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On partial cubes and graphs with convex intervals

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Abstract. A graph is called a partial cube if it admits an isometric embedding into a hypercube. Subdivisions of wheels are considered with respect to such embeddings and with respect to the convexity of their intervals. This allows us to answer in negative a question of Chepoi and Tardif from 1994 whether all bipartite graphs with convex intervals are partial cubes. On a positive side we prove that a graph which is bipartite, has convex intervals, and is not a partial cube, always contains a subdivision of K_4.

Keywords: isometric embeddings, hypercubes, partial cubes, convex intervals, subdivisions

Classification: 05C12, 05C75

1. Introduction

Isometric subgraphs of Hamming graphs (called *partial Hamming graphs*) and related classes of graphs have been considered by several authors over the last years. Isometric subgraphs of hypercubes (called *partial cubes*), which are precisely bipartite partial Hamming graphs, have been first investigated in the seventies by Graham and Pollak [10] who used them as a model for a communication network. Djoković [8], Avis [3], Winkler [19], Chepoi [4], and Wilkeit [18] followed with nice characterizations of these graphs. Recognition algorithms for partial cubes and for partial Hamming graphs of complexity $O(mn)$, where m is the number of edges and n the number of vertices, were developed in [2] and [1], respectively. Interestingly, no faster algorithms are known by now, cf. [11], [12], so it seems that even more insight into the structure of these graphs is needed in order to either improve this complexity or to prove an appropriate lower bound. Partial cubes have also found several applications, cf. [5], [6], [9], [14], [15].

Clearly, partial cubes are bipartite and it is not difficult to see that they have convex intervals. (In fact, just observe that hypercubes have convex intervals, and use the definition of partial cubes.) During the 1994 Bielefeld conference on "Discrete Metric Spaces", Chepoi and Tardif [7] asked whether the converse could also be true. This question appeared as Conjecture 2.45 in [13] under the name "Chepoi-Tardif conjecture", and it was the main motivation for the present paper. More precisely, calling graphs with convex intervals *interval monotone graphs*, the

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question was whether every bipartite interval monotone graph is a partial cube. (Interval monotonicity versus interval-regularity was studied in [16].)

In order to answer it we first study subdivisions of wheels and the convexity of their intervals. Let W_k be the k-wheel, and let $W_k(m,n)$ be the graph obtained from W_k by subdividing every edge incident to the central vertex of W_k by n vertices and every other edge by m vertices. Then we characterize interval monotone graphs and partial cubes among the family of graphs $W_k(m,n)$, $k \geq 3$, $n, m \geq 0$. As a consequence we obtain that $W_3(m,n)$ is a bipartite, interval monotone graph, which does not admit an isometric embedding into a hypercube, provided that $n \geq 2$, m is an odd integer, and $m \leq 2n$. We also prove that a graph which is bipartite, has convex intervals, and is not a partial cube, contains a subdivision of K_4.

For a graph G, the *distance* $d_G(u,v)$ (or briefly $d(u,v)$) between vertices u and v is defined as the number of edges on a shortest u,v-path. The *interval* $I(u,v)$ between vertices u and v consists of all vertices on shortest paths between u and v. A subgraph H of G is *convex*, if for any u, $v \in V(H)$, $I(u,v) \subseteq V(H)$. A subgraph H of G is called *isometric* if $d_H(u,v) = d_G(u,v)$ for all $u, v \in V(H)$. Isometric subgraphs of hypercubes are called *partial cubes*. An important subclass of partial cubes are *median graphs*, that is, the graphs G in which for every triple of vertices u, v, and w of G we have $|I(u,v) \cap I(u,w) \cap I(v,w)| = 1$. For an edge ab of a graph G let

$$W_{ab} = \{x \in V(G) : d(x,a) < d(x,b)\}.$$

We will also use W_{ab} to denote the corresponding induced subgraph of G. Djoković [8] characterized partial cubes in the following way.

Theorem 1.1. *A graph G is a partial cube if and only if it is bipartite and if for any edge ab of G the subgraph W_{ab} is convex.*

In [8] relation Θ was defined as follows: Edges $xy, ab \in E(G)$ are in relation Θ, if $x \in W_{ab}$ and $y \in W_{ba}$. For bipartite graphs this is equivalent to the next definition: $xy, ab \in E(G)$ are in relation Θ if

$$d(x,a) + d(y,b) \neq d(x,b) + d(y,a).$$

Using this definition Winkler proved in [19]:

Theorem 1.2. *A graph G is a partial cube if and only if it is bipartite and Θ is transitive.*

Besides the above two characterizations of partial cubes we will also make use of the following one due to Wilkeit [18]:

Theorem 1.3. *A graph G is a partial cube if and only if it is bipartite and if for any edges ab and xy, $xy\Theta ab$ implies $W_{ab} = W_{xy}$.*

Let W_k be the *k-wheel*, that is, the graph obtained as a join of the one vertex graph K_1 and the k-cycle C_k. In the rest of the paper we will denote the central vertex of W_k by u and the remaining vertices by $w_1, w_2, \ldots, w_k$, where adjacencies are defined in a natural way. The cycle of W_k induced by the vertices $w_1, w_2, \ldots, w_k$ will also be called the *outer cycle* of W_k. These notions will also be used for subdivided wheels, in particular for the graphs $W_k(m,n)$.

2. Interval monotone subdivisions of wheels

In this section we characterize interval monotone graphs among the subdivided wheels $W_k(m,n)$. We begin with graphs that are obtained from W_3 (i.e. from K_4). Let $W(m_1,m_2,m_3;n_1,n_2,n_3)$ be the graph obtained by subdividing edges of K_4, where m_i is the number of vertices added on the edges of the outer cycle, and n_i the number of vertices added on the inner edges, so that numbers n_i and m_i correspond to two nonincident edges of K_4 ($i = 1,2,3$). Then we have:

Lemma 2.1. *Assume that for $W(m_1,m_2,m_3;n_1,n_2,n_3)$ the following properties hold:*

(A) $m_i + m_j \geq m_k$, *for all permutations (i,j,k) of $(1,2,3)$;*
(B) $m_i \leq n_j + n_k$, *for all permutations (i,j,k) of $(1,2,3)$;*
(C) $m_i - m_j = n_j - n_i$, *for all $1 \leq i < j \leq 3$;*
(D) $n_i \leq m_j + n_k$, *for all permutations (i,j,k) of $(1,2,3)$.*

Then $W(m_1,m_2,m_3;n_1,n_2,n_3)$ is interval monotone.

Proof: Note first that if x and y are vertices of the outer cycle then (B) implies that $I(x,y)$ is contained in the outer cycle, hence convex.

Let x and y be two vertices of the inner subdivided edges. If they both lie on a path between some w_i and u, then from (D) we deduce that $I(x,y)$ is a path. Suppose next that they are in different subdivided inner edges, say on w_iu and w_ju. Then $I(x,y)$ is a path, if they are both close enough to u (e.g., if they are both neighbors of u). If they lie far from u, then use (A) and (D) to observe that $I(x,y)$ can be either a path which goes through w_i and w_j, or the cycle $w_i \to \ldots \to w_j \to \ldots \to u \to \ldots \to w_i$. Again use (A) and (D) for the convexity of this cycle.

It remains to check the case when x is on the outer cycle, say between w_1 and w_2, and y is one of the inner vertices. The case when y is on w_1u or w_2u is essentially the same as above. Thus the last case to consider is when y is on w_3u. If $I(x,w_3)$ is not equal to the outer cycle we can argue as above. So let $I(x,w_3)$ be the outer cycle, i.e., x is the vertex on the outer cycle at the largest distance from w_3. But then we deduce from (C) that $I(x,u)$ is the whole cycle $w_1 \to \ldots \to w_2 \to \ldots \to u \to \ldots \to w_1$, hence $I(x,y)$ is either the whole graph or one of the cycles together with some short path from u to y, or from w_3 to y. $\square$

Theorem 2.2. *Let $k \geq 3$. Then $W_k(m,n)$ is interval monotone if and only if*

(i) *$k = 3$ and $m \leq 2n$; or*
(ii) *$k \geq 3$, $m \geq n = 0$.*

Proof: We distinguish several cases.

Case 1: $k \geq 4$, $m > n \geq 1$.

Let $x_1, x_2, \ldots, x_m$ be the vertices of $W_k(m,n)$ with which the edge w_1w_2 is subdivided and let $y_m, y_{m-1}, \ldots, y_1$ be the vertices with which w_2w_3 is subdivided, see Figure 1.

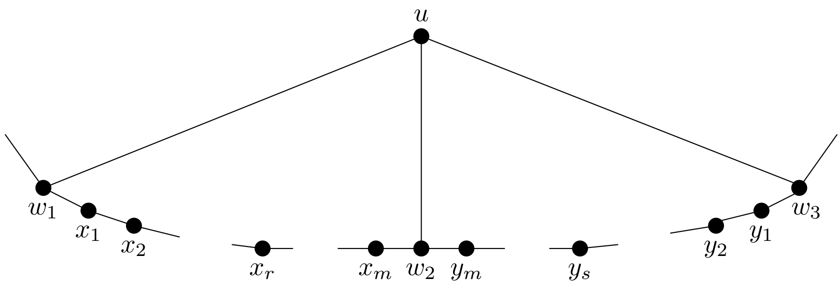

Figure 1: Subdivided vertices of Case 1.

Denote $x_0 = w_1$, set $r = \lfloor (m-n)/2 \rfloor$, $s = \lceil (m-n)/2 \rceil$, and consider the following paths between x_r and y_s:

$$\begin{aligned}
P_1 &: \; x_r, x_{r+1}, \ldots, x_m, w_2, y_m, \ldots, y_s;\\
P_2 &: \; x_r, x_{r-1}, \ldots, w_1, \ldots, u, \ldots, w_3, y_1, \ldots, y_s;\\
P_3 &: \; x_r, x_{r-1}, \ldots, w_1, \ldots, u, \ldots, w_2, y_m, \ldots, y_s;\\
P_4 &: \; x_r, x_{r+1}, \ldots, w_2, \ldots, u, \ldots, w_3, y_1, \ldots, y_s.
\end{aligned}$$

Then the lengths of these paths are:

$$\begin{aligned}
P_1 &: \; (m+1-\lfloor (m-n)/2 \rfloor) + (m+1-\lceil (m-n)/2 \rceil) = m+n+2;\\
P_2 &: \; \lfloor (m-n)/2 \rfloor + 2(n+1) + \lceil (m-n)/2 \rceil = m+n+2;\\
P_3 &: \; \lfloor (m-n)/2 \rfloor + 2(n+1) + (m+1-\lceil (m-n)/2 \rceil) \geq 2n+m+2;\\
P_4 &: \; (m+1-\lfloor (m-n)/2 \rfloor) + 2(n+1) + \lceil (m-n)/2 \rceil \geq 2n+m+2.
\end{aligned}$$

It follows that $d(x_r, y_s) = m+n+2$. Moreover, $u, w_2 \in I(x_r, y_s)$, but no interior vertex on the w_2, u-path of length $n+1$ belongs to $I(x_r, y_s)$. Hence, $I(x_r, y_s)$ is not convex.

Case 2: $k \geq 4$, $1 \leq m \leq n$ or $0 = m < n$.

Let $z_1, z_2, \dots, z_n$ be the vertices of $W_k(m,n)$ with which the edge uw_3 is subdivided, see Figure 2.

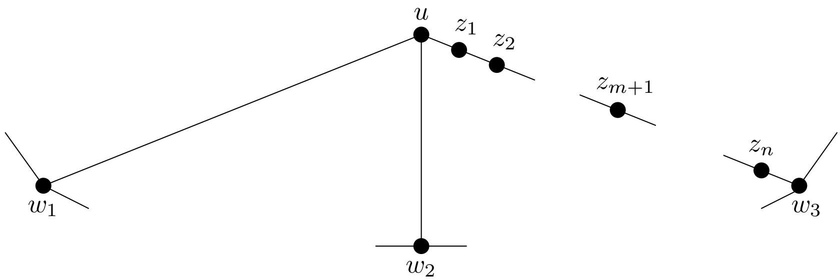

Figure 2: Subdivided vertices of Case 2.

Then the following two paths between w_1 and z_{m+1} (note that we allow $z_{m+1} = w_3$):

$P_1 \;:\; w_1, \dots, u, z_1, \dots, z_{m+1};$
$P_2 \;:\; w_1, \dots, w_2, \dots, w_3, z_n, \dots, z_{m+1},$

are shortest w_1, z_{m+1}-paths of length $n+m+2$. Hence $u, w_2 \in I(w_1, z_{m+1})$. Since no interior vertex of the w_2, u-path of length $n+1$ belongs to this interval, we are done also in this case.

Case 3: $k = 3$, $m \geq 2n+1 \geq 3$.

Let $x_1, x_2, \dots, x_m$, $y_m, y_{m-1}, \dots, y_1$, r, and s, be defined as in Case 1, cf. Figure 1. In addition, let P_1, P_2, P_3 and P_4 be the x_r, y_s-paths as defined in Case 1. Then the length of P_1 as well as of P_2 is $m+n+2$. Also, the lengths of P_3 and P_4 are at least $2n+m+2$. Finally, let P_5 be the path $x_r, x_{r-1}, \dots, w_1, \dots, w_3, y_1, \dots, y_s$. Its length is $r + (m+1) + s = 2m - n + 1$ and since $m \geq 2n+1$ the length of P_5 is at least $m+n+2$. We conclude that no interior vertex of the w_2, u-path of length $n+1$ belongs to $I(x_r, y_s)$.

Case 4: $k = 3$, $1 \leq m \leq 2n$.

Note that $W_3(m,n)$ is isomorphic to $W(m,m,m;n,n,n)$. Hence in this case $W_3(m,n)$ is interval monotone by Lemma 2.1.

The last case to consider is $n = 0$, more precisely:

Case 5: $k \geq 3, m \geq n = 0$.

We claim that $W_k(m,0)$ is interval monotone for $k \geq 3$, $m \geq 0$. For wheels, that is, for $m = 0$, this is clear. Also, it is well known that $W_k(1,0)$ are median graphs. Let $m \geq 2$. It clearly suffices to check interval monotonicity for vertices x, y of the outer cycle. In this case $d(w_i, w_j) = 2$ for $i \neq j$, while the distance between w_i and w_j in the outer cycle is at least three. From here we conclude by a simple case analysis that the claim is true also in the subcase $m \geq 2$. $\square$

3. Subdivisions of wheels and partial cubes

In this section we locate partial cubes among the graphs $W_k(m,n)$. Together with Theorem 2.2 this allows us to answer the question of Chepoi and Tardif. At the end we also show that the structure of bipartite, interval monotone graphs, which are not partial cubes, must be similar as in our examples.

We begin with the following two straightforward observations.

Lemma 3.1. *The graph $W_k(m,n)$ is bipartite if and only if m is odd.*

Lemma 3.2. *Let C be an isometric even cycle of a graph G and e an edge of C. Then the antipodal edge of e is the unique edge (different from e) of C that is in relation Θ with e.*

Theorem 3.3. *Let $k \geq 3$. Then $W_k(m,n)$ is a partial cube if and only if*

(i) *$k = 3$, $m = 1$, and $n = 1$; or*
(ii) *$k \geq 3$, $n = 0$, and m is odd.*

Proof: First we consider the case $n > 0$. Since partial cubes are interval monotone, Theorem 2.2 implies that then $k = 3$ and $m \leq 2n$. Let $m \geq 3$. Then $n \geq 2$, and let x be the neighbor of w_1 on the subdivision of w_1w_3. Since the outer cycle is isometric, by Lemma 3.2 there is a unique edge ab on the outer cycle which is in relation Θ with uv. In addition, it is clear that ab belongs to the subdivision of w_2w_3 and that $a \neq w_2, b \neq w_2$. There exists an isomorphism φ between the isometric cycles

$$C_1 : w_3 \to \ldots \to u \to \ldots \to w_1 \to \ldots \to w_3 \quad \text{and}$$
$$C_2 : w_3 \to \ldots \to u \to \ldots \to w_2 \to \ldots \to w_3$$

which preserves the shortest w_3,u-path, so that $\varphi(w_1) = w_2$. Note that xw_1 (resp. ab) is in relation Θ with precisely one edge e of C_1 (resp. f of C_2) whose endvertices are the unique vertices at the largest distance from x and w_1 (resp. a and b). Both e and f are on subdivided edge of uw_3, but $e \neq f$ because $\varphi(xw_1) \neq ab$. Hence Θ is not transitive and by Theorem 1.2 we infer that $W_3(m,n)$, $n > 0$, $m \geq 3$, is not a partial cube.

Let $m = 1$ and $n \geq 2$. We define cycles C_1 and C_2 as above, but this time let x be a neighbor of w_1 on the subdivision of w_1u. Applying Lemma 3.2 we infer that w_1x is in relation Θ with precisely two edges e and f of the cycle C_2, where e is on the subdivision of w_2u and f on the subdivision of w_3u, and endvertices of e and f are neither w_2 nor w_3. Since e and f are not in relation Θ, $W_3(m,n)$, $n \geq 2$, $m = 1$, is not a partial cube.

It is straightforward to check that $W_3(1,1)$ is a partial cube.

In the case where $n = 0$ we see that $W_k(1,0)$ are median graphs which makes them partial cubes. Finally, let $m \geq 3$ be odd. We claim that in this case

$W_k(m,0)$ is a partial cube. Since it is bipartite, by Theorem 1.1 it is enough to show that the sets W_{ab} are convex. First note that W_{w_1u} is a path of length $m+1$ and clearly convex. Likewise the set W_{uw_1} is easily seen to be convex. Consider now an arbitrary edge ab of the outer cycle and assume without loss of generality that $d(a,u) < d(b,u)$. But then we infer that W_{ba} induces a path, and we easily conclude that again W_{ba} and W_{ab} are convex. □

Combining Theorems 2.2 and 3.3 we can now answer a question of Chepoi and Tardif as follows.

Corollary 3.4. *Let $n \geq 2$, and let m be an odd integer, $m \leq 2n$. Then $W_3(m,n)$ is a bipartite, interval monotone graph, which does not admit an isometric embedding into a hypercube.*

We note that also nonsymmetric subdivided K_4's can be interval monotone, bipartite and not partial cubes. For example, consider the class of graphs

$$W(2k+1, 2k, 2k; 2k+1, 2k+2, 2k+2)$$

for all $k \geq 1$. We believe that there are more such cases. However, any interval monotone, bipartite graph, that is not a partial cube, contains a subdivision of K_4, as our final result claims.

Theorem 3.5. *Let G be a bipartite interval monotone graph. Then either G is a partial cube or it contains a subdivision of K_4.*

Proof: Let G be a bipartite graph in which all intervals are convex, and suppose that G is not a partial cube. Then by Theorem 1.3 there exist edges $ab, xy \in E(G)$ which are in relation Θ such that $W_{ab} \neq W_{xy}$. Hence, since G is bipartite, there exists a vertex $w \in W_{xy}$, such that also $w \in W_{ba}$. We select edges ab and xy so that $d(a,x)$ is as small as possible, and among such pairs let ab and xy be chosen in such a way that for some $w \in W_{xy} \cap W_{ba}$ the sum $d(x,w) + d(w,b)$ is as small as possible. Note that under these conditions vertex w can still be chosen in such a way that its neighbor on a shortest w,x-path is in W_{ab}.

Let $x' \in I(x,a) \cap I(x,w)$ be such that its neighbor x'' on a shortest x,w-path is not in $I(x,a)$ (clearly, such a neighbor x'' of x' exists, since w cannot be in $I(x,a)$). Then, it is easy to see that the remainder of the shortest path from x'' to w is disjoint with $I(x,a)$. Let b' be the first vertex on a shortest path from w to b which is in $I(y,b)$. Then obviously $I(b',b) \subseteq I(y,b)$. Let P be a path from x' to b' which is a concatenation of a shortest x',w-path, and a shortest w,b'-path. Since $w \notin I(y,a)$ and $I(y,a)$ is convex, it follows that P cannot be a shortest x',b'-path. We shall now prove that $a,b \in I(x',b')$ or $x,y \in I(x',b')$.

Suppose there is a shortest x',b'-path P' that avoids all four vertices x,y,a and b. Let v be the first vertex on P' which is in W_{ba} (such a vertex exists since $b' \in W_{ba}$). Hence its preceding neighbor u on P' is in W_{ab}, thus $ab\Theta uv$. We distinguish two cases.

Case 1: $xy\Theta uv$.

Note that in this case v cannot be closer to x than to y, because then we would derive that $v \in W_{ba} \cap W_{xy}$, and by the choice of P' we would have $d(x,v)+d(v,b) < d(x,w)+d(w,b)$. Hence $v \in W_{yx}$ and $u \in W_{xy}$. Now we have two possibilities: if $w \in W_{uv}$ then w is a vertex in $W_{uv} \cap W_{ba}$, where $d(a,x) > d(a,u)$, a contradiction to the choice of x and a being the vertices with the smallest distance such that $W_{ab} \neq W_{xy}$. On the other hand, if $w \in W_{vu}$ then w is a vertex in $W_{vu} \cap W_{xy}$, where $d(a,x) > d(u,x)$, again the same contradiction.

Case 2: $\neg(xy\Theta uv)$.

In this case both x and y are either in W_{uv} or in W_{vu}. If $x, y \in W_{uv}$ then $y \in W_{uv} \cap W_{ba}$, where $d(u,a) < d(x,a)$, again a contradiction with the choice of x and a. If $x, y \in W_{vu}$ then $x \in W_{vu} \cap W_{ab}$, where $d(u,a) < d(x,a)$, the same contradiction.

Hence $I(x',b')$ includes at least one pair of vertices x, y or a, b. Without loss of generality assume that $x, y \in I(x',b')$. We have noted in the beginning of the proof that w can be chosen in such a way that its neighbor w' on P is in W_{ab}. Then obviously $w' \in I(w,a)$ and $b' \in I(w,a)$. Now, if x' would also be in $I(w,a)$, then since G is interval monotone and $y \in I(x',b')$ that would imply $y \in I(w,a)$. This is possible only if $b' = y$ which leads straightforward to a contradiction with the choice of w as a vertex in $W_{xy} \cap W_{ba}$. Thereby $x' \notin I(w,a)$, and let a' be a nearest vertex to w in $I(w,a) \cap I(a,x)$, and w'' a vertex in $I(w,a') \cap P$ at the largest distance from w. We have thus obtained a subdivided K_4 in G with vertices w'', a', x' and b' and the proof is complete. $\square$

It would be interesting to see whether one can strengthen Theorem 3.5 to derive the existence of an isometric subdivided K_4 in G. Moreover, a characterization of partial cubes as bipartite interval monotone graphs with some (nice) additional condition(s) seems to be a challenging task.

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