Locally solid topologies on spaces of vector-valued continuous functions

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Abstract. Let X be a completely regular Hausdorff space and E a real normed space. We examine the general properties of locally solid topologies on the space $C_b(X,E)$ of all E-valued continuous and bounded functions from X into E. The mutual relationship between locally solid topologies on $C_b(X,E)$ and $C_b(X)$ $(= C_b(X,\mathbb{R}))$ is considered. In particular, the mutual relationship between strict topologies on $C_b(X)$ and $C_b(X,E)$ is established. It is shown that the strict topology $\beta_\sigma(X,E)$ (respectively $\beta_\tau(X,E)$) is the finest σ-Dini topology (respectively Dini topology) on $C_b(X,E)$. A characterization of σ-Dini and Dini topologies on $C_b(X,E)$ in terms of their topological duals is given.

Keywords: vector-valued continuous functions, strict topologies, locally solid topologies, Dini topologies

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0. Introduction

Let X be a completely regular Hausdorff space, βX its Stone-Čech compactification and let $(E, \|\cdot\|_E)$ be a real normed space. Let S_E stand for the closed unit sphere in E. Let $C_b(X,E)$ be the space of all bounded continuous functions f from X into E. We will write $C_b(X)$ instead of $C_b(X,\mathbb{R})$, where $\mathbb{R}$ is the field of all real numbers. For a function $u \in C_b(X)$, $\overline{u}$ denotes its unique continuous extension to βX. For a function $f \in C_b(X,E)$ we will write $\|f\|(x) = \|f(x)\|_E$ for all $x \in X$. Then $\|f\| \in C_b(X)$ and the space $C_b(X,E)$ can be equipped with a norm $\|f\|_\infty = \sup_{x\in X} \|f\|(x) = \|\,\|f\|\,\|_\infty$, where $\|u\|_\infty = \sup_{x\in X} |u(x)|$ for $u \in C_b(X)$.

A subset H of $C_b(X,E)$ is said to be *solid* whenever $\|f_1\| \le \|f_2\|$ (i.e. $\|f_1(x)\|_E \le \|f_2(x)\|_E$ for all $x \in X$) and $f_1 \in C_b(X,E)$, $f_2 \in H$ implies $f_1 \in H$. A linear topology τ on $C_b(X,E)$ is said to be *locally solid* if it has a local base at 0 consisting of solid sets (see [Ku], [KuO]). The so-called strict topologies on $C_b(X,E)$ and some subspaces of $C_b(X,E)$ have been considered by many authors (see [A], [F], [K$_1$], [K$_2$], [K$_3$], [Ku], [KuO], [KuV$_1$], [KuV$_2$]). It is well known that the strict topologies $\beta_t(X,E)$, $\beta_\tau(X,E)$, $\beta_\sigma(X,E)$, $\beta_\infty(X,E)$, $\beta_g(X,E)$ and $\beta_p(X,E)$ on $C_b(X,E)$ are locally solid (see [Ku, Theorem 8.1], [KuO, Theorem 6], [KuV$_1$, Theorem 5]).

In Section 1 we examine some general properties of solid sets in $C_b(X,E)$ and next, in Section 2, general properties of locally solid topologies on $C_b(X,E)$. It is shown that a locally convex topology τ on $C_b(X,E)$ is locally solid iff τ is generated by some family of solid seminorms defined on $C_b(X,E)$. Recall here that a seminorm ρ on $C_b(X,E)$ is called solid whenever $\rho(f_1) \leq \rho(f_2)$ if $f_1, f_2 \in C_b(X,E)$ and $\|f_1\| \leq \|f_2\|$. In Section 3 we introduce a general method which establishes a mutual relationship between locally solid topologies on $C_b(X)$ and $C_b(X,E)$. In particular, in Section 4, the mutual relationship between strict topologies defined on $C_b(X)$ and $C_b(X,E)$ is established. In Section 5 we distinguish some important classes of locally convex-solid topologies on $C_b(X,E)$. Namely, a locally convex-solid topology τ on $C_b(X,E)$ is said to be *a σ-Dini topology* whenever for a sequence (f_n) in $C_b(X,E)$, $\|f_n\| \downarrow 0$ (i.e. $\|f_n(x)\|_E \downarrow 0$ for each $x \in X$) implies $f_n \to 0$ for τ. Replacing sequences by nets in $C_b(X,E)$ we obtain *a Dini topology* on $C_b(X,E)$. It is shown that the strict topology $\beta_\sigma(X,E)$ (resp. $\beta_\tau(X,E)$) is the finest σ-Dini topology (resp. Dini topology) on $C_b(X,E)$. We obtain a characterization of both the σ-Dini and the Dini-topologies on $C_b(X,E)$ in terms of their topological duals.

1. The solid structure of spaces of vector-valued continuous functions

In this section we examine the solid structure of the space $C_b(X,E)$.

Definition 1.1 (see [Ku]). A subset H of $C_b(X,E)$ is said to be *solid* whenever $\|f_1\| \leq \|f_2\|$ and $f_1 \in C_b(X,E)$, $f_2 \in H$ implies $f_1 \in H$.

The following lemma will be of a key importance for an examination of the solid structure of $C_b(X,E)$.

Lemma 1.1 [The solid decomposition property]. *Assume that for $f, g_1, \dots, g_n \in C_b(X,E)$, $\|f\| \leq \|g_1 + \ldots + g_n\|$. Then there exist $f_1, \dots, f_n \in C_b(X,E)$ satisfying: $\|f_i\| \leq \|g_i\| \quad (i = 1,2,\dots,n)$ and $f = f_1 + \cdots + f_n$.*

PROOF: By using induction it is enough to establish the result for $n=2$. Thus assume first that $\|f(x)\|_E \leq \|g_1(x) + g_2(x)\|_E$ for all $x \in X$, where $f, g_1, g_2, \in C_b(X,E)$.

Let us put (for $i = 1,2$)

$$f_i(x) = \begin{cases} \frac{\|g_i\|(x)}{\|g_1\|(x)+\|g_2\|(x)} f(x) & \text{if } \|g_1\|(x) + \|g_2\|(x) > 0, \\ 0 & \text{if } \|g_1\|(x) + \|g_2\|(x) = 0. \end{cases}$$

It is seen that $f_i \in C_b(X,E)$ and $f_1 + f_2 = f$. To show that $\|f_i\| \leq \|g_i\|$ for

$i = 1, 2$, assume first that $\|g_1\|(x_0) + \|g_2\|(x_0) > 0$ for $x_0 \in X$. Then

$$\begin{aligned}\|f_i\|(x_0) &= \frac{\|g_i\|(x_0)}{\|g_1\|(x_0) + \|g_2\|(x_0)}\|f\|(x_0) \\ &\leq \frac{\|g_i\|(x_0)}{\|g_1\|(x_0) + \|g_2\|(x_0)}(\|g_1\|(x_0) + \|g_2\|(x_0)) = \|g_i\|(x_0).\end{aligned}$$

Next, let $\|g_1\|(x_0) + \|g_2\|(x_0) = 0$ for some $x_0 \in X$. Then $\|f_i\|(x_0) = 0 \leq \|g_i\|(x_0)$ $(i = 1, 2)$. Thus the proof is complete. □

Theorem 1.2. *The convex hull* (conv H) *of a solid subset H of $C_b(X, E)$ is solid.*

Proof: Let H be a solid subset of $C_b(X, E)$, and let $\|f\| \leq \|g\|$, where $f \in C_b(X, E)$ and $g \in \operatorname{conv} H$. Then there exist $g_1, \dots, g_n \in H$ and numbers $\alpha_1, \dots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$ such that $g = \sum_{i=1}^n \alpha_i g_i$. Hence by Lemma 1.1 there exist $f_1, \dots, f_n \in C_b(X, E)$, such that $\|f_i\| \leq \alpha_i \|g_i\|$ for $i = 1, 2, \dots, n$ and $f = \sum_{i=1}^n f_i$. Putting $h_i = \alpha_i^{-1} f_i$ we get $\|h_i\| \leq \|g_i\|$, so $h_i \in H$, $(i = 1, 2, \dots, n)$. But then $f = \sum_{i=1}^n f_i = \sum_{i=1}^n \alpha_i h_i \in \operatorname{conv} H$, so conv H is solid, as desired. □

2. Locally solid topologies on spaces of vector-valued continuous functions

We start this section with the definition of locally solid topologies on $C_b(X, E)$.

Definition 2.1 (see [Ku]). A linear topology τ on $C_b(X, E)$ is said to be *locally solid* if it has a local base at zero consisting of solid sets.

Theorem 2.1. *Let τ be a locally solid topology on $C_b(X, E)$. Then the τ-closure $\overline{H}$ of a solid subset H of $C_b(X, E)$ is solid.*

Proof: Let $\mathcal{B}_\tau$ be a local base at 0 for τ consisting of solid sets. Then $\overline{H} = \bigcap\{H + V : V \in \mathcal{B}_\tau\}$. Assume that $\|f\| \leq \|g\|$, where $f \in C_b(X, E)$, $g \in \overline{H}$, and let $V_0 \in \mathcal{B}_\tau$. Then $g = g_1 + g_2$ where $g_1 \in H$ and $g_2 \in V_0$. Since $\|f\| \leq \|g\|$, by Lemma 1.1 there exist $f_1, f_2 \in C_b(X, E)$ such that $f = f_1 + f_2$ and $\|f_i\| \leq \|g_i\|$ $(i = 1, 2)$. Hence $f_1 \in H$ and $f_2 \in V_0$, because both sets H and V_0 are solid. Thus $f \in H + V$ for every $V \in \mathcal{B}_\tau$, so $f \in \overline{H}$. This means that $\overline{H}$ is solid, as desired. □

Definition 2.2. A linear topology τ on $C_b(X, E)$ that is at the same time locally solid and locally convex will be called a *locally convex-solid topology* on $C_b(X, E)$.

In view of Theorems 1.2 and 2.1 we see that for a locally convex-solid topology on $C_b(X, E)$ the collection of all τ-closed, convex and solid τ-neighborhoods of zero forms a local base at 0 for τ.

Definition 2.3. A seminorm ρ on $C_b(X,E)$ is said to be *solid* whenever $\rho(f_1) \le \rho(f_2)$ if $f_1, f_2 \in C_b(X,E)$ and $\|f_1\| \le \|f_2\|$.

Theorem 2.2. *For a locally convex topology τ on $C_b(X,E)$ the following statements are equivalent:*

(i) *τ is generated by some family of solid seminorms;*
(ii) *τ is a locally convex-solid topology.*

Proof: (i) $\Rightarrow$ (ii). It is obvious.

(ii) $\Rightarrow$ (i). Let $\mathcal{B}_\tau = \{V_\alpha : \alpha \in \mathcal{A}\}$ be a basis of zero for τ consisting of τ-closed, solid and convex sets. Let ρ_α stand for the Minkowski functional generated by V_α, that is

$$\rho_\alpha(f) = \inf\{\lambda > 0 : f \in \lambda V_\alpha\} \quad \text{for} \quad f \in C_b(X,E).$$

Then ρ_α is a solid τ-continuous seminorm and $\{f \in C_b(X,E) : \rho_\alpha(f) < 1\} \subset V_\alpha = \{f \in C_b(X,E) : \rho_\alpha(f) \le 1\}$. This means that the family $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ generates the topology τ. $\square$

3. The relationship between topological structures of $C_b(X)$ and $C_b(X,E)$

In this section, using Theorem 2.2 we introduce a general method which establishes a mutual relationship between locally solid topologies on $C_b(X)$ and $C_b(X,E)$.

Recall that the algebraic tensor product $C_b(X) \otimes E$ is the subspace of $C_b(X,E)$ spanned by the functions of the form $u \otimes e$, $(u \otimes e)(x) = u(x)e$, where $u \in C_b(X)$ and $e \in E$.

Given a Riesz seminorm p on $C_b(X)$ let us set

$$p^\vee(f) := p(\|f\|) \quad \text{for all} \quad f \in C_b(X,E).$$

It is easy to verify that $p^\vee$ is a solid seminorm on $C_b(X,E)$.

From now on let $e_0 \in S_E$ be fixed. Given a solid seminorm ρ on $C_b(X,E)$, let us put

$$\rho^\wedge(u) := \rho(u \otimes e_0) \quad \text{for all} \quad u \in C_b(X).$$

It is seen that $\rho^\wedge$ is well defined because $\rho(u \otimes e_0)$ does not depend on $e_0 \in S_E$, due to solidness of ρ. It is easy to check that $\rho^\wedge$ is a Riesz seminorm on $C_b(X)$.

Lemma 3.1. (i) *If ρ is a solid seminorm on $C_b(X,E)$, then $(\rho^\wedge)^\vee(f) = \rho(f)$ for all $f \in C_b(X,E)$.*

(ii) *If p is a Riesz seminorm on $C_b(X)$, then $(p^\vee)^\wedge(u) = p(u)$ for $u \in C_b(X)$.*

Proof: (i) For $f \in C_b(X,E)$ we have $(\rho^\wedge)^\vee(f) = \rho^\wedge(\|f\|) = \rho(\|f\| \otimes e_0)$, where $\|(\|f\| \otimes e_0)(x)\|_E = \| \, \|f\|(x)e_0\|_E = \|f\|(x) = \|f(x)\|_E$ for all $x \in X$. In view of the solidness of ρ we get $(\rho^\wedge)^\vee(f) = \rho(f)$.

(ii) For $u \in C_b(X)$ we have $(p^\vee)^\wedge(u) = p^\vee(u \otimes e_0) = p(\|u \otimes e_0\|)$, where $\|u \otimes e_0\|(x) = \|(u \otimes e_0)(x)\|_E = \|u(x)e_0\|_E = |u(x)| = |u|(x)$ for $x \in X$. Since p is a Riesz seminorm, we get $(p^\vee)^\wedge(u) = p(|u|) = p(u)$. □

Let τ be a locally convex-solid topology on $C_b(X,E)$. Then in view of Theorem 2.2 τ is generated by some family $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ of solid seminorms on $C_b(X,E)$. By $\tau^\wedge$ we will denote the locally convex-solid topology on $C_b(X)$ generated by the family $\{\rho_\alpha^\wedge : \alpha \in \mathcal{A}\}$ of Riesz seminorms on $C_b(X)$. One can check that $\tau^\wedge$ does not depend on the choice of a family $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ of solid seminorms on $C_b(X,E)$ generating τ.

Next, let ξ be a locally convex-solid topology on $C_b(X)$. Then ξ is generated by some family $\{p_\alpha : \alpha \in \mathcal{A}\}$ of Riesz seminorms on $C_b(X)$ (see [AB, Theorem 6.3]). By $\xi^\vee$ we will denote the locally convex-solid topology on $C_b(X,E)$ generated by the family $\{p_\alpha^\vee : \alpha \in \mathcal{A}\}$ of solid seminorms on $C_b(X,E)$. One can verify that $\xi^\vee$ does not depend on the choice of a family $\{p_\alpha : \alpha \in \mathcal{A}\}$ of Riesz seminorms on $C_b(X)$ that generates ξ.

In view of Lemma 3.1 we can easily get:

Theorem 3.2. (i) *For a locally convex-solid topology τ on $C_b(X,E)$ we have: $(\tau^\wedge)^\vee = \tau$.*

(ii) *For a locally convex-solid topology ξ on $C_b(X)$ we have: $(\xi^\vee)^\wedge = \xi$.*

Theorem 3.3. *Let ξ be a locally convex-solid topology on $C_b(X)$ and let τ be a locally convex-solid topology on $C_b(X,E)$.*

(i) *For a net (f_σ) in $C_b(X,E)$ we have:*
$f_\sigma \xrightarrow{\tau} 0$ *if and only if* $\|f_\sigma\| \xrightarrow{\tau^\wedge} 0$.

(ii) *For a net (u_σ) in $C_b(X)$ we have:*
$u_\sigma \xrightarrow{\xi} 0$ *if and only if* $u_\sigma \otimes e_0 \xrightarrow{\xi^\vee} 0$.

Theorem 3.4. *Let τ_1 and τ_2 be locally convex-solid topologies on $C_b(X,E)$ and let ξ_1 and ξ_2 be locally convex-solid topologies on $C_b(X)$. Then*

(i) *if $\tau_1 \subset \tau_2$, then $\tau_1^\wedge \subset \tau_2^\wedge$;*

(ii) *if $\xi_1 \subset \xi_2$, then $\xi_1^\vee \subset \xi_2^\vee$.*

Proof: (i) Let $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ and $\{\rho_\beta : \beta \in \mathcal{B}\}$ be generating families of solid seminorms for τ_1 and τ_2 respectively. Since $\tau_1 \subset \tau_2$, for each $\alpha \in \mathcal{A}$ there exist $\beta_1, \dots, \beta_n \in \mathcal{B}$ such that $\rho_\alpha(f) \le a \max_{1\le i\le n} \rho_{\beta_i}(f)$ for some $a > 0$ and all $f \in C_b(X,E)$. It easily follows that $\rho_\alpha^\wedge(u) \le a \max_{1\le i\le n} \rho_{\beta_i}^\wedge(u)$ for all $u \in C_b(X)$, and this means that $\tau_1^\wedge \subset \tau_2^\wedge$.

(ii) Let $\{p_\alpha : \alpha \in \mathcal{A}\}$ and $\{p_\beta : \beta \in \mathcal{B}\}$ be generating families of Riesz seminorms for ξ_1 and ξ_2 respectively. Since $\xi_1 \subset \xi_2$ for each $\alpha \in \mathcal{A}$ there exist $\beta_1, \dots, \beta_n \in \mathcal{B}$ such that $p_\alpha(u) \le a \max_{1\le i\le n} p_{\beta_i}(u)$ for some $a > 0$ and all

$u \in C_b(X)$. It follows that $p_\alpha^\wedge(f) \le a \max_{1\le i\le n} p_{\beta_i}^\wedge(f)$ for all $f \in C_b(X,E)$, and this means that $\xi_1^\vee \subset \xi_2^\vee$. $\square$

4. Strict topologies on spaces of continuous functions

In this section, by making use of the results of Section 3, we establish a mutual relationship between strict topologies on $C_b(X)$ and $C_b(X,E)$ which allows us to examine in a unified manner strict topologies on $C_b(X,E)$ by means of strict topologies on $C_b(X)$.

First we recall some definitions (see [S], [W], [Ku], [KuO], [KuV$_1$]). For a compact subset Q of $\beta X \setminus X$ let $C_Q(X) = \{v \in C_b(X) : \overline{v}|Q \equiv 0\}$. For each $v \in C_Q(X)$ let

$$p_v(u) = \sup_{x\in X} |v(x)u(x)| \quad \text{for} \quad u \in C_b(X)$$

and

$$\rho_v(f) = \sup_{x\in X} |v(x)|\, \|f\|(x) \quad \text{for} \quad f \in C_b(X,E).$$

Then p_v is a Riesz seminorm on $C_b(X)$ and ρ_v is a solid seminorm on $C_b(X,E)$. For each $u \in C_b(X)$ and a fixed $e_0 \in S_E$ we have:

$$\rho_v^\wedge(u) = \rho_v(u \otimes e_0) = \sup_{x\in X} |v(x)|\,|u(x)| = p_v(u) \tag{4.1}$$

and moreover, for each $f \in C_b(X,E)$ we get:

$$p_v(\|f\|) = \sup_{x\in X} |v(x)|\, \|f\|(x) = \rho_v(f). \tag{4.2}$$

Let $\beta_Q(X)$ be the locally convex-solid topology on $C_b(X)$ defined by $\{p_v : v \in C_Q(X)\}$ and let $\beta_Q(X,E)$ be the locally convex-solid topology on $C_b(X,E)$ defined by $\{\rho_v : v \in C_Q(X)\}$.

Thus $\beta_Q(X) = \beta_Q(X,\mathbb{R})$ and by (4.1) and (4.2) we get:

$$\beta_Q(X)^\vee = \beta_Q(X,E) \tag{4.3}$$

and

$$\beta_Q(X,E)^\wedge = \beta_Q(X). \tag{4.4}$$

Now let $\mathcal{C}$ be some family of compact subsets of $\beta X \setminus X$. The *strict topology* $\beta_{\mathcal{C}}(X,E)$ on $C_b(X,E)$ determined by $\mathcal{C}$ is the greatest lower bound (in the class of locally convex topologies) of the topologies $\beta_Q(X,E)$, as Q runs over $\mathcal{C}$. Thus $\beta_{\mathcal{C}}(X,E)$ is an inductive limit topology, and we denote it by LIN $\{\beta_Q(X,E) : Q \in \mathcal{C}\}$.

We will shortly write $\beta_{\mathcal{C}}(X)$ instead of $\beta_{\mathcal{C}}(X,\mathbb{R})$. It is well known that the strict topology $\beta_{\mathcal{C}}(X)$ on $C_b(X)$ is locally solid (see [W, Theorem 11.6]). Observe that the strict topology $\beta_{\mathcal{C}}(X,E)$ on $C_b(X,E)$ has a local base at 0 consisting of all sets of the form:

$$(+)\qquad \text{abs conv}\Big(\bigcup\nolimits_{Q\in\mathcal{C}} W_{v_Q}: \text{ for some } v_Q\in C_Q(X)\Big)$$

where for $v_Q\in C_Q(X)$, $W_{v_Q}=\{f\in C_b(X,E):\rho_{v_Q}(f)\le 1\}$.

By making use of Lemma 1.1 it is easy to check that the sets of the form (+) are solid. Thus we get:

Theorem 4.1. *The strict topologies $\beta_{\mathcal{C}}(X,E)$ on $C_b(X,E)$ are locally solid.*

Remark. The property of local solidness of strict topologies $\beta_{\mathcal{C}}(X,E)$ on $C_b(X,E)$ for some important classes $\mathcal{C}_\tau$, $\mathcal{C}_\sigma$ (see definition below) was obtained in a different way in [Ku].

The following theorem establishes a mutual relationship between strict topologies $\beta_{\mathcal{C}}(X,E)$ on $C_b(X,E)$ and $\beta_{\mathcal{C}}(X)$ on $C_b(X)$.

Theorem 4.2. *We have:*

$$\beta_{\mathcal{C}}(X)^{\vee}=\beta_{\mathcal{C}}(X,E)\quad\textit{and}\quad\beta_{\mathcal{C}}(X,E)^{\wedge}=\beta_{\mathcal{C}}(X).$$

Proof: By the definition of strict topologies and (4.3) and (4.4) we get

$$\beta_{\mathcal{C}}(X)\subset\beta_Q(X)=\beta_Q(X,E)^{\wedge}\ \text{ and }\ \beta_{\mathcal{C}}(X,E)\subset\beta_Q(X,E)=\beta_Q(X)^{\vee}.$$

Hence by Theorem 3.2 and Theorem 3.3 for each $Q\in\mathcal{C}$ we have

$$\beta_{\mathcal{C}}(X)^{\vee}\subset(\beta_Q(X,E)^{\wedge})^{\vee}=\beta_Q(X,E),\quad\text{so}\quad\beta_{\mathcal{C}}(X)^{\vee}\subset\beta_{\mathcal{C}}(X,E)$$

and

$$\beta_{\mathcal{C}}(X,E)^{\wedge}\subset(\beta_Q(X)^{\vee})^{\wedge}=\beta_Q(X),\quad\text{so}\quad\beta_{\mathcal{C}}(X,E)^{\wedge}\subset\beta_{\mathcal{C}}(X).$$

Thus

$$\beta_{\mathcal{C}}(X,E)=(\beta_{\mathcal{C}}(X,E)^{\wedge})^{\vee}\subset\beta_{\mathcal{C}}(X)^{\vee}\subset\beta_{\mathcal{C}}(X,E),\quad\text{so}\quad\beta_{\mathcal{C}}(X,E)=\beta_{\mathcal{C}}(X)^{\vee}$$

and

$$\beta_{\mathcal{C}}(X)=(\beta_{\mathcal{C}}(X)^{\vee})^{\wedge}\subset\beta_{\mathcal{C}}(X,E)^{\wedge}\subset\beta_{\mathcal{C}}(X),\quad\text{so}\quad\beta_{\mathcal{C}}(X)=\beta_{\mathcal{C}}(X,E)^{\wedge}.$$

Thus the proof is complete. □

As an application of Theorem 4.1, Theorem 4.2 and Theorem 3.3 we get:

Corollary 4.3. (i) *For a net* (f_σ) *in* $C_b(X,E)$ *we have:* $f_\sigma \to 0$ *for* $\beta_{\mathcal{C}}(X,E)$ *if and only if* $\|f_\sigma\| \to 0$ *for* $\beta_{\mathcal{C}}(X)$.

(ii) *For a net* (u_σ) *in* $C_b(X)$ *we have:* $u_\sigma \to 0$ *for* $\beta_{\mathcal{C}}(X)$ *if and only if* $u_\sigma \otimes e_0 \to 0$ *for* $\beta_{\mathcal{C}}(X,E)$.

Now we distinguish some important families of compact subsets of $\beta X \setminus X$. Let

$\mathcal{C}_\tau =$ the family of all compact subsets of $\beta X \setminus X$.

$\mathcal{C}_\sigma =$ the family of all zero subsets of $\beta X \setminus X$.

The strict topologies $\beta_\tau(X,E)$ and $\beta_\sigma(X,E)$ on $C_b(X,E)$ are now obtained by choosing $\mathcal{C}_\tau$ and $\mathcal{C}_\sigma$ as $\mathcal{C}$ appropriately (see [W, Definition 7.8, Definition 10.13], [Ku]). In particular, in view of Theorem 4.2 we get:

Corollary 4.4. *We have:*

$$\beta_\tau(X)^\vee = \beta_\tau(X,E), \quad \beta_\sigma(X)^\vee = \beta_\sigma(X,E),$$

and

$$\beta_\tau(X,E)^\wedge = \beta_\tau(X), \quad \beta_\sigma(X,E)^\wedge = \beta_\sigma(X).$$

Remark. The statement (i) of Corollary 4.3 was obtained in a different way for topologies $\beta_\tau(X,E)$ and $\beta_\sigma(X,E)$ in [Ku, Lemma 2.4].

Remark. The important classes of strict topologies $\beta_s(X,E)$, $\beta_p(X,E)$ and $\beta_g(X,E)$ on $C_b(X,E)$ can also be defined as inductive limit topologies by taking appropriate classes $\mathcal{C}$ of subsets of $\beta X \setminus X$ (see [W, Definitions 10.13, 10.15], [KuV], [KuO]).

5. Dini topologies on spaces of vector-valued continuous functions

The well known Dini's theorem is telling us that whenever a topological space X is pseudocompact then for a net (u_σ) in $C_b(X)$, $u_\sigma \downarrow 0$ (i.e., $u_\sigma(x) \downarrow 0$ for each $x \in X$) implies $\|u_\sigma\|_\infty \to 0$. F.D. Sentilles (see [S, Theorem 6.3]) showed that a Dini type theorem holds for topologies $\beta_\sigma(X)$ and $\beta_\tau(X)$ for X being a completely regular Hausdorff space, that is, $\beta_\sigma(X)$ (resp. $\beta_\tau(X)$) is the finest of all locally convex topologies ξ on $C_b(X)$ such that $u_n \downarrow 0$ implies $u_n \xrightarrow{\xi} 0$ (resp. $u_\sigma \downarrow 0$ implies $u_\sigma \xrightarrow{\xi} 0$). These properties of strict topologies justify the following definition of σ-Dini and Dini topologies in the vector-valued setting.

Definition 5.1. (i) A locally convex-solid topology τ on $C_b(X,E)$ is said to be *a σ-Dini topology* whenever for a sequence (f_n) in $C_b(X,E)$, $\|f_n\| \downarrow 0$ (i.e., $\|f_n\|(x) \downarrow 0$ for each $x \in X$) implies $f_n \to 0$ for τ.

(ii) A locally convex-solid topology τ on $C_b(X,E)$ is said to be *a Dini topology* whenever for a net (f_σ) in $C_b(X,E)$, $\|f_\sigma\| \downarrow 0$ (i.e., $\|f_\sigma\|(x) \downarrow 0$ for each $x \in X$) implies $f_\sigma \to 0$ for τ.

Thus $\beta_\sigma(X)$ (resp. $\beta_\tau(X)$) is the finest σ-Dini (resp. Dini) topology on $C_b(X)$.

In this section, by making use of the results of Sections 3 and 4 we show that $\beta_\sigma(X,E)$ (resp. $\beta_\tau(X,E)$) is the finest σ-Dini (resp. Dini) topology on $C_b(X,E)$.

We need the following technical results.

Lemma 5.1. (i) *If ξ is a σ-Dini topology (resp. a Dini topology) on $C_b(X)$, then $\xi^\vee$ is a σ-Dini topology (resp. a Dini topology) on $C_b(X,E)$.*

(ii) *If τ is a σ-Dini topology (resp. a Dini topology) on $C_b(X,E)$, then $\tau^\wedge$ is a σ-Dini topology (resp. a Dini topology) on $C_b(X)$.*

Proof: (i) Assume that ξ is a σ-Dini topology on $C_b(X)$ generated by a family $\{p_\alpha : \alpha \in \mathcal{A}\}$ of Riesz seminorms on $C_b(X)$. Then for a sequence (f_n) in $C_b(X,E)$ with $\|f_n\| \downarrow 0$ we get $p_\alpha^\vee(f_n) \to 0$, because $p_\alpha^\vee(f_n) = p_\alpha(\|f_n\|)$ for each $\alpha \in \mathcal{A}$ and $n \in \mathbb{N}$. This means that $f_n \to 0$ for $\xi^\vee$, as desired.

Similarly we get $f_\sigma \to 0$ for $\xi^\vee$ whenever ξ is a Dini topology.

(ii) Assume that τ is a σ-Dini topology on $C_b(X,E)$ generated by a family $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ of solid seminorms on $C_b(X,E)$. Then for a sequence (u_n) in $C_b(X)$ with $u_n \downarrow 0$ and a fixed $e_0 \in S_E$ we get $\|u_n \otimes e_0\| \downarrow 0$, because $\|u_n \otimes e_0\|(x) = \|u_n(x)e_0\|_E = |u_n(x)|$. Since $\rho_\alpha^\wedge(u_n) = \rho_\alpha(u_n \otimes e_0)$ for each $\alpha \in \mathcal{A}$ and $n \in \mathbb{N}$, we have that $u_n \to 0$ for $\tau^\wedge$, as desired.

Similarly, we obtain that $u_\sigma \to 0$ for $\tau^\wedge$ whenever τ is a Dini topology. $\square$

The next theorem is an extension of the Sentilles results (see [S, Theorem 6.3], [W, Corollary 11.16, Corollary 11.28]).

Theorem 5.2. (i) *The strict topology $\beta_\sigma(X,E)$ is the finest σ-Dini topology on $C_b(X,E)$.*

(ii) *The strict topology $\beta_\tau(X,E)$ is the finest Dini topology on $C_b(X,E)$.*

Proof: (i) Since $\beta_\sigma(X)$ is a σ-Dini topology on $C_b(X)$, by Lemma 5.1 and Corollary 4.4 we obtain that $\beta_\sigma(X,E)$ is a σ-Dini topology on $C_b(X,E)$. Now assume that τ is a σ-Dini topology on $C_b(X,E)$. Then by Lemma 5.1 $\tau^\wedge$ is a σ-Dini topology on $C_b(X)$. Hence $\tau^\wedge \subset \beta_\sigma(X)$, because $\beta_\sigma(X)$ is the finest σ-Dini topology on $C_b(X)$ (see [S, Theorem 6.3]). By making use of Theorem 3.2, Theorem 3.4 and Corollary 4.4 we get $\tau = (\tau^\wedge)^\vee \subset \beta_\sigma(X)^\vee = \beta_\sigma(X,E)$, as desired.

(ii) Similarly as in (i). $\square$

Now we are going to characterize σ-Dini topologies and Dini topologies on $C_b(X,E)$ in terms of their topological duals.

For a linear topology τ on $C_b(X,E)$ by $(C_b(X,E),\tau)'$ we denote the topological dual of $(C_b(X,E),\tau)$. In particular, let $C_b(X,E)'$ stand for the topological dual of $(C_b(X,E), \|\cdot\|_\infty)$.

We shall need the following definitions.

Definition 5.2. (i) A functional $\Phi \in C_b(X,E)'$ is said to be *σ-additive* whenever for a sequence (f_n) in $C_b(X,E)$, $\|f_n\| \downarrow 0$ implies $\Phi(f_n) \to 0$. The set consisting of all σ-additive functionals on $C_b(X,E)$ will be denoted by $L_\sigma(C_b(X,E))$.

(ii) A functional $\Phi \in C_b(X,E)'$ is said to be *τ-additive* whenever for a net (f_σ) in $C_b(X,E)$, $\|f_\sigma\| \downarrow 0$ implies $\Phi(f_\sigma) \to 0$. The set consisting of all τ-additive functionals on $C_b(X,E)$ will be denoted by $L_\tau(C_b(X,E))$.

Now we are in position to state our desired result.

Theorem 5.3. *For a locally convex-solid Hausdorff topology τ on $C_b(X,E)$ the following statements are equivalent:*

(i) $(C_b(X,E),\tau)' \subset L_\sigma(C_b(X,E))$;
(ii) *τ is a σ-Dini topology.*

PROOF: (ii) $\Rightarrow$ (i). It is obvious.

(i) $\Rightarrow$ (ii). Let $\{\rho_\alpha : \alpha \in \mathcal{A}\}$ be the family of solid seminorms on $C_b(X,E)$ that generates τ (see Theorem 2.2), and let $\tau^\wedge$ denote the locally convex-solid topology generated by the family $\{\rho_\alpha^\wedge : \alpha \in \mathcal{A}\}$ of Riesz seminorms on $C_b(X)$, where $\rho_\alpha^\wedge(u) = \rho(u \otimes e_0)$ for some fixed $e_0 \in S_E$ and $u \in C_b(X)$.

We shall first show that $(C_b(X),\tau^\wedge)' \subset L_\sigma(C_b(X))$. Indeed, let $\varphi \in (C_b(X),\tau^\wedge)'$ and let $u_n \downarrow 0$ (i.e. $u_n(x) \downarrow 0$ for all $x \in X$), where $u_n \in C_b(X)$. Define a linear functional Φ_φ on a subspace $C_b(X)(e_0)$ $(= \{u \otimes e_0 : u \in C_b(X)\})$ of $C_b(X,E)$ by putting $\Phi_\varphi(u \otimes e_0) = \varphi(u)$. Since $\varphi \in (C_b(X),\tau^\wedge)'$ there exist $c > 0$ and $\alpha_1,\dots,\alpha_n \in \mathcal{A}$ such that $|\Phi_\varphi(u \otimes e_0)| = |\varphi(u)| \le c \max_{1 \le i \le n} \hat\rho_{\alpha_i}(u) = c \max_{1 \le i \le n} \rho_{\alpha_i}(u \otimes e_0)$ for all $u \in C_b(X)$. This means that
$\Phi_\varphi \in (C_b(X)(e_0), \tau|_{C_b(X)(e_0)})'$, so by the Hahn-Banach extension theorem there is $\overline{\Phi}_\varphi \in (C_b(X,E),\tau)'$ such that $\overline{\Phi}_\varphi(u \otimes e_0) = \varphi(u)$ for all $u \in C_b(X)$. By our assumption $\overline{\Phi}_\varphi \in L_\sigma(C_b(X,E))$, so $\overline{\Phi}_\varphi(u_n \otimes e_0) \to 0$, because $\|u_n \otimes e_0\| = u_n \downarrow 0$. It follows that $\varphi(u_n) \to 0$, so $\varphi \in L_\sigma(C_b(X))$.

Thus in view of [K_2, Theorem 5.6] (applied to a Banach lattice $E = \mathbb{R}$), $\tau^\wedge$ is a σ-Dini topology on $C_b(X)$, so by Lemma 5.1 $(\tau^\wedge)^\vee$ is a σ-Dini topology on $C_b(X,E)$. But by Theorem 3.2 $\tau = (\tau^\wedge)^\vee$, and the proof is complete. □

We have an analogous result for Dini topologies with a similar proof.

Theorem 5.4. *For a locally convex-solid Hausdorff topology τ on $C_b(X,E)$ the following statements are equivalent:*

(i) $(C_b(X,E),\tau)' \subset L_\tau(C_b(X,E))$;
(ii) *τ is a Dini topology.*

Remark. In case E is a Banach lattice, the spaces $C_b(X,E)$ and $C_{rc}(X,E)$ (= the space of all $f \in C_b(X,E)$ for which $f(X)$ is relatively compact in E) became vector lattices under the natural ordering: $f \le g$ whenever $f(x) \le g(x)$ in E for all $x \in X$. Thus one can consider the concepts of solidness and a locally solid topology for $C_b(X,E)$ and $C_{rc}(X,E)$ in terms of the theory of Riesz spaces (see [AB]). Moreover, in [K_2, Section 5] a functional $\Phi \in C_{rc}(X,E)'$ is called σ-additive if $\Phi(f_n) \to 0$ for a sequence (f_n) in $C_{rc}(X,E)$ such that $f_n(x) \downarrow 0$ in E for all $x \in X$. Similarly τ-additive functionals on $C_{rc}(X,E)$ are defined. The above Theorems 5.3 and 5.4 are analogous to [K_2, Theorem 5.6, Theorem 5.5].

References

[A] Aguayo-Garrido J., *Strict topologies on spaces of continuous functions and u-additive measure spaces*, J. Math. Anal. Appl. **220** (1998), 77–89.

[AB] Aliprantis C.D., Burkinshaw O., *Locally Solid Topologies*, Academic Press, New York, San Francisco, London, 1978.

[F] Fontenot R.A., *Strict topologies for vector-valued functions*, Canad. J. Math. **26** (1974), 841–853.

[K_1] Katsaras A.K., *Spaces of vector measures*, Trans. Amer. Math. Soc. **206** (1975), 313–328.

[K_2] Katsaras A.K., *Some locally convex spaces of continuous vector-valued functions over a completely regular space and their duals*, Trans. Amer. Math. Soc. **216** (1976), 367–387.

[K_3] Katsaras A.K., *Locally convex topologies on spaces of continuous vector functions*, Math. Nachr. **71** (1976), 211–226.

[Ku] Khurana S.S., *Topologies on spaces of vector-valued continuous functions*, Trans. Amer. Math. Soc. **241** (1978), 195–211.

[KuO] Khurana S.S., Othman S.I., *Grothendieck measures*, J. London Math. Soc. (2) **39** (1989), 481–486.

[KuV_1] Khurana S.S., Vielma J.E., *Strict topology and perfect measures*, Czechoslovak Math. J. **40** (1990), 1–7.

[KuV_2] Khurana S.S., Vielma J.E., *Weak sequential convergence and weak compactness in spaces of vector-valued continuous functions*, J. Math. Anal. Appl. **195** (1995), 251–260.

[S] Sentilles F.D., *Bounded continous functions on a completely regular space*, Trans. Amer. Math. Soc. **168** (1972), 311–336.

[W] Wheeler R.F., *Survey of Baire measures and strict topologies*, Exposition Math. **2** (1983), 97–190.

[W_1] Wheeler R.F., *The strict topology, separable measures, and paracompactness*, Pacific J. Math. **47** (1973), 287–302.

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