Characterization of ω-limit sets of continuous maps of the circle

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Abstract. In this paper we extend results of Blokh, Bruckner, Humke and Smítal [Trans. Amer. Math. Soc. **348** (1996), 1357–1372] about characterization of ω-limit sets from the class $\mathcal{C}(I, I)$ of continuous maps of the interval to the class $\mathcal{C}(\mathbb{S}, \mathbb{S})$ of continuous maps of the circle. Among others we give geometric characterization of ω-limit sets and then we prove that the family of ω-limit sets is closed with respect to the Hausdorff metric.

Keywords: dynamical system, circle map, ω-limit set

Classification: Primary 37E10, 37B99, 26A18

1. Introduction

Continuous maps of the interval and continuous maps of the circle have many properties in common. Some of them are proved in [6]. In this paper we extend results proved in [3] from the class $\mathcal{C}(I, I)$ of continuous maps of the interval to the class $\mathcal{C}(\mathbb{S}, \mathbb{S})$ of continuous maps of the circle by using the same technique used in [6]. Other results concerning continuous maps of the circle can be found in [1] or [5].

Throughout the paper, the symbols I and $\mathbb{S}$ denote the unit interval $[0, 1]$ and the circle $\{z \in \mathbb{C}; |z| = 1\}$, respectively, and X denotes either I or $\mathbb{S}$. Denote by $\mathbb{S}_b$ the circle cut at a point $b \in \mathbb{S}$, i.e. $\mathbb{S}_b = \mathbb{S} \setminus \{b\}$. Let $e : \mathbb{R} \to \mathbb{S}$ be the natural projection defined by $e(x) = \exp(2\pi i x)$. Note that the map $\tilde{e} : (v, v+1) \to \mathbb{S}_{e(v)}$ obtained by restricting e to the interval $(v, v+1)$, is a homeomorphism. It is clear that if we define a map $h_v(x) := e(x + v)$, where $v \in \mathbb{R}$, then $\tilde{h}_v := h_v|_{(0,1)}$ is a homeomorphism from $(0, 1)$ onto $\mathbb{S} \setminus \{e(v)\}$ (see Lemma 3.1.3 in [1]). We say that $\tilde{h}_v(x) \le \tilde{h}_v(y)$ whenever $x \le y$. For an interval $A \subset \mathbb{S}_{e(v)}$ a point a is called the *left endpoint*, resp. the *right endpoint*, of A if $a \le x$, resp. $x \le a$, for every $x \in A$. Recall that the *trajectory* of a point x under a map f is the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where f^n is the n-th iteration of f. The set of limit points of the trajectory of

This research was supported, in part, by the contract No. 201/00/0859 from the Grant Agency of the Czech Republic and the contract No. CEZ:J10/98:192400002 from the Czech Ministry of Education. Support of these institutions is gratefully acknowledged.

x is called *ω-limit set* and we denote the set by $\omega_f(x)$. A set $\{U_0, \dots, U_{n-1}\}$ of mutually disjoint intervals is called a *cycle of intervals* if $f(U_i) = U_{i+1}$ for $i = 0, 1, \dots, n-2$ and $f(U_{n-1}) = U_0$. The map f is *transitive* if for every two non-empty open sets V, W there is a positive integer n such, that $f^n(V) \cap W \neq \emptyset$. Two maps $f : Y_1 \to Y_1$ and $g : Y_2 \to Y_2$ are *topologically conjugate* if there exists a homeomorphism $\varphi : Y_1 \to Y_2$ such that $\varphi \circ f(x) = g \circ \varphi(x)$ for any $x \in Y_1$. For more terminology see standard books like [1] or [2].

Now we introduce some notions used in [3] and modified for maps from $\mathcal{C}(\mathbb{S}, \mathbb{S})$. We say that a set $A \subset \mathbb{S}$ is *T-side* or *T-unilateral neighborhood* (T means either "left" or "right") of an $x \in \mathbb{S}$ if the set A is a closed interval and the point x is T endpoint of the set A.
Let $U = U_0 \cup \dots \cup U_{N-1}$ be a union of pairwise disjoint non-degenerate closed intervals and $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$. For any set $K \subset U$ let $f_U(K) = f(K) \cap U$ (this may be empty). Inductively define $f_U^n(K) = f_U(f_U^{n-1}(K))$. Define $\tilde{K} \equiv \tilde{K}(U) = \bigcup_{i=1}^{\infty} f_U^i(K)$; although $\tilde{K}$ depends on U, to avoid convoluted notation we use $\tilde{K}$ whenever the set U is evident. Let $A \subset \mathbb{S}$ be a closed set and $x \in A$. We say that a side T of a point x is *A-covering* if for any union of finitely many closed intervals U such that $A \subset \mathrm{Int}(U)$ and any closed T-unilateral neighborhood $V(x)$ there are finitely many components of $\tilde{V}(x)$ such that the closure of their union covers A. If T is an A-covering side of x then any T-unilateral neighborhood $V(x)$ is also said to be *A-covering*. We call the set A *locally expanding* according to the map f if every $x \in A$ has an A-covering side.

The main theorems of this paper are the following.

Theorem 1.1. *Let f be a map in $\mathcal{C}(X, X)$. A closed set $A \subset X$ is an ω-limit set if and only if it is locally expanding.*

Theorem 1.2. *Let $\{\omega_n\}_{n=1}^{\infty} = \{\omega_f(x_n)\}_{n=1}^{\infty}$ be a sequence of ω-limit sets of a continuous map $f \in \mathcal{C}(X, X)$ and let a point a have a side T, such that for any T-unilateral neighborhood V of a, there exists a positive integer N such that for each $n \geq N$, the orbit of x_n enters V infinitely many times. Then $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n}$ is an ω-limit set.*

Theorem 1.3. *Let f be a map in $\mathcal{C}(X, X)$. Then the family of all ω-limit sets of f endowed with the Hausdorff metric is compact.*

2. Proof of the main theorems

Let $b \in \mathbb{S}$ and $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$. We denote by $e^{-1}(b)$ the point $x \in [0, 1)$ such that $e(x) = b$. In the rest of the paper by h we denote the map $\tilde{h}_{e^{-1}(b)}$ whenever the point $b \in \mathbb{S}$ is evident and by A^* we mean the preimage of the set $A \subset \mathbb{S}_b$ under the map $\tilde{h}_{e^{-1}(b)}$. Denote by S the set $\mathbb{S} \setminus \bigcup_{n=0}^{\infty} f^{-n}(b)$. Now we can define a map $f^* \in \mathcal{C}(S^*, S^*)$ as

$$f^* := h^{-1} \circ f \circ h|_{S^*}.$$

The map f^* is defined only on the subset S^* of the interval $(0,1)$, but we overcome this difficulty using Lemma 2.1.

Lemma 2.1. *Let $f \in \mathcal{C}(X,X)$ and $A \subset X$ be a locally expanding set according to the map f. Then the set A is invariant, i.e. $f(A) \subset A$.*

PROOF: In the case when $X = I$ the lemma is proved in [3] (Lemma 2.5). It remains to consider the case $X = \mathbb{S}$. The case $A = \mathbb{S}$ is trivial. Let $A \subset \mathbb{S}_b$, $x \in A$ and $f(x) \notin A$. Then there exists a union of finitely many intervals $U = U_0 \cup \ldots \cup U_{n-1}$, $U \supset A$ such that for any sufficiently small neighborhood V of x we have $f(V) \cap U = \emptyset$. The definition of $\tilde{V}$ implies that $\tilde{V} = \emptyset$ which is a contradiction. $\square$

Lemma 2.2. *A set $A \subset \mathbb{S}$ is a T-side of a point $x \in \mathbb{S}$ if and only if the set A^* is a T-side of the point x^*.*

The proof is omitted.

Lemma 2.3. *If the whole circle $\mathbb{S}$ is locally expanding with respect to a map $f \in \mathcal{C}(\mathbb{S},\mathbb{S})$ then f is transitive.*

PROOF: Take two nonempty open sets V, W. Since a point $x \in \mathrm{Int}(V)$ has $\mathbb{S}$ covering side then $\tilde{V} = \mathbb{S}$ and hence there is a positive integer n such that $f^n(V) \cap W \neq \emptyset$. This proves that the map f is transitive. $\square$

Lemma 2.4. *Let f be a map in $\mathcal{C}(\mathbb{S},\mathbb{S})$. A closed set $A \subset \mathbb{S}_b$ is locally expanding according to the map f if and only if the set $A^* \subset (0,1)$ is locally expanding according to the map f^*.*

PROOF: First assume that the set A^* is locally expanding. Hence the sets A^*, A are closed and by Lemma 2.1 the set A^* is invariant and $A^* \subset S^*$. Take a point $x \in A$. Since the set A^* is locally expanding the point x^* has an A^*-covering side T^*. By Lemma 2.2 the set T is a side of the point x. Take a union of finitely many closed intervals $U \subset \mathbb{S}_b$ such that $A \subset \mathrm{Int}(U)$ and any closed T-unilateral neighborhood $V(x)$. Using the assumptions there are finitely many components of $\tilde{W}$ where $W = V(x)^*$ such that the closure of their union covers A^* and clearly $\tilde{W} \subset (0,1)$. Hence the set $\tilde{V}(x)$ has finitely many components such that the closure of their union covers A as well. Thus the set A is locally expanding.

The proof of the converse is analogous. $\square$

Lemma 2.5. *A set $A \subset \mathbb{S}_b$ is an ω-limit set of the map $f \in \mathcal{C}(\mathbb{S},\mathbb{S})$ if and only if the set A^* is an ω-limit set of the map f^*.*

PROOF: First consider the closed set $A \subset \mathbb{S}_b$ to be an ω-limit set. There is a point $x_0 \in \mathbb{S}$ such that $\omega_f(x_0) = A$. If there are two positive integers $m_1 < m_2$ such that $f^{m_1}(x_0) = f^{m_2}(x_0) = b$ then the ω-limit set A is finite and $b \in A$ which is a contradiction. We may assume that $f^n(x_0) \neq b$ for every positive integer n (in the case when there is just one positive integer m such that $f^m(x_0) = b$ we replace x_0 by $f^{m+1}(x_0)$) and thus $\{f^n(x_0)\}_{n=0}^{\infty} \subset S$. Hence $(\{f^n(x_0)\}_{n=0}^{\infty})^* \subset S^*$ and we have $\omega_{f^*}(x_0^*) = (\omega_f(x_0))^* = A^*$.

The proof of the converse is analogous. □

Before stating the next lemma, let us recall one of Blokh's results from [4].

Proposition 2.6. *Suppose that $f \in \mathcal{C}(\mathbb{S}, \mathbb{S})$ is a transitive map. Then there is a positive integer m, such that $\mathbb{S} = \bigcup_{i=0}^{m-1} K_i$, where all the K_i are connected compact sets, $K_i \cap K_j$ is finite for $i \neq j$, $f(K_i) = K_{i+1}$, $i = 0, 1, \ldots, m-2$, $f(K_{m-1}) = K_0$ and two cases are possible:*

(1) *$P(f) \neq \emptyset$; then $f^{mq}|_{K_i}$ is transitive for any $i = 0, 1, \ldots, m-1$ and any positive integer q,*
(2) *$P(f) = \emptyset$; then $m = 1$, $K_0 = \mathbb{S}$ and f is conjugate to an irrational rotation.*

Lemma 2.7 (Lemma 2.6 in [3] for $\mathcal{C}(I, I)$)**.** *Let f be a map in $\mathcal{C}(\mathbb{S}, \mathbb{S})$ and $A \subset \mathbb{S}$ be a locally expanding set according to the map f with non-empty interior. Then A is a cycle of intervals and $f|_A$ is transitive.*

Proof: Suppose that $A = \mathbb{S}$. By Lemma 2.3 the map f is transitive and by Proposition 2.6 the set A must be a cycle of intervals. Suppose that $A \subset \mathbb{S}_b$. Since A is locally expanding then by Lemma 2.1 $A \subset S$ and by Lemma 2.4 the set $A^* \subset S^*$ is locally expanding. By Lemma 2.6 in [3] the set A^* is a cycle of intervals $A_0^*, \ldots, A_{n-1}^*$ and $f^*|_{A^*}$ is transitive. The map h is a homeomorphism and hence the set $A = h(A^*) = h(A_0^*) \cup \ldots \cup h(A_{n-1}^*)$ and

$$
\begin{aligned}
f(A_i) = \left(h \circ f^* \circ h^{-1}|_S\right)(A_i) &= \left(h \circ f^* \circ h^{-1}|_S\right)(h(A_i^*)) \\
&= h(f^*(A_i^*)) = h(A_{i+1}^*) = A_{i+1},
\end{aligned}
$$

where $A_j = h(A_j^*)$ and j is taken modulo n. This means that A is a cycle of intervals. It remains to show that $f|_A$ is transitive when $A \subset \mathbb{S}_b$. Take two open sets $V, W \subset A$. Then the sets $V^*, W^* \subset S^*$ are open sets and so there is a positive integer n such that $(f^*)^n(V^*) \cap W^* \neq \emptyset$. Hence

$$
f^n(V) \cap W = (h \circ (f^*)^n \circ h^{-1}|_S)(V) \cap W = h((f^*)^n(V^*) \cap W^*) \neq \emptyset.
$$

□

Lemma 2.8 (Lemma 2.7 in [3] for $\mathcal{C}(I, I)$)**.** *Let f be a map in $\mathcal{C}(\mathbb{S}, \mathbb{S})$ and $A \subset \mathbb{S}$ be a locally expanding or an ω-limit set. Then $f(A) = A$.*

Proof: The case of an ω-limit set is trivial and well known. Let A be a locally expanding set. When $A = \mathbb{S}$ then f is transitive (Lemma 2.3) and the lemma is proved. It remains to consider the case when $A \subset \mathbb{S}_b$. By Lemma 2.1 $A \subset S$, and by Lemma 2.7 in [3] we have $f^*(A^*) = A^*$. Clearly

$$f(A) = \left(h \circ f^* \circ h^{-1}|_S\right)(A) = h(f^*(A^*)) = h(A^*) = A.$$

□

We continue by proving the main theorems.

Proof of Theorem 1.1: In the case when $X = I$ the theorem is proved in [3] (Theorem 2.12). It remains to consider the case when $X = \mathbb{S}$. First we show that if $A = \omega_f(x)$ is an ω-limit set then A is locally expanding. In the case $A \subset \mathbb{S}_b$, A^* is an ω-limit set by Lemma 2.5, hence A^* is locally expanding (see Theorem 2.12 in [3]) and by Lemma 2.4, the set A is locally expanding as well. It remains to consider the case when $A = \mathbb{S}$. Since A is an ω-limit set and it has a non-empty interior, A is a cycle of intervals (see Theorem 1.1 in [6]). From this it follows that if $W \subset A$ is an interval, then W has a dense orbit in A and hence there is an $n \in \mathbb{N}$ such that $f^n(W) \cap W \neq \emptyset$. Therefore the union $\bigcup_{i=1}^{\infty} f^i(W)$ is dense in A and has finitely many component intervals. As this is true for every such interval W, it follows that A is locally expanding.

Assume that A is locally expanding. In the case $A \subset \mathbb{S}_b$ we can again prove the theorem by using our Lemmas 2.4 and 2.5, and Theorem 2.12 in [3]. It remains to consider the case when $A = \mathbb{S}$. By Lemma 2.7 the set A is a cycle of intervals and $f|_A$ is transitive. Thus the set A is an ω-limit set. □

Proof of Theorem 1.2: In the case when $X = I$ the theorem is proved in [3] (Theorem 3.1). It remains to consider the case when $X = \mathbb{S}$. We will prove this in several steps.

Case 1. Assume that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \subset \mathbb{S}_b$. Using our Lemma 2.5 and Theorem 3.1 in [3] we get that the set $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n^*}$ is an ω-limit set. By Lemma 2.5 the set $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n}$ is an ω-limit set as well.

Case 2. Next assume that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} = \mathbb{S}$. Then it suffices to show that f is transitive. Take two non-empty open sets $V, W \subset \mathbb{S}$.

Subcase 2.1. If there is an m such that ω_m intersects both V and W we are done since there are positive integers $p < q$ such that $f^p(x_m) \in V$ and $f^q(x_m) \in W$ and consequently, $f^{q-p}(V) \cap W \neq \emptyset$.

Subcase 2.2. If there is no such m, then $V \cap W = \emptyset$. Let $\{\omega_{n_i}\}_{i=1}^{\infty}$ be the subsequence of $\{\omega_n\}_{n=1}^{\infty}$ consisting of ω-limit sets intersecting V. Then $\omega_V = \bigcap_{k=1}^{\infty} \overline{\bigcup_{i=k}^{\infty} \omega_{n_i}} \subset \mathbb{S}_b$ for any $b \in W$, hence, according to the first part, $\omega_V = \omega_f(v)$ is an ω-limit set, $\omega_f(v) \cap W = \emptyset$, and $a \in \omega_f(v)$ is its cluster point from the side T. Similarly, for some w, $\omega_f(w)$ is an ω-limit set intersecting W such that $\omega_f(w) \cap V = \emptyset$ and a is its cluster point from the side T. Let $A = \omega_f(v) \cup \omega_f(w)$.

Subcase 2.2.1. If $A \neq \mathbb{S}$ then $A \subset \mathbb{S}_b$ for some b. We apply the result by Sharkovsky [7] which is also stated in [3]: If, for a map in $\mathcal{C}(I, I)$, two ω-limit sets have a common cluster point from the same side then their union is an ω-limit set. So, by Lemma 2.5 A is an ω-limit set since both $(\omega_V)^*$ and $(\omega_W)^*$ are and have a point a^* as a common cluster point from side T. We have the situation described in Subcase 2.1.

Subcase 2.2.2. $A = \omega_f(v) \cup \omega_f(w) = \mathbb{S}$. Since any ω-limit set in $\mathbb{S}$ is either nowhere dense or a finite union of non-degenerate intervals, and since $\omega_f(v) \cap W = \emptyset = \omega_f(w) \cap V$, both $\omega_f(v)$ and $\omega_f(w)$ are finite unions of intervals. If $\omega_f(v) \cap \omega_f(w)$ is infinite then the two ω-limit sets have an interval in common and the transitivity is easily proven. If the intersection $\omega_f(v) \cap \omega_f(w)$ would be finite then the condition with the T-side must be violated since the intersection contains a. □

Proof of Theorem 1.3: In the case when $X = I$ the theorem is proved in [3] (Theorem 3.2). It remains to consider the case when $X = \mathbb{S}$. Let $\{\omega_1, \omega_2, \dots\}$ be a sequence of ω-limit sets converging in the Hausdorff metric to a set A. Choosing a subsequence (if necessary) we may also assume that there exists a point a, a side T of a and points $a_n \in \omega_n$, $a_n \neq a$ converging to a from T. As the original sequence converges to A, the subsequence does as well. To finish the proof it remains to use Theorem 1.2 and to show that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} = A$. Since we consider Hausdorff metric and all the sets ω_n are closed then the set A is closed as well. Hence it is clear that $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \supset A$. Consider the sequence of open sets $\{A_{1/n}\}_{n=1}^{\infty}$ where $A_\varepsilon := \{x \in X; \operatorname{dist}(x, A) < \varepsilon\}$, $\operatorname{dist}(x, A) := \inf\{d(x, a); a \in A\}$ and d is the metric on X, and note that for every m there is a positive integer k such that $\overline{\bigcup_{n=k}^{\infty} \omega_n} \subset A_{1/m}$. Therefore $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \omega_n} \subset A$. □

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(*Received* February 2, 2001, *revised* April 18, 2002)