{n} B^n(t^*, t) + A^n(t^*, t)|}{\gamma^{*2} n (t-t^*)^4} = o_P(1) \quad \text{as} \quad n \to \infty. \tag{14}$$

Using the law of iterated logarithm we may show

$$\max_{|t-t^*| \geq \frac{r_n}{n^{1/4}}} \frac{\left|\sum_{i=\min(k,k^*)+1}^{\max(k,k^*)} e_i\right|}{n\,|\gamma^*(t-t^*)^2|} = o_P(1),$$

$$\max_{|t-t^*| \geq \frac{r_n}{n^{1/4}}} \frac{\left|\sum_{i=\min(k,k^*)+1}^{\max(k,k^*)} (t^* - i/n)\, e_i\right|}{n\,|\gamma^*(t-t^*)^3|} = o_P(1),$$

$$\max_{|t-t^*| \geq \frac{r_n}{n^{1/4}}} \frac{\left|\sum_{i=\min(k,k^*)+1}^{\max(k,k^*)} (t^* - i/n)^2 e_i\right|}{n\,|\gamma^*(t-t^*)^4|} = o_P(1),$$

as $n \to \infty$. Now (14) follows from the preceding assertions and (12). □

Lemma 3. *For any sequence* $\{r_n\}$ *such that* $r_n = o\big(n^{1/4}/\ln\ln n\big)$

$$\max_{|t-t^*|\le \frac{r_n}{n^{1/4}}} \frac{|\boldsymbol{d}(t^*,t)\big(\boldsymbol{e}^n(t)\big)^T - e_2^n(t^*) - (t-t^*)^2 X_n(t^*)|}{(t-t^*)^2} = o_P(1).$$

as $n \to \infty$*, where*

$$X_n(t^*) = -\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt^*]} e_i + \frac{4}{t^*}\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt^*]} (t^* - i/n)e_i - \frac{10}{3\,t^{*2}}\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt^*]} (t^* - i/n)^2 e_i.$$

Proof: As $n \to \infty$

$$\max_{|t-t^*|\le \frac{r_n}{n^{1/4}}} \frac{\Big|\sum_{i=\min(k,k^*)+1}^{\max(k,k^*)} e_i\Big|}{\sqrt{n}} = o_P(1),$$

$$\max_{|t-t^*|\le \frac{r_n}{n^{1/4}}} \frac{\Big|\sum_{i=\min(k,k^*)+1}^{\max(k,k^*)} (t^*-i/n)\, e_i\Big|}{\sqrt{n}\,|t-t^*|} = o_P(1),$$

$$\max_{|t-t^*|\le \frac{r_n}{n^{1/4}}} \frac{\Big|\sum_{i=\min(k,k^*)+1}^{\max(k,k^*)} (t^*-i/n)^2\, e_i\Big|}{\sqrt{n}\,(t-t^*)^2} = o_P(1).$$

Using (12) we get the assertion of Lemma 3. □

Notice that asymptotically $X_n(t^*)$ has the same distribution as X, i.e. a normal distribution $N(0, t^*/9)$.

Lemma 4. *For any arbitrary sequence* $\{r_n\}$ *such that* $r_n = o\big(n^{1/4}/\ln\ln n\big)$

$$\gamma^{*2}\, n\, C^n(t^*,t) + 2\,\gamma^*\sqrt{n}\, B^n(t^*,t) + A^n(t^*,t) =$$

$$\gamma^{*2}\,\Big(-\frac{t^*}{9}\Big)\, n\,(t-t^*)^4\big(1+o(1)\big) + 2\,\gamma^* X_n(t^*)\,\sqrt{n}\,(t-t^*)^2\big(1+o_P(1)\big) + O_P(|\,t-t^*|)$$

as $n \to \infty$ *uniformly for* $|t-t^*| \le r_n/n^{1/4}$.

PROOF: Lemma 4 is a consequence of Lemma 1, Lemma 3 and the fact that

$$\max_{|t-t^*|\le \frac{r_n}{n^{1/4}}} \frac{A^n(t^*,t)}{t-t^*} = O_P(1).$$

□

As $x = n^{1/2}(\widehat{t^*} - t^*)^2$ is the value maximizing the quadratic function $x\left(-({\gamma^*}^2 t^*/9)\,x + 2\gamma^* X_n(t^*)\right)$ over the set $\{x,\, x>0\}$ the assertion (6) of Theorem B is proved. Since

$$(\widehat{\beta^*},\widehat{\gamma^*}) = \gamma^* \boldsymbol{d}^n(t^*,\widehat{t^*}) + \frac{1}{\sqrt{n}}\boldsymbol{e}^n(\widehat{t^*})\big(\boldsymbol{R}^n(\widehat{t^*},\widehat{t^*})\big)^{-1}$$

and

$$\begin{aligned}
&\boldsymbol{e}^n(\widehat{t^*})\big(\boldsymbol{R}^n(\widehat{t^*},\widehat{t^*})\big)^{-1} - \boldsymbol{e}^n(t^*)\big(\boldsymbol{R}^n(t^*,t^*)\big)^{-1} = o_P(1),\\
&n^{1/4}\widehat{\beta^*} - \left(-2\,\gamma^*\, n^{1/4}\,(\widehat{t^*}-t^*)\right) = o_P(1),\\
&n^{1/2}(\widehat{\gamma^*}-\gamma^*) - \Bigg(-\frac{10}{3}\frac{1}{{t^*}^2}\gamma^*\, n^{1/2}\,(\widehat{t^*}-t^*)^2 -\\
&\frac{60}{{t^*}^4}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{k^*}(t^*-i/n)\,e_i\Big) + \frac{80}{{t^*}^5}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{k^*}(t^*-i/n)^2 e_i\Big)\Bigg) = o_P(1),\\
&S_n\big(t^*,\widetilde{\beta_{t^*}},\widetilde{\gamma_{t^*}}\big) - S_n\big(\widehat{t^*},\widehat{\beta^*},\widehat{\gamma^*}\big) - \big(\gamma^*\, n^{1/2}\,(\widehat{t^*}-t^*)^2\big)^2 (t^*/9) = o_P(1),
\end{aligned}$$

(7), (8) and (9) hold true.

Remark 1. In Theorem B we supposed that γ^* is a fixed value but the assertion of Theorem B remains true even if $\gamma_n^* \to 0$ in a way that

$$\frac{(\ln\ln n)^2}{n^{1/2}\gamma_n^*} \to 0.$$

If we choose the sequence $\{r_n\}$ tending to infinity such that

$$\frac{r_n^2\,(\ln\ln n)^2}{n^{1/2}\gamma_n^*} \to 0,$$

we get

$$\max_{|t-t^*|\le \frac{r_n}{n^{1/4}|\gamma_n^*|^{1/2}}} \frac{\left|\sum_{i=\min(k,k^*)+1}^{\max(k,k^*)} e_i\right|}{\sqrt{n}} = O_P\Big(\frac{(\ln\ln n)^{1/2} r_n^{1/2}}{n^{1/8}|\gamma_n^*|^{1/4}}\Big) = o_p(1),$$

$$\max_{|t-t^*|\geq \frac{r_n}{n^{1/4}|\gamma_n^*|^{1/2}}} \frac{|\sum_{i=\min(k,k^*)+1}^{\max(k,k^*)} e_i|}{n\,|\gamma_n^*|(t-t^*)^2} =$$

$$O_P\Big(\frac{(\ln\ln n)^{1/2}}{n^{1/2}} \frac{1}{|\gamma_n^*|\big(\frac{r_n}{n^{1/4}|\gamma_n^*|^{1/2}}\big)^{3/2}}\Big) = o_P(1).$$

Then the proof of Theorem B is analogous as for a fixed γ^*. □

Remark 2. Our simulation study showed that the limit distribution of (6) underestimates the variability of $\widehat{t^*}$. As a consequence the symmetric $(1-\alpha)\,100\%$ confidence interval for t^* based on (6), i.e.

$$\Big(-\Big(\frac{3\,u_{1-\alpha}}{\gamma^*}\Big)^{1/2}\frac{1}{t^{*\,1/4}} \leq n^{1/4}(\widehat{t^*}-t^*) \leq \Big(\frac{3\,u_{1-\alpha}}{\gamma^*}\Big)^{1/2}\frac{1}{t^{*\,1/4}}\Big),$$

contains less than $(1-\alpha)\,100\%$ realizations of $n^{1/4}(\widehat{t^*}-t^*)$. In our simulation study (10 000 repetitions) for $n = 5\,000$, $t^* = 0.5$ and $\gamma^* = 60$ only 88.2% of realizations of $n^{1/2}(\widehat{t^*}-t^*)^2$ have fallen into the 95% symmetric confidence interval. Figure 1 shows the histogram of $\widehat{t^*}$ for $n = 5\,000$, $t^* = 0.5$ and $\gamma^* = 60$.

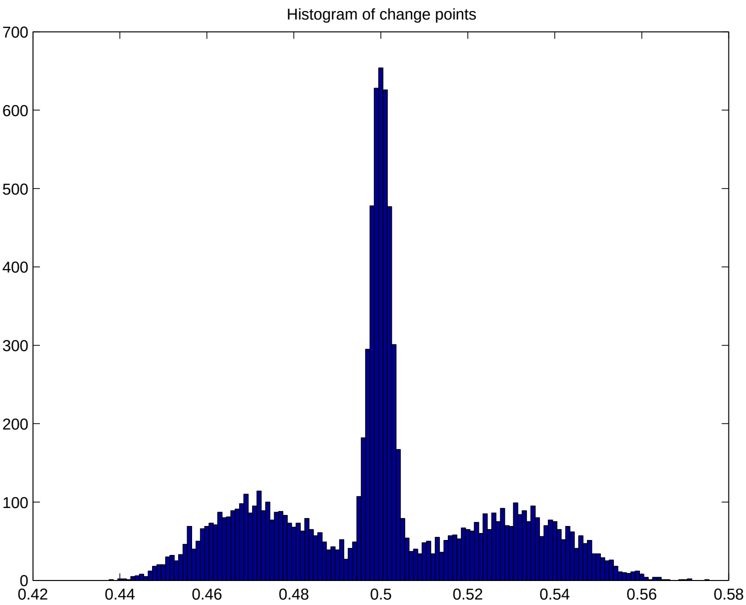

Table 1 presents relative frequencies of $n^{1/2}(\widehat{t^*}-t^*)^2\frac{\gamma^*\sqrt{t^*}}{3}$ with $n=5\,000$, $t^*=0.3,0.5,0.7,0.9$ and $\gamma^*=60$ obtained from simulation study (10 000 repetitions) and probabilities calculated from the distribution of $\max(0,N(0,1))$ according to (6).

	rel. fr.	rel. fr.	rel. fr.	rel. fr.	prob.
interval	$t^*=0.3$	$t^*=0.5$	$t^*=0.7$	$t^*=0.9$	
0.00 – 0.25	0.4439	0.4890	0.5118	0.5270	0.5987
0.25 – 0.75	0.1937	0.1753	0.1752	0.1680	0.1747
0.75 – 1.25	0.1356	0.1437	0.1354	0.1352	0.1210
1.25 – 1.75	0.0961	0.0869	0.0957	0.0922	0.0656
1.75 – 2.25	0.0602	0.0591	0.0476	0.0427	0.0278
2.25 – 2.75	0.0352	0.0293	0.0230	0.0261	0.0092
2.75 – 3.25	0.0159	0.0110	0.0073	0.0064	0.0024
3.25 – 3.75	0.0089	0.0040	0.0029	0.0017	0.0005
3.75 – ∞	0.0105	0.0017	0.0011	0.0007	0.0001

Table 1. Relative frequencies and corresponding probabilities of the statistic $\sqrt{n}(\widehat{t^*}-t^*)^2\gamma^*\sqrt{t^*}/3$ for $n=5\,000$, $t^*=0.5$ and $\gamma^*=60$.

Remark 3. Getting back to the model (2) with $x_i=i/n$ it can be derived analogously as in Theorem A and B that if $\beta^*\neq 0$ then the asymptotic distribution of $n^{1/2}(\widehat{t^*}-t^*)$ is zero mean normal with the variance $(4+5t^*)/\big(\beta^{*2}t^*(1-t^*)\big)$. If $\beta^*=0$ and $\gamma^*\neq 0$ then $n^{1/2}(\widehat{t^*}-t^*)^2$ is asymptotically distributed as U_+ where U has a zero mean normal distribution with the variance $(4+5t^*)/\big(\gamma^{*2}t^*(1-t^*)\big)$.

Remark 4. Theorem B gives the limit distribution of the unknown coefficients t^*, β^* and γ^* in the model (4) in the case $\beta^*=0$. If we consider the least squares estimator $\widehat{t^*}$ of the parameter t^* in the model

$$Y_i=\gamma^*\,(t^*-i/n)_+^2+e_i,\quad i=1,\dots,n,\quad \gamma^*\neq 0,$$

then $n^{1/2}(\widehat{t^*}-t^*)$ has asymptotically a zero-mean normal distribution with the variance $12/\big(\gamma^{*2}t^{*3}\big)$. Moreover, $S_n\big(t^*,\widetilde{\gamma_{t^*}}\big)-S_n\big(\widehat{t^*},\widehat{\gamma^*}\big)$ has asymptotically χ^2 distribution with one degree of freedom. The asymptotic distribution of the change point estimator $\widehat{t^*}$ in the model

$$Y_i=a+b\,(i/n)+\gamma^*\,(i/n-t^*)_+^2+e_i,\quad i=1,\dots,n,\quad \gamma^*\neq 0,$$

was given in Jarušková (1998).

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