Liftings of vector fields to 1-forms on the r-jet prolongation of the cotangent bundle

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Abstract. For natural numbers r and $n \geq 2$ all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ transforming vector fields from n-manifolds M into 1-forms on $J^rT^*M = \{j^r_x(\omega) \mid \omega \in \Omega^1(M),\ x \in M\}$ are classified. A similar problem with fibered manifolds instead of manifolds is discussed.

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0. Introduction

Let n and r be natural numbers.

In [4], we studied how a vector field X on an n-dimensional manifold M can induce a 1-form $A(X)$ on the r-cotangent bundle $T^{r*}M = J^r(M, \mathbb{R})_0$ of M. This problem is reflected in the concept of natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$. We proved that for $n \geq 2$ the set of all natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*T^{r*}$ is a free $2r$-dimensional $\mathcal{C}^\infty(\mathbb{R}^r)$-module, and we constructed explicitly the basis of this module. In particular, we reobtain a result from [1] saying that every canonical 1-form on T^*M is a constant multiple of the well-known Liouville 1-form λ.

In the present paper we study a similar problem with the r-jet prolongation $J^rT^*M = \{j^r_x\omega \mid \omega \in \Omega^1(M),\ x \in M\}$ of the cotangent bundle T^*M instead of $T^{r*}M$. We investigate how a vector field X on an n-manifold M can induce a 1-form $A(X)$ on J^rT^*M. This problem is reflected in the concept of natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ in the sense of Kolář, Michor and Slovák [2]. We prove that for $n \geq 2$ the set of all natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ is a free $(3r+2)$-dimensional $\mathcal{C}^\infty(\mathbb{R}^{r+1})$-module, and we construct explicitly the basis of this module.

A similar problem with fibered manifolds instead of manifolds is discussed.

Analyzing constant natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ we reobtain a result from [3] saying that every canonical 1-form on J^rT^*M is a constant multiple of $\lambda^r = (\pi^r_0)^*\lambda$, where $\pi^r_0 : J^rT^*M \to T^*M$ is the jet projection and λ is the Liouville 1-form on T^*M.

Some natural operators transforming functions, vector fields, forms on some natural bundles F are used practically in all papers in which problem of prolongation of geometric structures is considered. That is why such natural operators have been studied, see [2].

From now on $x^1, \dots, x^n$ denote the usual coordinates on $\mathbb{R}^n$, and $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, \dots, n$ are the canonical vector fields on $\mathbb{R}^n$.

All manifolds and maps are assumed to be of class $\mathcal{C}^\infty$.

1. The r-jet prolongation of the cotangent bundle

For every n-dimensional manifold M we have the vector bundle $J^rT^*M = \{j^r_x\omega \mid \omega \in \Omega^1(M),\ x \in M\}$ over M. It is called the r-jet prolongation of the cotangent bundle T^*M. Every embedding $\varphi : M \to N$ of two n-manifolds induces a vector bundle map $J^rT^*\varphi : J^rT^*M \to J^rT^*N$, $J^rT^*\varphi(j^r_x\omega) = j^r_{\varphi(x)}(\varphi_*\omega)$, $\omega \in \Omega^1(M)$, $x \in M$. The correspondence $J^rT^* : \mathcal{M}f_n \to \mathcal{VB}$ is a vector natural bundle over n-manifolds in the sense of [2].

2. Examples of natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$

Example 1. Let X be a vector field on an n-manifold M. For every $q = 0, \dots, r$ we have a map $\overset{(q)}{X} : J^rT^*M \to \mathbb{R}$, $\overset{(q)}{X}(j^r_x\omega) := X^q\omega(X)(x)$, $\omega \in \Omega^1(M)$, $x \in M$, where $X^q = X \circ \dots \circ X$ (q-times). Then for every $q = 0, \dots, r$ we have a 1-form $d\overset{(q)}{X}$ on J^rT^*M. The correspondence $\overset{(q)}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$, $X \to d\overset{(q)}{X}$, is a natural operator.

Example 2. Let X be a vector field on an n-manifold M. For every $p = 0, \dots, r-1$ we have a 1-form $\overset{<p>}{X} : TJ^rT^*M \to \mathbb{R}$ on J^rT^*M, $\overset{<p>}{X}(v) = \; < d_x(X^p\omega(X)), T\pi(v) >$, $v \in (TJ^rT^*)_xM$, $x \in M$, $\omega \in \Omega^1(M)$, $p^T(v) = j^r_x\omega$, $p^T : TJ^rT^*M \to J^rT^*M$ is the tangent bundle projection, $\pi : J^rT^*M \to M$ is the bundle projection. The correspondence $\overset{<p>}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$, $X \to \overset{<p>}{X}$, is a natural operator.

Example 3. Let X be a vector field on an n-manifold M. For every $q = 0, \dots, r$ we have a 1-form $\overset{<<q>>}{X} : TJ^rT^*M \to \mathbb{R}$ on J^rT^*M, $\overset{<<q>>}{X}(v) = \; < (L_X)^q\omega, T\pi(v) >$, $v \in (TJ^rT^*)_xM$, $x \in M$, $\omega \in \Omega^1(M)$, $p^T(v) = j^r_x\omega$, where $(L_X)^q = L_X \circ \dots \circ L_X$ (q-times), L_X is the Lie derivative with respect to X. The correspondence $\overset{<<q>>}{A} : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$, $X \to \overset{<<q>>}{X}$, is a natural operator.

3. A classification theorem

The set of all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ is a module over the algebra $\mathcal{C}^\infty(\mathbb{R}^{r+1})$. Actually, if $f \in \mathcal{C}^\infty(\mathbb{R}^{r+1})$ and $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ is a natural operator, then $fA : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ is given by $(fA)(X) = f(\overset{(0)}{X}, \dots, \overset{(r)}{X})A(X)$, $X \in \mathcal{X}(M)$, $M \in \mathrm{Obj}\,(\mathcal{M}f_n)$.

The main result of this paper is the following classification theorem.

Theorem 1. *For natural numbers r and $n \geq 2$ the $\mathcal{C}^\infty(\mathbb{R}^{r+1})$-module of all natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$ is free and $(3r+2)$-dimensional. The natural operators $\overset{(q)}{A}$, $\overset{<p>}{A}$ and $\overset{<<q>>}{A}$ for $q = 0, \dots, r$ and $p = 0, \dots, r-1$ form the basis over $\mathcal{C}^\infty(\mathbb{R}^{r+1})$ of this module.*

The proof of Theorem 1 will occupy Sections 4 and 5.

From now on we consider a natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^rT^*)$.

4. Some preparations

Since the operators $\overset{(0)}{A}, \dots, \overset{(r)}{A}$, $\overset{<0>}{A}, \dots, \overset{<r-1>}{A}$ and $\overset{<<0>>}{A}, \dots, \overset{<<r>>}{A}$ are $\mathcal{C}^\infty(\mathbb{R}^{r+1})$-linearly independent, we prove only that A is a linear combination of $\overset{(0)}{A}, \dots, \overset{(r)}{A}$, $\overset{<0>}{A}, \dots, \overset{<r-1>}{A}$ and $\overset{<<0>>}{A}, \dots, \overset{<<r>>}{A}$ with $\mathcal{C}^\infty(\mathbb{R}^{r+1})$-coefficients.

The following lemma shows that A is uniquely determined by the restriction $A(\partial_1)|(TJ^rT^*)_0\mathbb{R}^n$.

Lemma 1. *If $A(\partial_1)|(TJ^rT^*)_0\mathbb{R}^n = 0$, then $A = 0$.*

Proof: The proof is standard. We use the naturality of A and the fact that any non-vanishing vector field is locally ∂_1. □

So, we will study the restriction $A(\partial_1)|(TJ^rT^*)_0\mathbb{R}^n$.

Lemma 2. *There are maps $f_0, \dots, f_r \in \mathcal{C}^\infty(\mathbb{R}^{r+1})$ such that*

$$\left(A - \sum_{q=0}^{r} f_q \overset{(q)}{A}\right)(\partial_1)|(VJ^rT^*)_0\mathbb{R}^n = 0,$$

where $VJ^rT^*M \subset TJ^rT^*M$ denotes the π-vertical subbundle.

Proof: We have $(VJ^rT^*)_0\mathbb{R}^n \tilde{=} (J^rT^*)_0\mathbb{R}^n \times (J^rT^*)_0\mathbb{R}^n$,

$$\frac{d}{dt}_{|t=0}(u + tw) \tilde{=} (u, w), \;\; u, w \in (J^rT^*)_0\mathbb{R}^n.$$

For $q = 0, \dots, r$ we define $f_q : \mathbb{R}^{r+1} \to \mathbb{R}$,

$$f_q(a) = A(\partial_1)\Big(j^r_0\Big(\sum_{l=0}^{r} \frac{1}{l!} a_l (x^1)^l dx^1\Big), j^r_0\Big(\frac{1}{q!}(x^1)^q dx^1\Big)\Big),$$

where $a = (a_0, \dots, a_r) \in \mathbb{R}^{r+1}$.

We prove the assertion of the lemma. For simplicity denote

$$\tilde{A} := A - \sum_{q=0}^{r} f_q \overset{(q)}{A}.$$

Consider $\omega, \eta \in \Omega^1(\mathbb{R}^n)$. Define $a = (a_0, \dots, a_r) \in \mathbb{R}^{r+1}$ by

$$j^r_0((x^1, 0, \dots, 0)^* \omega) = j^r_0\Big(\sum_{l=0}^{r} \frac{1}{l!} a_l (x^1)^l dx^1\Big).$$

Define $b = (b_0, \dots, b_r) \in \mathbb{R}^{r+1}$ by

$$j^r_0((x^1, 0, \dots, 0)^* \eta) = j^r_0\Big(\sum_{l=0}^{r} \frac{1}{l!} b_l (x^1)^l dx^1\Big).$$

Using the naturality of $\tilde{A}$ with respect to the homotheties $(x^1, tx^2, \dots, tx^n)$ for $t \neq 0$ and putting $t \to 0$ we get

$$\tilde{A}(\partial_1)(j^r_0\omega, j^r_0\eta) = \tilde{A}(\partial_1)(j^r_0((x^1, 0, \dots, 0)^*\omega), j^r_0((x^1, 0, \dots, 0)^*\eta)).$$

Then $\tilde{A}(\partial_1)(j^r_0\omega, j^r_0\eta) = \sum_{q=0}^{r} b_q f_q(a) - \sum_{q=0}^{r} f_q(a) b_q = 0.$ □

5. Proof of Theorem 1

Replacing A by $A - \sum_{q=0}^{r} f_q \overset{(q)}{A}$ we can assume that

$$A(\partial_1)|(VJ^rT^*)_0\mathbb{R}^n = 0.$$

It remains to show that there exist maps $g_0, \dots, g_{r-1}, h_0, \dots, h_r \in \mathcal{C}^\infty(\mathbb{R}^{r+1})$ such that

$$(*) \qquad A = \sum_{p=0}^{r-1} g_p \overset{<p>}{A} + \sum_{q=0}^{r} h_q \overset{<<q>>}{A}.$$

For $p = 0, \dots, r-1$ define $g_p : \mathbb{R}^{r+1} \to \mathbb{R}$,

$$g_p(a) = A(\partial_1)\Big(J^rT^*\partial_2\Big(j_0^r\Big(\sum_{l=0}^{r} \frac{1}{l!}a_l(x^1)^l dx^1 + \frac{1}{p!}(x^1)^p x^2 dx^1\Big)\Big)\Big),$$

where $a = (a_0, \dots, a_r) \in \mathbb{R}^{r+1}$. For $q = 0, \dots, r$ define $h_q : \mathbb{R}^{r+1} \to \mathbb{R}$,

$$h_q(a) = A(\partial_1)\Big(J^rT^*\partial_2\Big(j_0^r\Big(\sum_{l=0}^{r} \frac{1}{l!}a_l(x^1)^l dx^1 + \frac{1}{q!}(x^1)^q dx^2\Big)\Big)\Big),$$

where $a = (a_0, \dots, a_r) \in \mathbb{R}^{r+1}$. We inform that J^rT^*X denotes the complete lifting (flow operator) of a vector field $X \in \mathcal{X}(M)$ to J^rT^*M.

We are going to prove $(*)$. By Lemma 1 and $A(\partial_1)|(VT^{r*})_0\mathbb{R}^n = 0$ it is sufficient to show

$$A(\partial_1)(J^rT^*\partial(j_0^r\omega)) = \Big(\sum_{p=0}^{r-1} g_p \overset{<p>}{A} + \sum_{q=0}^{r} h_q \overset{<<q>>}{A}\Big)(\partial_1)(J^rT^*\partial(j_0^r\omega))$$

for any $\omega \in \Omega^1(\mathbb{R}^n)$ and any linearly independent on ∂_1 constant vector field ∂ on $\mathbb{R}^n$.

For simplicity denote

$$\tilde{A} = \sum_{p=0}^{r-1} g_p \overset{<p>}{A} + \sum_{q=0}^{r} h_q \overset{<<q>>}{A}.$$

Using the naturality of A and $\tilde{A}$ with respect to linear isomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ preserving ∂_1 we can assume that $\partial = \partial_2$.

Consider $\omega \in \Omega^1(\mathbb{R}^n)$.

Define $a = (a_0, \dots, a_r) \in \mathbb{R}^{r+1}$ by

$$a_q = \partial_1^q\omega(\partial_1)(0), \ q = 0, \dots, r.$$

Define $b = (b_0, \dots, b_{r-1}) \in \mathbb{R}^r$ by

$$b_p = \partial_2\partial_1^p\omega(\partial_1)(0), \ p = 0, \dots, r-1.$$

Define $c = (c_0, \dots, c_r) \in \mathbb{R}^{r+1}$ by

$$c_q = \partial_1^q\omega(\partial_2)(0), \ q = 0, \dots, r.$$

Using the naturality of A with respect to $(x^1, tx^2, \tau x^3 \dots, \tau x^n) : \mathbb{R}^n \to \mathbb{R}^n$ for $t, \tau \neq 0$ we get the homogeneity condition

$$tA(\partial_1)(J^r T^* \partial_2 j_0^r(\omega)) = A(\partial_1)(J^r T^* \partial_2 (j_0^r(x^1, tx^2, \tau x^3, \dots, \tau x^n)^* \omega)).$$

This type of homogeneity gives

$$A(\partial_1)(J^r T^* \partial_2 (j_0^r \omega)) = \sum_{p=0}^{r-1} g_p(a) b_p + \sum_{q=0}^{r} h_q(a) c_q$$

because of the homogeneous function theorem [2].

On the other hand

$$\tilde{A}(\partial_1)(J^r T^* \partial_2 (j_0^r \omega)) = \sum_{p=0}^{r-1} g_p(a) b_p + \sum_{q=0}^{r} h_q(a) c_q.$$

The proof of Theorem 1 is complete. □

6. Corollaries

Using the homogeneous function theorem, we have the following corollary of Theorem 1.

Corollary 1. *Let r and $n \geq 2$ be natural numbers. Then for every linear natural operator $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^* J^r(T^*)$ there exist real numbers $\alpha, \beta, \gamma, \delta$ such that*

$$A(X) = \alpha \overset{(0)}{A}(X) + \beta \overset{<0>}{A}(X) + \gamma \overset{<1>}{A}(X) + \delta \overset{(0)}{X} \overset{<<0>>}{A}(X)$$

for any vector field $X \in \mathcal{X}(M)$.

The operator $\overset{<<0>>}{A}$ can be considered as the well-known canonical 1-form λ^r on $J^r T^*$, the pull-back $(\pi_0^r)^* \lambda$ of the Liouville 1-form λ on T^* with respect to the jet projection $\pi_0^r : J^r T^* \to T^*$. Considering the values of natural operators $T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^r T^*)$ at $X = 0$ we obtain the next corollary of Theorem 1.

Corollary 2 ([3])**.** *For natural numbers r and $n \geq 2$ every canonical 1-form on $J^r T^*$ is a constant multiple of λ^r.*

Corollary 3 ([5])**.** *For natural numbers r and $n \geq 2$ there is no canonical simplectic structure on $J^r T^*$.*

Proof: Using Corollary 2 and the Poincaré lemma it is easy to see that any canonical closed 2-form on $J^r T^* M$ is a constant multiple of $d\lambda^r$. □

7. A generalization to fibered manifolds

Given a fibered manifold $Y \to M$ we say that a 1-form ω on Y is horizontal if $\omega|VY = 0$, where $VY \subset TY$ is the vertical bundle of $Y \to M$. By $\Omega^1_{\text{hor}}(Y)$ we denote the space of all horizontal 1-forms on Y.

Let s, r be two natural numbers with $s \geq r$. We say that two horizontal 1-forms $\omega, \eta \in \Omega^1_{\text{hor}}(Y)$ on a fibered manifold $\tilde{p} : Y \to M$ determine the same (r,s)-jet $j^{r,s}_y\omega = j^{r,s}_y\eta$ at $y \in Y$ if $j^r_y\omega = j^r_y\eta$ and $j^s_y(\omega|Y_x) = j^s_y(\eta|Y_x)$, see [2]. Here Y_x is the fiber of Y over $x = \tilde{p}(y)$.

Let m, n, r, s be natural numbers, $s \geq r$. For every (m,n)-dimensional fibered manifold $Y \to M$ ($\dim(M) = m$, $\dim(Y) = m + n$) we have a vector bundle $J^{r,s}T^*_{\text{hor}}Y = \{j^{r,s}_y\omega \mid \omega \in \Omega^1_{\text{hor}}(Y),\ y \in Y\}$ over Y. Every fibered embedding $\varphi : Y \to Z$ of two (m,n)-dimensional fibered manifolds induces a vector bundle map $J^{r,s}T^*_{\text{hor}}\varphi : J^{r,s}T^*_{\text{hor}}Y \to J^{r,s}T^*_{\text{hor}}Z$, $J^{r,s}T^*_{\text{hor}}\varphi(j^{r,s}_y\omega) = j^{r,s}_{\varphi(y)}(\varphi_*\omega)$, $\omega \in \Omega^1_{\text{hor}}(Y)$, $y \in Y$. The correspondence $J^{r,s}T^*_{\text{hor}} : \mathcal{FM}_{m,n} \to \mathcal{VB}$ is a vector natural bundle on the category $\mathcal{FM}_{m,n}$ of (m,n)-dimensional fibered manifolds and their fibered embeddings.

Let m, n, r, s be natural numbers with $s \geq r$.

Example 1'. Let X be a projectable vector field on an (m,n)-dimensional fibered manifold $\tilde{p} : Y \to M$. (We say that a vector field X on Y is projectable if there exists a $\tilde{p}$-related with X vector field X_o on M.) For every $q = 0, \dots, r$ we have a map $\overset{(q)}{X} : J^{r,s}T^*_{\text{hor}}Y \to \mathbb{R}$, $\overset{(q)}{X}(j^{r,s}_y\omega) := X^q\omega(X)(y)$, $\omega \in \Omega^1_{\text{hor}}(Y)$, $y \in Y$, where $X^q = X \circ \dots \circ X$ (q-times). Then for every $q = 0, \dots, r$ we have a 1-form $d\overset{(q)}{X}$ on $J^{r,s}T^*_{\text{hor}}Y$. The correspondence $\overset{(q)}{A} : T_{\text{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s}T^*_{\text{hor}})$, $X \to d\overset{(q)}{X}$, is a natural operator.

Example 2'. Let X be a projectable vector field on an (m,n)-dimensional fibered manifold Y. For every $p = 0, \dots, r-1$ we have a 1-form $\overset{<p>}{X} : TJ^{r,s}T^*_{\text{hor}}Y \to \mathbb{R}$ on $J^{r,s}T^*_{\text{hor}}Y$, $\overset{<p>}{X}(v) =< d_x(X^p\omega(X)), T\pi(v) >$, where $v \in (TJ^{r,s}T^*_{\text{hor}})_yY$, $y \in Y$, $\omega \in \Omega^1_{\text{hor}}(Y)$, $p^T(v) = j^{r,s}_y\omega$, $p^T : TJ^{r,s}T^*_{\text{hor}}Y \to J^{r,s}T^*_{\text{hor}}Y$ is the tangent bundle projection, $\pi : J^{r,s}T^*_{\text{hor}}Y \to Y$ is the bundle projection. The correspondence $\overset{<p>}{A} : T_{\text{proj}|\ \mathcal{FM}m,n} \rightsquigarrow T^*(J^{r,s}T^*_{\text{hor}})$, $X \to \overset{<p>}{X}$, is a natural operator.

Example 3'. Let X be a projectable vector field on an (m,n)-dimensional fibered manifold Y. For every $q = 0, \dots, r$ we have a 1-form $\overset{<<q>>}{X} : TJ^{r,s}T^*_{\text{hor}}Y \to \mathbb{R}$ on $J^{r,s}T^*_{\text{hor}}Y$, $\overset{<<q>>}{X}(v) =< (L_X)^q\omega, T\pi(v) >$, where $v \in (TJ^{r,s}T^*_{\text{hor}})_yY$, $y \in Y$, $\omega \in \Omega^1_{\text{hor}}(Y)$, $p^T(v) = j^r_y\omega$, $(L_X)^q = L_X \circ \dots \circ L_X$ (q-times), L_X is the Lie derivative with respect to X. The correspondence $\overset{<<q>>}{A} : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s}T^*_{\mathrm{hor}})$, $X \to \overset{<<q>>}{X}$, is a natural operator.

The set of all natural operators $T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s}T^*_{\mathrm{hor}})$ is a module over the algebra $\mathcal{C}^\infty(\mathbb{R}^{r+1})$. Actually, if $f \in \mathcal{C}^\infty(\mathbb{R}^{r+1})$ and $A : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s}T^*_{\mathrm{hor}})$ is a natural operator, then $fA : T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s}T^*_{\mathrm{hor}})$ is given by $(fA)(X) = f(\overset{(0)}{X}, \dots, \overset{(r)}{X})A(X)$, $X \in \mathcal{X}_{\mathrm{proj}}(Y)$, $Y \in \mathrm{Obj}\,(\mathcal{FM}_{m,n})$.

Theorem 1'. *For natural numbers r, s, m and n with $m \geq 2$ and $s \geq r$ the $\mathcal{C}^\infty(\mathbb{R}^{r+1})$-module of all natural operators $T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s}T^*_{\mathrm{hor}})$ is free and $(3r+2)$-dimensional. The natural operators $\overset{(q)}{A}$, $\overset{<p>}{A}$ and $\overset{<<q>>}{A}$ for $q = 0, \dots, r$ and $p = 0, \dots, r-1$ form a basis over $\mathcal{C}^\infty(\mathbb{R}^{r+1})$ of this module.*

The proof of Theorem 1' is a simple modification of the proof of Theorem 1. It is left to the reader. We propose to use the fact that every projectable vector field on Y with non-vanishing underlying vector field is locally $\frac{\partial}{\partial x^1}$ in some fibered manifold coordinates $x^1, \dots, x^m, y_1, \dots, y^n$ on Y.

8. Exercises

Exercise 1. Let s, r, t be natural numbers with $s \geq r \leq t$. We say that two 1-forms $\omega, \eta \in \Omega^1(Y)$ on a fibered manifold $\tilde{p} : Y \to M$ determine the same (r, s, t)-jet $j_y^{r,s,t}\omega = j_y^{r,s,t}\eta$ at $y \in Y$ if $j_y^r\omega = j_y^r\eta$, $j_y^t\omega^R = j_y^t\eta^R$ and $j_y^s(\omega|Y_x) = j_y^s(\eta|Y_x)$. Here Y_x is the fiber of Y over $x = \tilde{p}(y)$ and $\omega^R : Y \to (VY)^*$ is given by the restriction $\omega_y|V_yY$ for any $y \in Y$. Define a bundle functor $J^{r,s,t}T^* : \mathcal{FM}_{m,n} \to \mathcal{VB}$ by using (r, s, q)-jets of 1-forms instead of (r, s)-jets. Classify natural operators $A : T_{\mathrm{proj}\,\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s,t}T^*)$.

Answer: For natural numbers r, s, t, m and n with $m \geq 2$ and $s \geq r \leq t$ all natural operators $T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s,t}T^*)$ form a free, $(3r+2)$-dimensional module over $C^\infty(\mathbb{R}^{r+1})$. The (similar as in Examples 1', 2' and 3') natural operators $\overset{(q)}{A}$, $\overset{<p>}{A}$ and $\overset{<<q>>}{A}$ for $q = 0, \dots, r$ and $p = 0, \dots, r-1$ form the basis over $\mathcal{C}^\infty(\mathbb{R}^{r+1})$ of this module.

Exercise 2. Let s, r, t, u be natural numbers with $s \geq r$, $u \geq t$, $t \geq r$ and $u \geq s$. We say that two 1-forms $\omega, \eta \in \Omega^1(Y)$ on a fibered manifold $\tilde{p} : Y \to M$ determine the same (r, s, t, u)-jet $j_y^{r,s,t,u}\omega = j_y^{r,s,t,u}\eta$ at $y \in Y$ if $j_y^r\omega = j_y^r\eta$, $j_y^t\omega^R = j_y^t\eta^R$, $j_y^s(\omega|Y_x) = j_y^s(\eta|Y_x)$ and $j_y^u(\omega^R|Y_x) = j_y^u(\eta^R|Y_x)$. ($Y_x$ and ω^R as in Exercise 1.) Define a bundle functor $J^{r,s,t,u}T^* : \mathcal{FM}_{m,n} \to \mathcal{VB}$ by using (r, s, q, u)-jets of 1-forms. Classify natural operators $A : T_{\mathrm{proj}\,\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s,t,u}T^*)$.

Answer: For natural numbers r, s, t, u, m and n with $m \geq 2$ and $s \geq r$, $u \geq t$, $t \geq r$ and $u \geq s$ all natural operators $T_{\mathrm{proj}\,|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{r,s,t,u}T^*)$ form a free, $(3r+2)$-dimensional module over $C^\infty(\mathbb{R}^{r+1})$. The (similar as in Examples 1', 2' and 3') natural operators $\overset{(q)}{A}$, $\overset{<p>}{A}$ and $\overset{<<q>>}{A}$ for $q = 0, \dots, r$ and $p = 0, \dots, r-1$ form the basis over $\mathcal{C}^\infty(\mathbb{R}^{r+1})$ of this module.

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