On algebra homomorphisms in complex almost f-algebras

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Abstract. Extensions of order bounded linear operators on an Archimedean vector lattice to its relatively uniform completion are considered and are applied to show that the multiplication in an Archimedean lattice ordered algebra can be extended, in a unique way, to its relatively uniform completion. This is applied to show, among other things, that any order bounded algebra homomorphism on a complex Archimedean almost f-algebra is a lattice homomorphism.

Keywords: vector lattice, order bounded operator, lattice ordered algebra, f-algebra, almost f-algebra

Classification: 06F20, 06F25

1. Introduction

In this paper, we show that any order bounded linear operator from an Archimedean vector lattice A into a uniformly complete vector lattice B has a unique order bounded extension to the relatively uniform completion of A. As an application, we show that the multiplication in an Archimedean lattice ordered algebra A can be extended, in a unique way, into a lattice ordered algebra multiplication on $\bar{A}$, the relatively uniform completion of A, in such a manner that A becomes a sub algebra of $\bar{A}$. Moreover, $\bar{A}$ is an f-algebra (respectively almost f-algebra, d-algebra) whenever A is an f-algebra (respectively almost f-algebra, d-algebra).

In Section 4 we are mainly concerned with generalizing a theorem, due to E. Scheffold (cf. [8, Theorem 2.2]), which states that if A is a Banach almost f-algebra (FF-algebra in his terminology), then any order bounded multiplicative functional on A_C, the complexification of A, is a lattice homomorphism. We show in a different manner that in fact the latter result holds for an arbitrary order bounded algebra homomorphism between A_C and B_C respectively the complexifications of the Archimedean almost f-algebras A and B, provided B is semiprime. Also, a Nagasawa-like theorem is proved for complex f-algebras in order to characterize algebra homomorphisms.

2. Preliminaries

For unexplained terminology and the basic results on vector lattices and f-algebras we refer to [4], [7] and [9]. All vector lattices and lattice ordered algebras under consideration are supposed to be Archimedean and the only topology we consider on these spaces is the relatively uniform topology (cf. [4, Sections 16 and 63]). If $\hat{A}$ is the Dedekind completion of the Archimedean vector lattice A, then $\bar{A}$, the closure of A in $\hat{A}$ with respect to the relatively uniform topology, is a relatively uniformly complete vector lattice which is, with respect to ([6, Definition 2.12]), the relatively uniform completion of A.

The real algebra A is called a lattice ordered algebra if A is a vector lattice with the following property

(i) $ab \in A^+$ for all $a, b \in A^+$.

The lattice ordered algebra A is called an *almost f-algebra* if A verifies the following property:

(ii) $ab = 0$ for all $a, b \in A$ such that $a \wedge b = 0$.

The lattice ordered algebra A is called an *f-algebra* if A verifies the following property:

(iii) $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A^+$.

The lattice ordered algebra A is called a *d-algebra* if A verifies the following property:

(iv) $a \wedge b = 0$ implies that $c(a \wedge b) = ca \wedge cb$ and $(a \wedge b)c = ac \wedge bc$ for all $c \in A^+$.

The lattice ordered algebra A is said to be *semiprime* if 0 is the only nilpotent element of A.

The linear mapping T defined on the vector lattice A with values in the vector lattice B is called *order bounded* (notation $T \in L_b(A, B)$ or $T \in L_b(A)$ if $A = B$) if T maps order intervals into order intervals.

The mapping $T \in L_b(A)$ is said to be an *orthomorphism* if it follows from $|a| \wedge |b| = 0$ that $|T(a)| \wedge |b| = 0$. The collection $\mathrm{Orth}(A)$ of all orthomorphisms on A is, with respect to the usual vector spaces operations and composition as multiplication, an Archimedean f-algebra with the identity mapping I_A on A as a unit element. If A is a semiprime Archimedean f-algebra, then the mapping $\rho : A \longrightarrow \mathrm{Orth}(A)$, which assigns to $a \in A$ the operator T_a defined on A by $T_a(b) = ab$ for all $b \in A$, is an injective f-algebra homomorphism.

Throughout this paper, a semiprime Archimedean f-algebra will be identified with $\rho(A)$ in $\mathrm{Orth}(A)$. If A is in addition relatively uniformly complete then A verifies the so-called "Stone condition"; i.e., $a \wedge I \in A$ holds for all $a \in A^+$. We shall denote by A_b the subalgebra of all bounded elements in A, i.e.,

$$A_b = \{a \in A : |a| \leq \lambda I_A,\ \lambda \in \mathbf{R}^+\}.$$

Let A be an Archimedean semiprime f-algebra, as agreed upon, we consider A as a sub algebra of $\mathrm{Orth}(A)$ and we denote by $A_C = A + iA$ the complexification of A. If, in addition, A is relatively uniformly complete then A_C and $\mathrm{Orth}(A)_C$ are complex f-algebras (cf. [1, Section 5]).

Note, in this connection, that even if A is not relatively uniformly complete, A_C can be regarded as a sub algebra of the complex f-algebra $\mathrm{Orth}(\bar{A})_C$ and hence $|a|$ exists in $\mathrm{Orth}(\overline{A})_C$ for all $a \in A_C$.

The linear mapping $T = T_1 + iT_2 : A_C \longrightarrow B_C$ is called *order bounded* if $T_1, T_2 \in L_b(A, B)$. T is said to be *contractive* if $|T(a)|^2 \leq |T(a)|$ whenever $|a|^2 \leq |a|$ or equivalently if $|T(a)| \leq I_B$ whenever $|a| \leq I_A$ (the absolute value is taken, if necessarily in $\mathrm{Orth}(\overline{A})_C$ or in $\mathrm{Orth}(\bar{B})_C$). A bijective operator T is said to be *bicontractive* if T and T^{-1} are contractive.

3. An extension theorem

Let A and B be Archimedean vector lattices and assume, in addition, that B is relatively uniformly complete and let $T \in L_b(A, B)$. In this section we show that T has a unique extension into an element $T' \in L_b(\bar{A}, B)$, where $\bar{A}$ is the relatively uniform completion of A. Moreover, we use this result to prove that the relatively uniform completion of an Archimedean lattice ordered algebra is again a lattice ordered algebra.

To a given $0 \leq T \in L_b(A, B)$ we associate the mappings U and V defined on $\bar{A}$ (the relatively uniform completion of A) with values in $\widehat{B}$, the Dedekind completion of B, as follows:

$U(x) = \sup\{T(a) : a \in A,\ a \leq x \text{ for all } x \in \bar{A}\}$;

$V(x) = \inf\{T(a) : a \in A,\ x \leq a \text{ for all } x \in \bar{A}\}$.

In the following lemma we collect some properties of U and V, the easy proof of which we leave to the reader.

Lemma 3.1. (1) *U and V are increasing mappings and $U(x) \leq V(x)$ for all $x \in \bar{A}$.*

(2) *$U(a) = V(a) = T(a)$ for all $a \in A$.*

(3) *$U(x + y) \geq U(x) + U(y)$ and $V(x + y) \leq V(x) + V(y)$ for all $x, y \in \bar{A}$.*

(4) *$U(\lambda x) = \lambda U(x)$ for all $0 \leq \lambda \in \mathbf{R}$ and $U(\lambda x) = \lambda V(x)$ for all $\lambda < 0$.*

Next we show that $U = V$ and that U is a positive extension of T to $\bar{A}$.

Lemma 3.2. *$U \in L_b(\bar{A}, \widehat{B})$ and it is the (unique) positive extension of T to $\bar{A}$.*

Proof: Since A is relatively uniformly dense in $\bar{A}$ and since $U(a) = V(a)$ for all $a \in A$, in order to show that $U(x) = V(x)$ for all $x \in \bar{A}$, it is therefore sufficient to prove that U and V are relatively uniformly continuous on $\bar{A}$. To this end, let $\{x_n : n = 1, 2, \dots\}$ be a relatively uniformly convergent sequence in $\bar{A}$ with limit x. Then, there is $v \in A^+$ such that, for every $\varepsilon > 0$ there exists a natural number N_ε such that $|x_n - x| \leq \varepsilon v$ for all $n \geq N_\varepsilon$, i.e., $x - \varepsilon v \leq x_n \leq x + \varepsilon v$. We have to show that $U(x_n)$ and $V(x_n)$ converge respectively to $U(x)$ and $V(x)$. $U(x - \varepsilon v) = \sup\{T(a) : a \in A,\ a \leq x - \varepsilon v\}$. If we put $b = a + \varepsilon v$ then we have

$b \in A$ and $b \leq x$. It follows that

$$\{T(a) : a \in A, a \leq x - \varepsilon v\} = \{T(b - \varepsilon v) : b \in A,\ b \leq x\}.$$

Hence,

$$\begin{aligned} U(x - \varepsilon v) &= \sup\{T(b) - \varepsilon T(v) : b \in A, b \leq x\} \\ &= \sup\{T(b) : b \in A, b \leq x\} - \varepsilon T(v) \\ &= U(x) - eT(v). \end{aligned}$$

Analogously we have that $U(x + \varepsilon v) = U(x) + \varepsilon T(v)$.

It follows from Lemma 3.1(1) that

$$U(x) - \varepsilon T(u) \leq U(x_n) \leq U(x) + \varepsilon T(v).$$

In other words, $U(x_n)$ converges relatively uniformly to $U(x)$ and hence U is relatively uniformly continuous. Similarly, V is relatively uniformly continuous on $\bar{A}$ and it follows that $U(x) = V(x)$ for all $x \in \bar{A}$. From Lemma 3.1(2), (3) and (4) we deduce easily that U is a positive linear mapping in $L_b(\bar{A}, \widehat{B})$ which extends T to $\bar{A}$. Finally it is not difficult to see that U is the unique positive extension of T to $\bar{A}$. This completes the proof. □

Now, let T be an element in $L_b(A, B)$. T can be regarded as an element of $L_b(A, \widehat{B})$. Hence, there exist two positive elements T_1 and T_2 in $L_b(A, \widehat{B})$ such that $T = T_1 - T_2$. By the preceding lemma, T_1 and T_2 have unique positive extensions T_1' and T_2' to $\bar{A}$ the relatively uniform completion of A. Hence, $T' = T_1' - T_2' \in L_b(\bar{A}, \widehat{B})$ is the unique order bounded extension of T to $\bar{A}$. Moreover, the relatively uniform continuity of T implies that the range of T' is contained in B since B is relatively uniformly closed in $\widehat{B}$.

Summarizing, we have the following theorem:

Theorem 3.3. *Let A and B be two Archimedean vector lattices. Assume in addition that B is relatively uniformly complete. Then any order bounded linear operator $T : A \longrightarrow B$ has a unique order bounded linear extension $T' : \bar{A} \longrightarrow B$, where $\bar{A}$ is the relatively uniform completion of A.*

Let A be an Archimedean lattice ordered algebra and let $a \in A^+$. If we define $T_a : A \longrightarrow \bar{A}$ by $T_a(y) = ay$, then $T_a \in L_b(A, \bar{A})$ and hence, by the preceding theorem, T_a extends to $T_a' \in L_b(\bar{A})$. For a fixed $y \in (\bar{A})^+$ we put $ay = T_a(y)$. Now, for a fixed $y \in (\bar{A})^+$ we define $R_y : A \longrightarrow \bar{A}$ by $R_y(a) = T_a'(y)$. Then $R_y \in L_b(A, \bar{A})$ and hence, R_y extends into $R_y' \in L_b(\bar{A})$.

If for arbitrary $x, y \in (\bar{A})^+$ we define on $\bar{A}$ a multiplication by putting $xy = R_y'(x)$, then it is an easy task to verify that this multiplication is the unique lattice ordered algebra multiplication in $\bar{A}$ which extends the multiplication in A in such a manner that A becomes a subalgebra of $\bar{A}$ with respect to this multiplication. Moreover, it is easy to verify that such a multiplication is an f-algebra (respectively, almost f-algebra, d-algebra) multiplication whenever A is an f-algebra (respectively, almost f-algebra, d-algebra).

Thus we obtain the following theorem:

Theorem 3.4. *Let A be an Archimedean lattice ordered algebra. Then the multiplication in A can be extended in a unique way into a lattice ordered algebra multiplication on $\bar{A}$ in such a manner that A becomes a sub algebra of $\bar{A}$. Moreover, $\bar{A}$ is an f-algebra (respectively almost f-algebra, d-algebra) whenever A is an f-algebra (respectively almost f-algebra, d-algebra).*

4. The main results

Let A and B be Archimedean semiprime f-algebras with unit elements e_A and e_B. The linear operator $T : A_C \longrightarrow B_C$ is contractive if $|T(a)| \leq e_B$ whenever $|a| \leq e_A$. In order to make things surveyable, we first show that if $T \in L_b(A_C, B_C)$ is contractive and verifies $T(e_A) = e_B$, then T is positive (i.e., $T(a) \in B^+$ for all $a \in A^+$). To this end, we need the following inequalities:

$(*)$ If $T \in L_b(A_C, B_C)$ and $u \in A^+$ then

$$|T(u)| \leq |T|(u) \leq \sup\{|T(a)| : a \in A_C; |a| \leq u\}$$

(cf. [9, Lemma 92.5 and Theorem 92.6]).

Proposition 4.1. *If $T \in L_b(A_C, B_C)$ is contractive and verifies $T(e_A) = e_B$, then T is positive.*

Proof: Note first that there exist $T_1, T_2 \in L_b(A, B)$ such that $T = T_1 + iT_2$. So, $T_1(e_A) = e_B$ since $T(e_A) = e_B$. Consider, now, T as an element of $L_b(A_C, \widehat{B}_C)$. Then, the contractivity of T together with the inequalities $(*)$ imply that

$$|T|(e_A) = e_B = T_1(e_A).$$

It follows from the fact that $T_1 \leq |T|$, that

$$0 \leq (|T| - T_1)(a) \leq (|T| - T_1)(e_A) = 0 \text{ for all } a \in A^+ \text{ verifying } a \leq e_A.$$

Consequently, the restriction of $|T| - T_1$ to A_b is the null operator on A_b. The relatively uniform continuity of $|T| - T_1$ together with the relatively uniform density of A_b in A imply now that $|T| - T_1$ is the null operator on A, i.e., $|T| = T_1$. Finally, the identity $|T(a)|^2 = T_1(a)^2 + T_2(a)^2$ for all $a \in A^+$ ([1, Theorem 5.2]) already implies that $T_2 = 0$ and hence $|T| = T$, which is the desired result. □

Before proving our main theorem we need the following lemma

Lemma 4.2. *If* $(a+\alpha I_A) \in A_C^{\#}$ *verifies* $|a+\alpha I_A| \le I_A$ *then* $|\alpha| \le 1$.

PROOF: It follows from the inequalities $|a| - |\alpha| I_A \le |a+\alpha I_A| \le I_A$ and $|\alpha| I_A - |a| \le |a+\alpha I_A| \le I_A$ that $(|\alpha|-1)I_A \le |a| \le (1+|\alpha|)I_A$. As a consequence of the Stone condition we have that $|\alpha| \le 1$. Indeed, $|\alpha| > 1$ would lead to $I_A < (|\alpha|-1)^{-1}|a|$ and hence $I_A = (|\alpha|-1)^{-1}|a| \wedge I_A \in A$, contradictory to our assumption that $I_A \notin A$. Hence, we necessarily have $|\alpha| \le 1$. ☐

We are now in a position to prove the main result of this paper which extends Theorem 5.3 of [3], proved in the real case under the additional assumption that A has a unit element.

Theorem 4.3. *Let* A *and* B *be Archimedean semiprime* f*-algebras. If* $T : A_C \longrightarrow B_C$ *is an order bounded algebra homomorphism then* T *is a lattice homomorphism* (*i.e.,* T *is real and the restriction of* T *to* A *is a lattice homomorphism*).

PROOF: First, we show that T is contractive. To this end, let $a \in A_C$ be such that $|a| \le I_A$. This implies that $0 \le |a^n| \le |a|$ for $n = 1, 2, \ldots$, so we have that $|a^n| \in [0, |a|]$ and it follows from the order boundedness of T that there exists $b \in B^+$ such that

$$|T(a^n)| = |Ta|^n \le b \quad \text{for all } n.$$

Now, if we put

$$|T(a)| \vee I_B = I_B + h$$

then h is positive and we have

$$0 \le |T(a^n)| \vee I_B = (|T(a)| \vee I_B)^n = (I_B + h)^n \le b \vee I_B \quad \text{for all } n,$$

from this we deduce that

$$0 \le I_B + nh \le b \vee I_B \quad \text{for all } n.$$

Multiplying these inequalities by $\frac{1}{n}$ we find

$$0 \le \frac{1}{n} I_B + h \le \frac{1}{n}(b \vee I_B) \text{ for all } n$$

and so

$$0 \le h \le \frac{1}{n}(b \vee I_B - I_B).$$

Therefore, it follows from the Archimedean assumption that $h = 0$. Consequently $|T(a)| \le I_B$ and hence T is contractive.

Now we consider two cases.

First case: A has a unit element e_A.

In this case $T(e_A)$ is an idempotent in B. Moreover, it follows from the fact that

$$T(a) = T(ae_A) = T(a)T(e_A) \quad \text{for all} \quad a \in A$$

that $T(a)$ is an element of the band F generated by $T(e_A)$ in B. F is an f-algebra with unit element $T(e_A)$, so if we consider T as an algebra homomorphism from A_C into B_C it follows from Proposition 4.1 that T is positive and hence T is a lattice homomorphism (see e.g. [3, Section 5]).

Second case: A does not possess a unit element.

Observe first that if $\bar{A}$ is the relatively uniform completion of A then it follows from Theorem 3.3 that T has a unique order bounded linear extension T' to $\bar{A}_C$ with values in $\bar{B}_C$. Moreover, it takes a little effort to verify that T' is an algebra homomorphism. So, we shall assume without loss of generality that A and B are relatively uniformly complete. Assume now that A does not possess a unit element and let A_b be the f-algebra of all bounded elements in A. Since T is contractive, the restriction of T to $(A_b)_C$ (which we shall denote again by T) is a contractive algebra homomorphism from $(A_b)_C$ into B_C. Consider now $T^{\#}$ the extension of T to $(A_b^{\#})_C$, then, an easy computation shows that $T^{\#}$ is an algebra homomorphism. Moreover, if $|a + \alpha I_A| \leq I_A$ then it follows from Lemma 4.2 that $|a| \leq 2I_A$ and hence that

$$|T^{\#}(a + \alpha I_A)| \leq |T(a)| + |\alpha| I_B \leq 3I_B.$$

Consequently, $T^{\#}$ is an order bounded algebra homomorphism. Again, $T^{\#}$ is contractive and since $T^{\#}(I_A) = I_B$, it follows from Proposition 4.1 that $T^{\#}$ is positive and, hence, the restriction of T to A_b is positive. Now the relatively uniform continuity of T together with the relatively uniform density of A_b in A imply that T is positive on A and, hence, that T is a lattice homomorphism, which is the desired result. $\square$

Remark 4.1. Example 5.2 of [3] shows that the condition that T is order bounded cannot be dropped in Theorem 4.3.

Next, we give a characterization of algebra isomorphisms in terms of contractive operators. This result is, in some respect, an f-algebra version of the well known Nagasawa's result for Banach algebras ([5, Theorem 1]).

Theorem 4.4. *Let A and B be Archimedean f-algebras with unit elements e_A and e_B and let $T : A_C \to B_C$ be an order bounded bijection such that $T(e_A) = e_B$. The following properties are equivalent:*

(i) *T is an algebra isomorphism;*
(ii) *T is a lattice isomorphism;*
(iii) *T is bicontractive.*

Proof: (i)⇔(ii). Follows from Theorem 4.3 and ([3]; Theorem 5.4).

(ii)⇒(iii). Obvious.

(iii)⇒(ii). Since T is contractive and $T(e_A) = e_B$, it follows from Proposition 4.1 that T is positive. T', the restriction of T to A_b, is a bijection onto B_b, so, T'^{-1} is an order bounded and contractive operator on B_b such that $T'^{-1}(e_B) = e_A$. It follows again from Proposition 4.1 that T'^{-1} is positive and, hence, T' is a lattice homomorphism. Now, let a and b be elements in A such that $a \wedge b = 0$ and define the sequences a_n and b_n by putting:

$$a_n = a \wedge ne_A \quad \text{and} \quad b_n = b \wedge ne_A \quad \text{for} \quad n = 1, 2, \ldots\ .$$

a_n and b_n are elements of A_b and converge relatively uniformly respectively to a and b. Moreover, we have that $a_n \wedge b_n = 0$, so,

$$T(a_n \wedge b_n) = T'(a_n \wedge b_n) = 0.$$

The relative uniform continuity of T now implies that $T(a \wedge b) = 0$. This shows that T is a lattice isomorphism and we are done. □

Assume now that A and B are Archimedean almost f-algebras and that B is semiprime. Let $T : A_C \longrightarrow B_C$ be an order bounded algebra homomorphism. $N(A)$, the set of all nilpotent elements in A, is a relatively uniformly closed algebra ideal and order ideal in A. It follows that $A/N(A)$ is a semiprime Archimedean f-algebra.

Let $s : A_C \longrightarrow (A/N(A))_C$ be the canonical surjection. Then s is a lattice and algebra homomorphism.

The linear operator $T' : (A/N(A))_C \longrightarrow B_C$ defined by $T'(s(a)) = T(a)$ for all $a \in A_C$ is an order bounded algebra homomorphism from $(A/N(A))_C$ into B_C.

So, by Theorem 4.3, T' is a lattice homomorphism and hence $T = T' \circ s$ is a lattice homomorphism since s is a lattice homomorphism.

This leads to the above mentioned generalization of Scheffold's result (cf. [8, Theorem 2.2]) which we state as a theorem.

Theorem 4.5. *Let A and B be Archimedean almost f-algebras. Assume, in addition, that B is semiprime. If $T : A_C \longrightarrow B_C$ is an order bounded algebra homomorphism then T is a lattice homomorphism.*

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(*Received* February 8, 2001)