Natural affinors on $(J^{r,s,q}(.,\mathbb{R}^{1,1})_0)^*$

Włodzimierz M. Mikulski

Abstract. Let $r,s,q,m,n \in \mathbb{N}$ be such that $s \geq r \leq q$. Let Y be a fibered manifold with m-dimensional basis and n-dimensional fibers. All natural affinors on $(J^{r,s,q}(Y,\mathbb{R}^{1,1})_0)^*$ are classified. It is deduced that there is no natural generalized connection on $(J^{r,s,q}(Y,\mathbb{R}^{1,1})_0)^*$. Similar problems with $(J^{r,s}(Y,\mathbb{R})_0)^*$ instead of $(J^{r,s,q}(Y,\mathbb{R}^{1,1})_0)^*$ are solved.

Keywords: bundle functors, natural transformations, natural affinors

Classification: 58A20, 53A55

0. Let us recall the following definitions (see e.g. [3]).

Let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a functor from the category $\mathcal{FM}_{m,n}$ of all fibered manifolds with m-dimensional bases and n-dimensional fibers and their local fibered diffeomorphisms into the category $\mathcal{FM}$ of fibered manifolds and fibered maps. Let $\mathcal{B} : \mathcal{FM} \to \mathcal{M}f$ be the base functor from $\mathcal{FM}$ into the category $\mathcal{M}f$ of manifolds. Let $\mathcal{T} : \mathcal{FM} \to \mathcal{M}f$ be the total space functor.

A *bundle functor* over $\mathcal{FM}_{m,n}$ is a (covariant) functor F satisfying $\mathcal{B} \circ F = \mathcal{T}_{|\mathcal{FM}_{m,n}}$ and the localization condition: for every inclusion of an open subset $i_U : U \to Y$, FU is the restriction $p_Y^{-1}(U)$ of $p_Y : FY \to Y$ over U and Fi_U is the inclusion $p_Y^{-1}(U) \to FY$.

An *affinor* D on a manifold M is a tensor type $(1,1)$, i.e. a linear morphism $D : TM \to TM$ over id_M.

A *natural affinor* on a bundle functor F is a system of affinors $D : TFY \to TFY$ on FY for every $\mathcal{FM}_{m,n}$-object Y satisfying $TFf \circ D = D \circ TFf$ for every local $\mathcal{FM}_{m,n}$-diffeomorphism $f : Y \to \overline{Y}$.

A *connection* on a fibre bundle Z is an affinor $\Gamma : TZ \to TZ$ on Z such that $\Gamma \circ \Gamma = \Gamma$ and $\mathrm{im}(\Gamma) = VZ$, the vertical bundle of Z.

A *natural connection* on a bundle functor F is a system of connections $\Gamma : TFY \to TFY$ on FY for every $\mathcal{FM}_{m,n}$-object Y which is (additionally) a natural affinor on F.

In [5] it was shown how natural affinors Q on some bundle functor FY can be used to study the torsion $\tau = [\Gamma, Q]$ of connections Γ on FY. That is why, natural affinors have been classified in many papers, [1], [2], [7]–[11]. For example, in [2] natural affinors on the r-th order vector tangent bundle $(J^r(M,\mathbb{R})_0)^*$ over m-manifolds $M \in \mathrm{obj}(\mathcal{FM}_{m,0})$ were classified.

In this paper we fix numbers $r,s,q,m,n \in \mathbb{N}$ such that $s \geq r \leq q$ and consider the bundle functor $F = T^{(r,s,q)}_{|\mathcal{FM}_{m,n}}$, where $T^{(r,s,q)} = (J^{r,s,q}(.,\mathbb{R}^{1,1})_0)^* : \mathcal{FM} \to \mathcal{FM}$ is the (introduced in [4]) bundle functor associating to every fibered manifold Y the vector bundle $(J^{r,s,q}(Y,\mathbb{R}^{1,1})_0)^*$ over Y. We prove that the set of all natural affinors on $T^{(r,s,q)}_{|\mathcal{FM}_{m,n}}$ is a 3-dimensional vector space over $\mathbb{R}$ and we construct explicitly the basis of this vector space.

We also solve the similar problem with $T^{(r,s)} = (J^{r,s}(.,\mathbb{R})_0)^* : \mathcal{FM} \to \mathcal{FM}$ instead of $T^{(r,s,q)}$.

As an application of the obtained results we deduce that there are no natural connections on $T^{(r,s,q)}$ and $T^{(r,s)}$.

The above results extend [2].

Throughout this paper $r,s,q,m,n \in \mathbb{N}$ are numbers with $s \geq r \leq q$.

The usual fiber coordinates on $\mathbb{R}^{m,n}$, the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n$ over $\mathbb{R}^m$, are denoted by $x^1,\dots,x^m,y^1,\dots,y^n$.

All manifolds and maps are assumed to be of class C^∞.

1. The concept of classical r-jets can be generalized as follows. Let $Y \to M$ and $Z \to N$ be fibered manifolds. We recall that two $\mathcal{FM}$-morphisms $f,g : Y \to Z$ with base maps $\underline{f},\underline{g} : M \to N$ determine the same (r,s,q)-jet $j^{r,s,q}_y f = j^{r,s,q}_y g$ at $y \in Y_x$, $x \in M$, if $j^r_y f = j^r_y g$, $j^s_y(f|Y_x) = j^s_y(g|Y_x)$ and $j^q_x \underline{f} = j^q_x \underline{g}$. The space of all (r,s,q)-jets of Y into Z is denoted by $J^{r,s,q}(Y,Z)$. The composition of $\mathcal{FM}$-morphisms induces the composition of (r,s,q)-jets ([3, p. 126]).

The space $T^{r,s,q*}Y = J^{r,s,q}(Y,\mathbb{R}^{1,1})_0$, $0 \in \mathbb{R}^2$, has an induced structure of a vector bundle over Y. Every $\mathcal{FM}$-morphism $f : Y \to Z$, $f(y) = z$, induces a linear map $\lambda(j^{r,s,q}_y f) : T^{r,s,q*}_z Z \to T^{r,s,q*}_y Y$ by means of the jet composition. If we denote by $T^{(r,s,q)}Y$ the dual vector bundle of $T^{r,s,q*}Y$ and define $T^{(r,s,q)}f : T^{(r,s,q)}Y \to T^{(r,s,q)}Z$ by using the dual maps to $\lambda(j^{r,s,q}_y f)$, we obtain (similarly as in [3, p. 123]) a vector bundle functor $T^{(r,s,q)}$ on $\mathcal{FM}$, see [4].

2. In this section all natural transformations $T^{(r,s,q)} \to T^{(r,s,q)}$ over $\mathcal{FM}_{m,n}$ will be classified. This extends [6].

A *natural transformation* $T^{(r,s,q)} \to T^{(r,s,q)}$ over $\mathcal{FM}_{m,n}$ is a system of fibered maps $A : T^{(r,s,q)}Y \to T^{(r,s,q)}Y$ covering the identity id_Y for every $\mathcal{FM}_{m,n}$-object Y satisfying $T^{(r,s,q)}f \circ A = A \circ T^{(r,s,q)}f$ for every local $\mathcal{FM}_{m,n}$-map $f : Y \to \overline{Y}$.

Example 1. Let Y be an $\mathcal{FM}_{m,n}$-object. For a fibered map $\gamma = (\gamma^1,\gamma^2) : Y \to \mathbb{R}^{1,1}$ we have fibered maps $\gamma^{\langle 1\rangle} = (\gamma^1, 0)$, $\gamma^{\langle 2\rangle} = (0,\gamma^2)$, $\gamma^{\langle 3\rangle} = (0,\gamma^1) : Y \to \mathbb{R}^{1,1}$. Clearly, $j^{r,s,q}_y\gamma^{\langle 1\rangle}$, $j^{r,s,q}_y\gamma^{\langle 2\rangle}$, $j^{r,s,q}_y\gamma^{\langle 3\rangle}$ depend linearly on $j^{r,s,q}_y\gamma$ for $y \in Y$. Define fibered maps $Pr^{\langle 1\rangle}, Pr^{\langle 2\rangle}, Pr^{\langle 3\rangle} : T^{(r,s,q)}Y \to T^{(r,s,q)}Y$ over id_Y by

$$\langle Pr^{\langle 1\rangle}(\omega), j^{r,s,q}_y\gamma\rangle = \langle \omega, j^{r,s,q}_y\gamma^{\langle 1\rangle}\rangle,$$
$$\langle Pr^{\langle 2\rangle}(\omega), j^{r,s,q}_y\gamma\rangle = \langle \omega, j^{r,s,q}_y\gamma^{\langle 2\rangle}\rangle,$$
$$\langle Pr^{\langle 3\rangle}(\omega), j^{r,s,q}_y\gamma\rangle = \langle \omega, j^{r,s,q}_y\gamma^{\langle 3\rangle}\rangle,$$

$\omega \in T^{(r,s,q)}_yY$, $y \in Y$, $\gamma = (\gamma_1,\gamma_2) : Y \to \mathbb{R}^{1,1}$ is fibered, $\gamma(y) = 0$. The families $Pr^{\langle 1\rangle}, Pr^{\langle 2\rangle}, Pr^{\langle 3\rangle} : T^{(r,s,q)} \to T^{(r,s,q)}$ are natural transformations over $\mathcal{FM}_{m,n}$.

Proposition 1. *Every natural transformation $A : T^{(r,s,q)} \to T^{(r,s,q)}$ over $\mathcal{FM}_{m,n}$ is a linear combination of $Pr^{\langle 1\rangle}$, $Pr^{\langle 2\rangle}$ and $Pr^{\langle 3\rangle}$.*

Proof: The elements $j^{r,s,q}_0(x^\alpha, 0)$ and $j^{r,s,q}_0(0, x^\beta y^\delta)$ for multiindices α and (β,δ) from obvious sets form the basis of $J^{r,s,q}_0(\mathbb{R}^{m,n},\mathbb{R}^{1,1})_0$.

By the fibered version of the rank theorem, $j^{r,s,q}_0(x^1, y^1)$ has dense orbit in $J^{r,s,q}_0(\mathbb{R}^{m,n},\mathbb{R}^{1,1})_0$. Then (by the naturality) A is uniquely determined by the contractions $\langle A(\omega), j^{r,s,q}_0(x^1,y^1)\rangle$ for all $\omega \in T^{(r,s,q)}_0\mathbb{R}^{m,n}$. So, it suffices to deduce that $\langle A(\cdot), j^{r,s,q}_0(x^1,y^1)\rangle : T^{(r,s,q)}_0\mathbb{R}^{m,n} \to \mathbb{R}$ is a linear combination of $j^{r,s,q}_0(x^1,0), j^{r,s,q}_0(0,x^1), j^{r,s,q}_0(0,y^1) : T^{(r,s,q)}_0\mathbb{R}^{m,n} \to \mathbb{R}$, i.e. that the vector space of all A as above has dimension ≤ 3.

By the naturality of A with respect to the homotheties $a_t = t\,\mathrm{id}_{\mathbb{R}^m\times\mathbb{R}^n} : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ for $t \neq 0$ and the homogeneous function theorem (see [3]), we deduce that $\langle A(\cdot), j^{r,s,q}_0(x^1,y^1)\rangle$ is a linear combination of $j^{r,s,q}_0(x^i,0)$, $j^{r,s,q}_0(0,x^i)$ and $j^{r,s,q}_0(0,y^j)$ for $i = 1,\dots,m$ and $j = 1,\dots,n$. Next, using the naturality of A with respect to the fibered maps $b_t = (x^1, tx^2,\dots,tx^n, y^1, ty^2,\dots,ty^n) : \mathbb{R}^{m,n} \to \mathbb{R}^{m,n}$ for $t \neq 0$ we finish the proof. □

3. In this section all linear natural transformations $TT^{(r,s,q)} \to T^{(r,s,q)}$ over $\mathcal{FM}_{m,n}$ will be classified.

A *natural transformation* $TT^{(r,s,q)} \to T^{(r,s,q)}$ over $\mathcal{FM}_{m,n}$ is a system of fibered maps $B : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y$ covering the identity id_Y for every $\mathcal{FM}_{m,n}$-object Y satisfying $T^{(r,s,q)}f \circ B = B \circ TT^{(r,s,q)}f$ for every local $\mathcal{FM}_{m,n}$-diffeomorphism $f : Y \to \overline{Y}$. The linearity of $B : TT^{(r,s,q)} \to T^{(r,s,q)}$ means that the restriction and corestriction $B_\omega : T_\omega T^{(r,s,q)}Y \to T_yY$ of $B : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y$ is linear for any $\omega \in T^{(r,s,q)}_yY$, $y \in Y$ and $Y \in \mathrm{obj}(\mathcal{FM}_{m,n})$.

Example 2. Given an $\mathcal{FM}_{m,n}$-object Y let $B^{\langle 1\rangle}, B^{\langle 2\rangle} : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y$ be fibered maps over id_Y such that

$$\langle B^{\langle 1\rangle}(v), j^{r,s,q}_y\gamma\rangle = d_y\gamma^1(T\pi(v)),$$
$$\langle B^{\langle 2\rangle}(v), j^{r,s,q}_y\gamma\rangle = d_y\gamma^2(T\pi(v)),$$

$v \in (TT^{(r,s,q)})_y Y$, $y \in Y$, $\gamma = (\gamma_1, \gamma_2) : Y \to \mathbb{R}^{1,1}$ is fibered, $\gamma(y) = 0$, where $\pi : T^{(r,s,q)}Y \to Y$ is the bundle projection, $T\pi : TT^{(r,s,q)}Y \to TY$ is its tangent map and $d_y\gamma_1 : T_yY \to \mathbb{R}$ is the differential of γ_1 at y. Then $B^{\langle 1\rangle}, B^{\langle 2\rangle} : TT^{(r,s,q)} \to T^{(r,s,q)}$ are linear natural transformations over $\mathcal{FM}_{m,n}$.

Proposition 2. *Every linear natural transformation $B : TT^{(r,s,q)} \to T^{(r,s,q)}$ over $\mathcal{FM}_{m,n}$ is a linear combination of $B^{\langle 1\rangle}$ and $B^{\langle 2\rangle}$.*

Proof: We use the notations from the proof of Proposition 1. Let $(j_0^{r,s,q}(x^\alpha, 0))^*$, $(j_0^{r,s,q}(0, x^\beta y^\delta))^* \in T_0^{(r,s,q)}\mathbb{R}^{m,n}$ be the basis dual to the one of $J_0^{r,s,q}(\mathbb{R}^{m,n}, \mathbb{R}^{1,1})_0$. Let

$$\begin{aligned}
\mathrm{pr}_1 &: \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \to \mathbb{R}^m \times \mathbb{R}^n,\\
\mathrm{pr}_2 &: \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \to T_0^{(r,s,q)}\mathbb{R}^{m,n},\\
\mathrm{pr}_3 &: \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \to T_0^{(r,s,q)}\mathbb{R}^{m,n}
\end{aligned}$$

be the projections.

Similarly as in the proof of Proposition 1, B is uniquely determined by the contractions $\langle B(v), j_0^{r,s,q}(x^1, y^1)\rangle$ for all $v \in (TT^{(r,s,q)})_0\,\mathbb{R}^{m,n} \tilde{=} \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \times T_0^{(r,s,q)}\mathbb{R}^{m,n}$, where $\tilde{=}$ is the standard identification. So, it remains to deduce that

$$\langle B(\cdot), j_0^{r,s,q}(x^1, y^1)\rangle : \mathbb{R}^m \times \mathbb{R}^n \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \times T_0^{(r,s,q)}\mathbb{R}^{m,n} \to \mathbb{R}$$

is a linear combination of $x^1 \circ \mathrm{pr}_1$ and $y^1 \circ \mathrm{pr}_1$.

Using similar arguments as in the proof of Proposition 1 (the naturality of B with respect to a_t and b_t and the homogeneous function theorem), we deduce that $\langle B(\cdot), j_0^{r,s,q}(x^1, y^1)\rangle$ is a linear combination of $x^1 \circ \mathrm{pr}_1$, $y^1 \circ \mathrm{pr}_1$, $j_0^{r,s,q}(x^1, 0) \circ \mathrm{pr}_2$, $j_0^{r,s,q}(0, x^1) \circ \mathrm{pr}_2$, $j_0^{r,s,q}(0, y^1) \circ \mathrm{pr}_2$, $j_0^{r,s,q}(x^1, 0) \circ \mathrm{pr}_3$, $j_0^{r,s,q}(0, x^1) \circ \mathrm{pr}_3$ and $j_0^{r,s,q}(0, y^1) \circ \mathrm{pr}_3$. Since B is linear, $\langle B(\cdot), j_0^{r,s,q}(x^1, y^1)\rangle$ is a linear combination of $x^1 \circ \mathrm{pr}_1$, $y^1 \circ \mathrm{pr}_1$, $j_0^{r,s,q}(x^1, 0) \circ \mathrm{pr}_3$, $j_0^{r,s,q}(0, x^1) \circ \mathrm{pr}_3$ and $j_0^{r,s,q}(0, y^1) \circ \mathrm{pr}_3$. Replacing B by $B - \lambda_1 B^{\langle 1\rangle} - \lambda_2 B^{\langle 2\rangle}$ we can assume that $\langle B(\cdot), j_0^{r,s,q}(x^1, y^1)\rangle$ is a linear combination of $j_0^{r,s,q}(x^1, 0) \circ \mathrm{pr}_3$, $j_0^{r,s,q}(0, x^1) \circ \mathrm{pr}_3$ and $j_0^{r,s,q}(0, y^1) \circ \mathrm{pr}_3$. (Then $\langle B(\partial_1^C{}_{|\omega}), j_0^{r,s,q}(x^1, y^1)\rangle = 0$ and $\langle B(\overline{\partial}_1^C{}_{|\omega}), j_0^{r,s,q}(x^1, y^1)\rangle = 0$ for any $\omega \in T_0^{(r,s,q)}\mathbb{R}^{m,n}$, where $\partial_1 = \frac{\partial}{\partial x_1}$, $\overline{\partial}_1 = \frac{\partial}{\partial y_1}$ and $()^C$ is the flow lift of projectable vector fields to $T^{(r,s,q)}$.) It remains to show

$$\langle B(0, 0, \tilde{\omega}), j_0^{r,s,q}(x^1, y^1)\rangle = 0 \tag{1}$$

for $\tilde{\omega} \in \{(j_0^{r,s,q}(x^1, 0))^*, (j_0^{r,s,q}(0, x^1))^*, (j_0^{r,s,q}(0, y^1))^*\}$. We consider 3 cases.

(I) Assume $\tilde{\omega} = (j_0^{r,s,q}(x^1,0))^*$. For showing (1), we prove

$$
\begin{aligned}
0 &= \langle A((\partial_1 + (x^1)^q\partial_1)^C_{|\omega}), j_0^{r,s,q}(x^1,y^1)\rangle \\
&= \langle A(((x^1)^q\partial_1)^C_{|\omega}), j_0^{r,s,q}(x^1,y^1)\rangle \\
&= \langle A(0,\omega,\tilde{\omega}+\dots), j_0^{r,s,q}(x^1,y^1)\rangle \\
&= \langle A(0,0,\tilde{\omega}), j_0^{r,s,q}(x^1,y^1)\rangle,
\end{aligned}
\tag{2}
$$

where $\omega = (j_0^{r,s,q}((x^1)^q,0))^*$ and the dots is the linear combination of the elements $\overline{\omega}$ from the dual basis of $T_0^{(r,s,q)}\mathbb{R}^{m,n}$ with $\overline{\omega} \notin \{(j_0^{r,s,q}(x^1,0))^*, (j_0^{r,s,q}(0,x^1))^*, (j_0^{r,s,q}(0,y^1))^*\}$.

The second equality of (2) is clear as $\langle B(\partial_1^C{}_{|\omega}), j_0^{r,s,q}(x^1,y^1)\rangle = 0$ and A is an affinor. The fourth equality of (2) is clear as $\langle B(\cdot), j_0^{r,s,q}(x^1,y^1)\rangle$ is a linear combination of $j_0^{r,s,q}(x^1,0)\circ \mathrm{pr}_3$, $j_0^{r,s,q}(0,x^1)\circ \mathrm{pr}_3$ and $j_0^{r,s,q}(0,y^1)\circ \mathrm{pr}_3$.

We can prove the first equality of (2) as follows. We consider for a moment ∂_1 and $\partial_1 + (x^1)^q\partial_1$ as the vector fields on $\mathbb{R}$. They have the same $(q-1)$-jets at $0\in\mathbb{R}$. Then there exists a diffeomorphism $\psi:\mathbb{R}\to\mathbb{R}$ such that $j_0^q\psi = \mathrm{id}$ and $\psi_*\partial_1 = \partial_1 + (x^1)^q\partial_1$ near $0\in\mathbb{R}$, see Lemma 42.4 in [3] (or [12]). Let $\varphi = \psi\times \mathrm{id}_{\mathbb{R}^{m-1}}\times \mathrm{id}_{\mathbb{R}^n}$. Then $\varphi:\mathbb{R}^{m,n}\to\mathbb{R}^{m,n}$ is an $\mathcal{FM}_{m,n}$-morphism such that $j_0^{r,s,q}\varphi = \mathrm{id}$ and $\varphi_*\partial_1 = \partial_1+(x^1)^q\partial_1$ near 0. Clearly, φ preserves $j_0^{r,s,q}(x^1,y^1)$ because of the jet argument. Then, using the naturality of A with respect to φ, from $\langle B(\partial_1^C{}_{|\omega}), j_0^{r,s,q}(x^1,y^1)\rangle = 0$ for any $\omega\in T_0^{(r,s,q)}\mathbb{R}^{m,n}$ it follows the first equality for any $\omega\in T_0^{(r,s,q)}\mathbb{R}^{m,n}$.

It remains to show the third equality of (2). Let φ_t be the flow of $(x^1)^q\partial_1$. Then

$$
\begin{aligned}
\langle ((x^1)^q\partial_1)^C_{|\omega}, j_0^{r,s,q}(x^1,0)\rangle &= \langle \frac{d}{dt}_{|t=0} T^{(r,s,q)}(\varphi_t)(\omega), j_0^{r,s,q}(x^1,0)\rangle \\
&= \langle \omega, j_0^{r,s,q}(\frac{d}{dt}_{|t=0}(x^1,0)\circ\varphi_t)\rangle \\
&= \langle \omega, j_0^{r,s,q}((x^1)^q,0)\rangle \\
&= 1
\end{aligned}
$$

because of the definition of ω. Similarly $\langle ((x^1)^q\partial_1)^C_{|\omega}, j_0^{r,s,q}(0,x^1)\rangle = 0$ and $\langle ((x^1)^q\partial_1)^C_{|\omega}, j_0^{r,s,q}(0,y^1)\rangle = 0$. Then $((x^1)^q\partial_1)^C_{|\omega} = (j_0^{r,s,q}(x^1,0))^* + \dots$ under the isomorphism $V_\omega T^{(r,s,q)}\mathbb{R}^{m,n}\tilde{=}T_0^{(r,s,q)}\mathbb{R}^{m,n}$, where the dots stand for a linear combination of the elements $\overline{\omega}$ from the dual basis of $T_0^{(r,s,q)}\mathbb{R}^{m,n}$ with $\overline{\omega}\notin\{(j_0^{r,s,q}(x^1,0))^*, (j_0^{r,s,q}(0,x^1))^*, (j_0^{r,s,q}(0,y^1))^*\}$. It implies the third equality of (2).

(II) Assume $\tilde{\omega} = (j_0^{r,s,q}(0,x^1))^*$. For showing (1), we prove (2), where $\omega = (j_0^{r,s,q}(0,(x^1)^q))^*$ and the dots stand for a linear combination of the elements $\overline{\omega}$ from the dual basis of $T_0^{(r,s,q)}\mathbb{R}^{m,n}$ with $\overline{\omega} \notin \{(j_0^{r,s,q}(x^1,0))^*, (j_0^{r,s,q}(0,x^1))^*, (j_0^{r,s,q}(0,y^1))^*\}$.

The proof of the third equality of (2) is almost the same as in case (I) (we have $\langle ((x^1)^q\partial_1)^C_{|\omega}, j_0^{r,s,q}(x^1,0)\rangle = 0$, $\langle ((x^1)^q\partial_1)^C_{|\omega}, j_0^{r,s,q}(0,x^1)\rangle = 1$ and $\langle ((x^1)^q\partial_1)^C_{|\omega}, j_0^{r,s,q}(0,y^1)\rangle = 0$). The proofs of the other equalities of (2) are the same as in case (I).

(III) Assume $\tilde{\omega} = (j_0^{r,s,q}(0,y^1))^*$. For showing (1), it suffices to prove

$$
\begin{aligned}
0 &= \langle A((\overline{\partial}_1 + (y^1)^s\overline{\partial}_1)^C_{|\omega}), j_0^{r,s,q}(x^1,y^1)\rangle \\
&= \langle A(((y^1)^s\overline{\partial}_1)^C_{|\omega}), j_0^{r,s,q}(x^1,y^1)\rangle \\
&= \langle A(0,\omega,\tilde{\omega}+\dots), j_0^{r,s,q}(x^1,y^1)\rangle \\
&= \langle A(0,0,\tilde{\omega}), j_0^{r,s,q}(x^1,y^1)\rangle,
\end{aligned}
\tag{2}'
$$

where $\omega = (j_0^{r,s,q}(0,(y^1)^s))^*$ and the dots stand for a linear combination of the elements $\overline{\omega}$ from the dual basis of $T_0^{(r,s,q)}\mathbb{R}^{m,n}$ with $\overline{\omega} \notin \{(j_0^{r,s,q}(x^1,0))^*, (j_0^{r,s,q}(0,x^1))^*, (j_0^{r,s,q}(0,y^1))^*\}$. The proof of (2)' is similar to that of (2) in case (II). We leave the details to the reader. □

4. In this section we classify all natural transformation $TT^{(r,s,q)} \to T$ over $\mathcal{FM}_{m,n}$. (The definition is similar to the one from Section 2.)

Example 3. Given an $\mathcal{FM}_{m,n}$-object Y, let $T\pi : TT^{(r,s,q)}Y \to TY$ be as in Section 3. Then $T\pi : TT^{(r,s,q)} \to T$ is a linear natural transformation over $\mathcal{FM}_{m,n}$.

Proposition 3. *Every linear natural transformation $C : TT^{(r,s,q)} \to T$ over $\mathcal{FM}_{m,n}$ is a constant multiple of $T\pi$.*

Proof: Using C, we construct a linear natural transformation $\tilde{C} : TT^{(r,s,q)} \to T^{(r,s,q)}$ over $\mathcal{FM}_{m,n}$ as follows. For any $Y \in \mathrm{obj}(\mathcal{FM}_{m,n})$ we define a fibered map $\tilde{C} : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y$ over id_Y by

$$\langle \tilde{C}(v), j_y^{r,s,q}\gamma\rangle = d_y\gamma_1(C(v)),$$

$v \in (TT^{(r,s,q)})_yY$, $y \in Y$, $\gamma = (\gamma_1,\gamma_2) : Y \to \mathbb{R}^{1,1}$ is fibered, $\gamma(y) = 0$.

Now, by Proposition 2, there exist numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\langle \tilde{C}(v), j_y^{r,s,q}\gamma\rangle = \lambda_1 \cdot d_y\gamma_1(T\pi(v)) + \lambda_2 \cdot d_y\gamma_2(T\pi(v))$$

for any $v \in (TT^{(r,s,q)})_yY$, $y \in Y$, $Y \in \mathrm{obj}(\mathcal{FM}_{m,n})$ and any fibered map $\gamma = (\gamma_1, \gamma_2) : Y \to \mathbb{R}^{1,1}$ with $\gamma(y) = 0$. Then $\lambda_2 = 0$ and $C = \lambda_1 \cdot T\pi$. □

5. In this section we prove the main result of this paper.

Example 4. For every $\mathcal{FM}_{m,n}$-object Y let $\mathrm{Id} : TT^{(r,s,q)}Y \to TT^{(r,s,q)}Y$ be the identity map and let $\tilde{B}^{\langle 1\rangle}, \tilde{B}^{\langle 2\rangle} : TT^{(r,s,q)}Y \to TT^{(r,s,q)}Y$ be affinors on $T^{(r,s,q)}Y$ such that

$$\tilde{B}^{\langle 1\rangle}(v) = (\omega, B^{\langle 1\rangle}(v)) \in T^{(r,s,q)}Y \times_Y T^{(r,s,q)}Y \tilde{=} VT^{(r,s,q)}Y \subset TT^{(r,s,q)}Y,$$
$$\tilde{B}^{\langle 2\rangle}(v) = (\omega, B^{\langle 2\rangle}(v)) \in TT^{(r,s,q)}Y,\ v \in T_\omega T^{(r,s,q)}Y,\ \omega \in T^{(r,s,q)}Y,$$

where $B^{\langle 1\rangle}, B^{\langle 2\rangle} : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y$ are as in Section 3. Then $\mathrm{Id}, \tilde{B}^{\langle 1\rangle}$ and $\tilde{B}^{\langle 2\rangle}$ are natural affinors on $T^{(r,s,q)}_{|\mathcal{FM}_{m,n}}$.

Theorem 1. *Every natural affinor D on $T^{(r,s,q)}_{|\mathcal{FM}_{m,n}}$ is a linear combination of* Id, $\tilde{B}^{\langle 1\rangle}$ *and* $\tilde{B}^{\langle 2\rangle}$.

Proof: The family $T\pi \circ D : TT^{(r,s,q)}Y \to TY$ for $Y \in \mathrm{obj}(\mathcal{FM}_{m,n})$ is a linear natural transformation $TT^{(r,s,q)} \to T$ over $\mathcal{FM}_{m,n}$. Then, by Proposition 3, there exists the real number λ such that $T\pi \circ D = \lambda \cdot T\pi$. Then $D - \lambda \cdot \mathrm{Id} : TT^{(r,s,q)}Y \to VT^{(r,s,q)}Y$ for any $\mathcal{FM}_{m,n}$-object Y. Let $\mathrm{pr} : VT^{(r,s,q)}Y \tilde{=} T^{(r,s,q)}Y \times_Y T^{(r,s,q)}Y \to T^{(r,s,q)}Y$ be the projection onto second factor for any Y as above. Then the family $\mathrm{pr} \circ (D - \lambda \cdot \mathrm{Id}) : TT^{(r,s,q)}Y \to T^{(r,s,q)}Y$ for any Y as above is a linear natural transformation over $\mathcal{FM}_{m,n}$. Now, by Proposition 2, there exist the numbers $\mu_1, \mu_2 \in \mathbb{R}$ such that $\mathrm{pr} \circ (D - \lambda \cdot \mathrm{Id}) = \mu_1 \cdot B^{\langle 1\rangle} + \mu_2 \cdot B^{\langle 2\rangle}$. Then $D = \lambda \cdot \mathrm{Id} + \mu_1 \cdot \tilde{B}^{\langle 1\rangle} + \mu_2 \cdot \tilde{B}^{\langle 2\rangle}$. □

6. We have the following corollary of Theorem 1.

Corollary 1. *There is no natural generalized connection on $T^{(r,s,q)}_{|\mathcal{FM}_{m,n}}$.*

Proof: Suppose that Γ is such a connection. By Theorem 1, there are numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that $\Gamma = \lambda_1 \cdot \mathrm{Id} + \lambda_2 \cdot \tilde{B}^{\langle 1\rangle} + \lambda_3 \cdot \tilde{B}^{\langle 2\rangle}$. Let Y be an $\mathcal{FM}_{m,n}$-object. Since $\mathrm{im}(\Gamma) = VT^{(r,s,q)}Y$ and $\mathrm{im}(\tilde{B}^{\langle 1\rangle}) \subset VT^{(r,s,q)}Y$ and $\mathrm{im}(\tilde{B}^{\langle 2\rangle}) \subset VT^{(r,s,q)}Y$, we get $\lambda_1 = 0$. It is easy to see that $VT^{(r,s,q)}Y \subset \ker(\tilde{B}^{\langle 1\rangle})$ and $VT^{(r,s,q)}Y \subset \ker(\tilde{B}^{\langle 2\rangle})$. Then $\Gamma \circ \Gamma = 0 \neq \Gamma$, a contradiction. □

7. We can solve similar problems with $T^{(r,s)} = (J^{r,s}(.,\mathbb{R})_0)^* : \mathcal{FM} \to \mathcal{FM}$ instead of $T^{(r,s,q)}$ as follows.

(i) Let $Y \to M$ be a fibered manifold and Q be a manifold. Two maps $f, g : Y \to Q$ determine the same (r,s)-jet $j^{r,s}_y f = j^{r,s}_y g$ at $y \in Y_x$, $x \in M$, if $j^r_y f = j^r_y g$, and $j^s_y(f|Y_x) = j^s_y(g|Y_x)$. The space of all (r,s)-jets of Y into Q is denoted by $J^{r,s}(Y,Q)$, see [3, p. 126].

The space $T^{r,s*}Y = J^{r,s}(Y,\mathbb{R})_0$ has an induced structure of a vector bundle over Y. Every $\mathcal{FM}$-morphism $h : Z \to Y$, $h(z) = y$, induces a linear map $\lambda(h)y, z : T^{r,s*}_y Y \to T^{r,s*}_z Z$, $j^{r,s}_y f \to j^{r,s}_z(f \circ h)$. If we denote by $T^{(r,s)}Y$ the dual vector bundle of $T^{r,s*}Y$ and define $T^{(r,s)}h : T^{(r,s,q)}Z \to T^{(r,s)}Y$ by using the dual maps to $\lambda(h)_{y,z}$, we obtain a vector bundle functor $T^{(r,s)}$ on $\mathcal{FM}$.

(ii) The family $\mathrm{id} : T^{(r,s)}Y \to T^{(r,s)}Y$ for any $\mathcal{FM}_{m,n}$-object Y is a natural transformation $T^{(r,s)} \to T^{(r,s)}$ over $\mathcal{FM}_{m,n}$.

Proposition 1′. *Every natural transformation $A : T^{(r,s)} \to T^{(r,s)}$ over $\mathcal{FM}_{m,n}$ is a constant multiple of the identity natural transformation.*

Proof: The proof is quite similar to the proof of Proposition 1. □

(iii) For every $\mathcal{FM}_{m,n}$-object Y let $B^{\langle\rangle} : TT^{(r,s)}Y \to T^{(r,s)}Y$ be a fibered map over id_Y such that $\langle B^{\langle\rangle}(v), j^{r,s}_y\gamma\rangle = d_y\gamma(T\pi(v))$, $v \in (TT^{(r,s)})_y Y$, $y \in Y$, $\gamma : Y \to \mathbb{R}$, $\gamma(y) = 0$, where $\pi : T^{(r,s)}Y \to Y$ is the bundle projection and $T\pi : TT^{(r,s)}Y \to TY$ is its tangent map. Then $B^{\langle\rangle} : TT^{(r,s)} \to T^{(r,s)}$ is a linear natural transformation over $\mathcal{FM}_{m,n}$.

Proposition 2′. *Every linear natural transformation $B : TT^{(r,s)} \to T^{(r,s)}$ over $\mathcal{FM}_{m,n}$ is a constant multiple of $B^{\langle\rangle}$.*

Proof: The proof is quite similar to the proof of Proposition 2. □

(iv) Given an $\mathcal{FM}_{m,n}$-object Y let $T\pi : TT^{(r,s)}Y \to TY$ be as in (iii). Then $T\pi : TT^{(r,s)} \to T$ is a linear natural transformation over $\mathcal{FM}_{m,n}$.

Proposition 3′. *Every linear natural transformation $C : TT^{(r,s)} \to T$ over $\mathcal{FM}_{m,n}$ is a constant multiple of $T\pi$.*

Proof: The proof is quite similar to the proof of Proposition 3. □

(v) For every $\mathcal{FM}_{m,n}$-object Y, let $\mathrm{Id} : TT^{(r,s)}Y \to TT^{(r,s)}Y$ be the identity map and let $\tilde{B}^{\langle\rangle} : TT^{(r,s)}Y \to TT^{(r,s)}Y$ be an affinor on $T^{(r,s)}Y$ such that $\tilde{B}^{\langle\rangle}(v) = (\omega, B^{\langle\rangle}(v)) \in T^{(r,s)}Y \times_Y T^{(r,s)}Y \tilde{=} VT^{(r,s)}Y \subset TT^{(r,s)}Y$, $v \in T_\omega T^{(r,s)}Y$, $\omega \in T^{(r,s)}Y$, where $B^{\langle\rangle} : TT^{(r,s)}Y \to T^{(r,s)}Y$ is as in Proposition 1′. Then Id and $\tilde{B}^{\langle\rangle}$ are natural affinors on $T^{(r,s)}_{|\mathcal{FM}_{m,n}}$.

Theorem 1$'$. *Every natural affinor D on $T^{(r,s)}_{|\mathcal{FM}_{m,n}}$ is a linear combination of* Id *and $\tilde{B}^{\langle\rangle}$.*

Proof: The proof is quite similar to the proof of Theorem 1. □

(vi) We have the following corollary of Theorem 1$'$.

Corollary 1$'$. *There is no natural generalized connection on $T^{(r,s)}_{|\mathcal{FM}_{m,n}}$.*

References

[1] Doupovec M., Kolář I., *Natural affinors on time-dependent Weil bundles*, Arch. Math. Brno **27** (1991), 205–209.

[2] Gancarzewicz J., Kolář I., *Natural affinors on the extended r-th order tangent bundles*, Suppl. Rendiconti Circolo Mat. Palermo **30** (1993), 95–100.

[3] Kolář I., Michor P. W., Slovák J., *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993.

[4] Kolář I., Mikulski W.M., *Contact elements on fibered manifolds*, to appear in Czech Math. J.

[5] Kolář I., Modugno M., *Torsions of connections on some natural bundles*, Differential Geom. Appl. **2** (1992), 1–16.

[6] Kolář I., Vosmanská G., *Natural transformations of higher order tangent bundles and jet spaces*, Čas. pěst. mat. **114** (1989), 181–185.

[7] Kurek J., *Natural affinors on higher order cotangent bundles*, Arch. Math. Brno **28** (1992), 175–180.

[8] Mikulski W.M., *Natural affinors on r-jet prolongation of the tangent bundles*, Arch. Math. Brno **34 (2)** (1998), 321–328.

[9] Mikulski W.M., *Natural affinors on $(J^rT^*)^*$*, Arch. Math. Brno **36 (4)** (2000), 261–267.

[10] Mikulski W.M., *The natural affinors on $\otimes^k T^{(r)}$*, Note di Matematica **19 (2)** (1999), 269–274.

[11] Mikulski W.M., *The natural affinors on generalized higher order tangent bundles*, to appear.

[12] Zajtz A., *On the order of natural operators and liftings*, Ann. Polon. Math. **49** (1988), 169–178.

Institute of Mathematics, Jagiellonian University, Reymonta 4, Kraków, Poland
E-mail: mikulski@im.uj.edu.pl

(*Received* March 27, 2001)