

## A new rank formula for idempotent matrices with applications

YONGGE TIAN, GEORGE P.H. STYAN

*Abstract.* It is shown that

$$\text{rank}(P^*AQ) = \text{rank}(P^*A) + \text{rank}(AQ) - \text{rank}(A),$$

where  $A$  is idempotent,  $[P, Q]$  has full row rank and  $P^*Q = 0$ . Some applications of the rank formula to generalized inverses of matrices are also presented.

*Keywords:* Drazin inverse, group inverse, idempotent matrix, inner inverse, rank, tripotent matrix

*Classification:* 15A03, 15A09

A complex square matrix  $A$  is said to be idempotent, or a projector, whenever  $A^2 = A$ ; when  $A$  is idempotent, and Hermitian (or real symmetric), it is often called an orthogonal projector, otherwise an oblique projector. Projectors are closely linked to generalized inverses of matrices. For example, for a given matrix  $A$  the product  $P_A = AA^+$  is well known as the orthogonal projector on the range (column space) of  $A$ , where  $A^+$  is the Moore-Penrose inverse of  $A$ ; which is the unique solution of the following four Penrose equations

$$(i) AA^+A = A, \quad (ii) A^+AA^+ = A^+, \quad (iii) (AA^+)^* = AA^+, \quad (iv) (A^+A)^* = A^+A.$$

In addition, the products  $AA^\#, AA^D$  and  $AA^-$  are also idempotent matrices, where  $A^\#, A^D$  and  $A^-$  are the group inverse, the Drazin inverse, and an inner inverse of  $A$ , respectively. In a recent paper by Drury, Liu, Lu, Puntanen and Styan [1], a rank formula for the orthogonal projector  $P_A$  is established as follows

$$(1) \quad \text{rank}(P^*AA^+Q) = \text{rank}(AP) + \text{rank}(AQ) - \text{rank}(A),$$

where  $A \in \mathbb{C}^{n \times n}$  is Hermitian nonnegative definite,  $P \in \mathbb{C}^{n \times p}$  and  $Q \in \mathbb{C}^{n \times q}$  such that  $[P, Q]$  has full row rank and  $P^*Q = 0$ . This formula can help to establish several useful rank equalities for block matrices and orthogonal projectors when  $X$  and  $Y$  are properly chosen, see Drury *et al.* [1] and Tian [2]. This work motivates us to consider in general the rank of  $P^*AQ$  and various related topics, where  $A$  is idempotent,  $[P, Q]$  has full row rank and  $P^*Q = 0$ . To do so, we need the following result.

**Lemma 1.** *Let  $A \in \mathbb{C}^{m \times n}$ ,  $B \in \mathbb{C}^{n \times k}$  and  $C \in \mathbb{C}^{k \times l}$  be given. Then*

$$(2) \quad \text{rank}(ABC) = \text{rank}(AB) + \text{rank}(BC) - \text{rank}(B)$$

*holds if and only if there are matrices  $X$  and  $Y$  such that  $BCX + YAB = B$ .*

In fact it is well known that the equation  $AX + YB = C$  is consistent if and only if

$$\text{rank} \begin{bmatrix} C & A \\ B & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

Applying this result to the equation  $BCX + YAB = B$ , we obtain Lemma 1.

Our main results are given below.

**Theorem 2.** *Let  $A \in \mathbb{C}^{m \times m}$  be an idempotent matrix, and let  $P \in \mathbb{C}^{m \times p}$  and  $Q \in \mathbb{C}^{m \times q}$  be any two matrices such that  $[P, Q]$  has full row rank and  $P^*Q = 0$ . Then*

$$(3) \quad \text{rank}(P^*AQ) = \text{rank}(P^*A) + \text{rank}(AQ) - \text{rank}(A).$$

PROOF: Since  $[P, Q]$  has full row rank and  $P^*Q = 0$ , it follows that

$$[P, Q]^+ = \begin{bmatrix} P^+ \\ Q^+ \end{bmatrix} \quad \text{and} \quad [P, Q][P, Q]^+ = PP^+ + QQ^+ = I_m.$$

Let  $X = Q^+A$  and  $Y = A(P^+)^*$ . Then we have

$$AQX + YP^*A = AQQ^+A + A(P^+)^*P^*A = A(I_m - PP^+)A + APP^+A = A.$$

Applying Lemma 1 to this equality yields (3). □

Now let  $P = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$  and  $Q = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$ . Then  $[P, Q]$  is of full row rank and  $P^*Q = 0$ . We derive from (3) the following result.

**Corollary 3.** *Suppose that*

$$(4) \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m \times m}, \quad A_{12} \in \mathbb{C}^{m \times k}, \quad A_{21} \in \mathbb{C}^{k \times m}, \quad A_{22} \in \mathbb{C}^{k \times k}$$

*is an idempotent matrix. Then the rank of  $A$  satisfies the following two rank equalities*

$$(5) \quad \text{rank}(A) = \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} + \text{rank}[A_{11}, A_{12}] - \text{rank}(A_{12}),$$

and

$$(6) \quad \text{rank}(A) = \text{rank} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} + \text{rank}[A_{21}, A_{22}] - \text{rank}(A_{21}).$$

Moreover, if

$$(7) \quad A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad A_{ij} \in \mathbb{C}^{t_i \times t_j}, \quad 1 \leq i, j \leq p$$

is idempotent, then the rank of  $A$  satisfies the rank equalities

$$(8) \quad \text{rank}(A) = \text{rank}(Q_{1i}) + \text{rank}(Q_{i+1,p}) - \text{rank}(Q_{i+1,i}), \quad i = 1, 2, \dots, p-1,$$

where

$$Q_{ij} = \begin{bmatrix} A_{i1} & \cdots & A_{ij} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pj} \end{bmatrix}, \quad 1 \leq i, j \leq p.$$

The rank formulas in (8) are derived from (6). If the matrix  $A$  in (4) is an orthogonal projector, then (5) becomes

$$\text{rank}(A) = \text{rank}(A_{11}) + \text{rank}(A_{22}) - \text{rank}(A_{12}).$$

If we replace the idempotent matrix  $A$  in (5) by the idempotent matrix  $I_{m+k} - A$ , and note that  $\text{rank}(I_{m+k} - A) = m + k - \text{rank}(A)$ , then (5) becomes

$$\text{rank}(A) = m + k + \text{rank}(A_{12}) - \text{rank}[I_m - A_{11}, A_{12}] - \text{rank} \begin{bmatrix} A_{12} \\ I_k - A_{22} \end{bmatrix}.$$

The Drazin inverse  $A^D$  of a square matrix  $A$  with  $\text{index}(A) = k$  is defined to be the unique solution of the three matrix equations

$$(i) \quad A^k X A = A^k, \quad (ii) \quad X A X = X, \quad (iii) \quad A X = X A.$$

When  $\text{index}(A) = 1$ , i.e.,  $\text{rank}(A^2) = \text{rank}(A)$ ,  $A^D$  is called the group inverse of  $A$  and denoted by  $A^\#$ . From  $A^D A A^D = A^D$  we see that  $A A^D A A^D = A A^D$ . Thus  $A A^D$  is idempotent. In addition,  $\text{rank}(A^D) = \text{rank}(A A^D) = \text{rank}(A^k)$ . In this case, applying Theorem 2 to  $P^* A A^D Q$  and  $P^* A A^\# Q$ , we get the following corollary.

**Corollary 4.** *Let  $A \in \mathbb{C}^{m \times m}$  be given with  $\text{index}(A) = k$ , let  $P \in \mathbb{C}^{m \times p}$  and  $Q \in \mathbb{C}^{m \times q}$  be any two matrices such that  $[P, Q]$  has full row rank and  $P^*Q = 0$ . Then*

$$(9) \quad \text{rank}(P^*AA^DQ) = \text{rank}(P^*A^k) + \text{rank}(A^kQ) - \text{rank}(A^k).$$

*In particular,*

$$(10) \quad \text{rank}(P^*AA^\#Q) = \text{rank}(P^*A) + \text{rank}(AQ) - \text{rank}(A).$$

Let  $P = \begin{bmatrix} I_m \\ 0 \end{bmatrix}$  and  $Q = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$  in (10). We also have the following corollary.

**Corollary 5.** *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m \times m}, \quad A_{12} \in \mathbb{C}^{m \times k}, \quad A_{21} \in \mathbb{C}^{k \times m}, \quad A_{22} \in \mathbb{C}^{k \times k}$$

*with  $\text{index}(A) = 1$ , and denote by  $(AA^\#)_{12}$  the upper-right  $m \times k$  block of the projector  $AA^\#$ . Then the rank of  $(AA^\#)_{12}$  is*

$$(11) \quad \text{rank}[(AA^\#)_{12}] = \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} + \text{rank}[A_{11}, A_{12}] - \text{rk}(A).$$

A square matrix  $A$  is called tripotent if  $A^3 = A$ . For the tripotent matrix  $A$ , its group inverse is  $A^\# = A$ . Now applying (9) to a tripotent matrix  $A$  and noting that

$$(AA^\#)_{12} = (A^2)_{12} = [A_{11}, A_{12}] \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix},$$

we obtain the following result.

**Corollary 6.** *Suppose that*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{m \times m}, \quad A_{12} \in \mathbb{C}^{m \times k}, \quad A_{21} \in \mathbb{C}^{k \times m}, \quad A_{22} \in \mathbb{C}^{k \times k}$$

*is a tripotent matrix. Then the rank of  $A$  satisfies the following two rank equalities*

$$(12) \quad \text{rank}(A) = \text{rank}[A_{11}, A_{12}] + \text{rank} \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} - \text{rank} \left( [A_{11}, A_{12}] \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} \right),$$

*and*

$$(13) \quad \text{rank}(A) = \text{rank}[A_{21}, A_{22}] + \text{rank} \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} - \text{rank} \left( [A_{21}, A_{22}] \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \right).$$

Finally, we present a result for a triangular inner inverse of an idempotent matrix. We will use the following result due to Tian [3, Corollary 4.3].

**Lemma 7.** *The block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad \text{where } A_{ij} \in \mathbb{C}^{s_i \times t_j}, \quad 1 \leq i, j \leq p,$$

*has an inner inverse with the upper triangular block form*

$$A^- = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ & S_{22} & \cdots & S_{2p} \\ & & \ddots & \vdots \\ & & & S_{pp} \end{bmatrix}, \quad S_{ij} \in \mathbb{C}^{t_i \times s_j}, \quad 1 \leq i, j \leq p$$

*if and only if*

$$\text{rank}(A) = \text{rank}(Q_{1i}) + \text{rank}(Q_{i+1,p}) - \text{rank}(Q_{i+1,i}), \quad i = 1, 2, \dots, p-1,$$

*where*

$$Q_{ij} = \begin{bmatrix} A_{i1} & \cdots & A_{ij} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pj} \end{bmatrix}, \quad 1 \leq i, j \leq p.$$

Applying this lemma to the idempotent block matrix  $A$  in (7) under (8), we immediately see that

**Theorem 8.** *If the block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}, \quad \text{where } A_{ij} \in \mathbb{C}^{t_i \times t_j}, \quad 1 \leq i, j \leq p,$$

*is idempotent, then it must have an inner inverse with the upper triangular block form*

$$A^- = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1p} \\ & S_{22} & \cdots & S_{2p} \\ & & \ddots & \vdots \\ & & & S_{pp} \end{bmatrix}, \quad S_{ij} \in \mathbb{C}^{t_i \times t_j}, \quad 1 \leq i, j \leq p.$$

*In particular, if*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} \in \mathbb{C}^{m \times m}, A_{12} \in \mathbb{C}^{m \times k}, A_{21} \in \mathbb{C}^{k \times m}, A_{22} \in \mathbb{C}^{k \times k},$$

is idempotent, then it must have an inner inverse with the upper triangular block form

$$A^- = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}, \quad G_{11} \in \mathbb{C}^{m \times m}, \quad G_{12} \in \mathbb{C}^{m \times k}, \quad G_{22} \in \mathbb{C}^{k \times k}.$$

For more rank equalities for idempotent matrices, see the authors' recent paper [4].

**Acknowledgment.** The research of both authors was supported in part by the Natural Sciences and Engineering Research Council of Canada.

#### REFERENCES

- [1] Drury S.W., Liu S., Lu C.Y., Puntanen S., Styan G.P.H., *Some comments on several matrix inequalities with applications to canonical correlations: historical background and recent developments*, Report A332 (December 2000), Dept. of Mathematics, Statistics and Philosophy, University of Tampere, Tampere, Finland, 63 pp. To be published in the special issue of *Sankhyā: The Indian Journal of Statistics, Series A* associated with "An International Conference in Honor of Professor C.R. Rao on the Occasion of his 80th Birthday, Statistics: Reflections on the Past and Visions for the Future, The University of Texas at San Antonio, March 2000".
- [2] Tian Y., *Two rank equalities associated with blocks of orthogonal projector*. Problem 25–4, Image, The Bulletin of the International Linear Algebra Society **25** (2000), p. 16 [Solutions by J.K. Baksalary & O.M. Baksalary, by H.J. Werner, and by S. Puntanen, G.P.H. Styan & Y. Tian, Image, The Bulletin of the International Linear Algebra Society **26** (2001), 6–9].
- [3] Tian Y., *Completing block matrices with maximal and minimal ranks*, Linear Algebra Appl. **321** (2000), 327–345.
- [4] Tian Y., Styan, G.P.H., *Some rank equalities for idempotent and involutory matrices*, Linear Algebra Appl. **335** (2001), 101–117.

DEPARTMENT OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON,  
ONTARIO K7L 3N6, CANADA

*E-mail:* ytian@mast.queensu.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, MONTRÉAL,  
QUÉBEC H3A 2K6, CANADA

*E-mail:* styan@together.net

(Received May 25, 2001, revised January 28, 2002)