A remark on the minimal displacement problem in spaces uniformly rotund in every direction

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Abstract. We give an example of uniformly rotund in every direction space for which the minimal displacement characteristic is maximal.

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1. Introduction

Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space with the closed unit ball B_X and the unit sphere S_X. For any $k \geq 0$, let $L(k)$ denote the class of Lipschitz mappings $T : B_X \to B_X$ with constant k. By d_T we will denote the minimal displacement of T

$$d_T = \inf_{x \in B_X} \|x - Tx\|.$$

Goebel [5] was the first who gave examples of Lipschitzian mappings with positive minimal displacement. He also introduced some useful functions which describe this problem. We will deal only with the minimal displacement characteristic of X which can be defined as

$$\psi_X(k) = \sup\left\{d_T : T \in L(k)\right\}, \quad k \geq 1.$$

It is known that for any space X

$$\psi_X(k) \leq 1 - \frac{1}{k}.$$

There are some "square" spaces like c_0, $C[0,1]$ for which $\psi_X(k) = 1 - \frac{1}{k}$. In the case of uniformly rotund spaces it is known that $\psi_X(k) < 1 - \frac{1}{k}$ for $k > 1$. In particular in Hilbert space the following estimate holds

$$\psi_H(k) \leq \left(1 - \frac{1}{k}\right)\sqrt{\frac{k}{k+1}}.$$

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For a long time it has been believed that: "If the unit ball in the space X is more rotund than the unit ball in the space Y then it should be that $\psi_X(k) \leq \psi_Y(k)$." We show that it is not true by giving an example of uniformly rotund in every direction space for which $\psi_X(k) = 1 - \frac{1}{k}$. On the other hand, in the case of the classical space l^1 it is known that

$$\psi_{l^1}(k) \leq \begin{cases} \frac{2+\sqrt{3}}{4}\left(1 - \frac{1}{k}\right) & \text{for } k \in \left[1, 3+2\sqrt{3}\right] \\ \frac{k+1}{k+3} & \text{for } k \in \left(3+2\sqrt{3}, \infty\right). \end{cases}$$

But the space l^1 is not even strictly convex. For a wider discussion of the topics on the minimal displacement problem we refer the reader to the book [6]. Newest results can be found in papers by the author.

Recall that the modulus of convexity in direction z, $\|z\| = 1$, is the function $\delta_z : [0,2] \to [0,1]$ defined as

$$\delta_z(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, x - y = \epsilon z\right\}.$$

If $\delta_z(\epsilon) > 0$ for any $\epsilon > 0$ and $\|z\| = 1$ then the space X is said to be uniformly rotund in every direction (URED). Before we give an example of a URED space with $\psi_X(k) = 1 - \frac{1}{k}$ we prove two technical lemmas for the classical space $C[0,1]$.

2. Results

Lemma 1. *For every $k \geq 1$ there exists a mapping $T : B_{C[0,1]} \to S_{C[0,1]}$, of class $L(k)$, with $d_T = 1 - \frac{1}{k}$, and such that*

$$(Tx)(0) = -1 \quad \text{and} \quad (Tx)(1) = 1$$

for every $x \in B_{C[0,1]}$.

Proof: Let us define a mapping $T_1 : B_{C[0,1]} \to C[0,1]$ as $(T_1 x)(t) = x(t) + 4t - 2$ and define the mapping $T : B_{C[0,1]} \to S_{C[0,1]}$ as a "composition", i.e. $(Tx)(t) = f((T_1 x)(t))$, where the function $f : \mathbb{R} \to [-1,1]$ is given for $k > 1$ as

$$f(t) = \begin{cases} -1 & \text{if } t \in \left(-\infty, -\frac{1}{k}\right) \\ kt & \text{if } t \in \left[-\frac{1}{k}, \frac{1}{k}\right] \\ 1 & \text{if } t \in \left(\frac{1}{k}, \infty\right). \end{cases}$$

The mapping T_1 is nonexpansive and the function f is Lipschitzian with constant k which implies that $T \in L(k)$. Observe that $(Tx)(0) = -1$ and $(Tx)(1) = 1$ for every $x \in B_{C[0,1]}$. Moreover observe that if $(T_1 x)\left(\frac{1}{2}\right) \geq 0$, then from the inequality $(T_1 y)(0) \leq -1$ valid for every $y \in B_{C[0,1]}$ we have that there exists $t_0 \in \left(0, \frac{1}{2}\right)$ such that $(T_1 x)(t_0) = -\frac{1}{k}$. From this equality we obtain that

$$x(t_0) > (T_1 x)(t_0) = -\frac{1}{k} > -1 = (Tx)(t_0),$$

which implies $\|x - Tx\| > 1 - \frac{1}{k}$. Analogously if $(T_1 x)\left(\frac{1}{2}\right) < 0$, the inequality $(T_1 x)(1) \geq 1$ implies that there exists $t_1 \in \left(\frac{1}{2}, 1\right)$, for which $(T_1 x)(t_1) = \frac{1}{k}$. This implies

$$x(t_1) < (T_1 x)(t_1) = \frac{1}{k} < 1 = (Tx)(t_1),$$

and further $\|x - Tx\| > 1 - \frac{1}{k}$. This, combined with the general fact that $\psi_X(k) \leq 1 - \frac{1}{k}$ for any Banach space, implies that $d_T = 1 - \frac{1}{k}$, which finishes the proof. □

Observe that we can prove slightly more, namely let for $k = 1$ the map T be given by a formula

$$(Tx)(t) = \max\{-1, \min[1, (T_1 x)(t)]\}.$$

This map is fixed point free because $(Tx)(t) > x(t)$ for some $t > \frac{1}{2}$ or $(Tx)(t) < x(t)$ for some $t < \frac{1}{2}$. We obtained, in both cases ($k > 1$ and $k = 1$), that the infimum in the definition of the minimal displacement is not attained for any $x \in B_{C[0,1]}$.

Now we can generalize Lemma 1.

Lemma 2. *Let $0 \leq a < b \leq 1$. For every $k \geq 1$ there exists a mapping $T_{[a,b]} : B_{C[0,1]} \to S_{C[0,1]}$ of class $L(k)$ such that for every $x \in B_{C[0,1]}$ the following conditions hold*

$$\left(T_{[a,b]} x\right)(t) = 0 \quad \textit{for every} \ \ t \in [0, a] \cup [b, 1]$$

and

$$\max_{t \in [a,b]} \left| x(t) - \left(T_{[a,b]} x\right)(t) \right| > 1 - \frac{1}{k}.$$

Proof: Let us choose c, d such that $a < c < d < b$. Because the spaces $C[0,1]$ and $C[c,d]$ are isometric then, according to the proof of previous lemma, for any $k \geq 1$ there exists a map $T : B_{C[c,d]} \to S_{C[c,d]}$, of class $L(k)$, such that $(Tx)(c) = -1$, $(Tx)(d) = 1$ and $\|x - Tx\|_{C[c,d]} > 1 - \frac{1}{k}$ for every $x \in B_{C[a,b]}$. Now let us define a map $T_{[a,b]} : B_{C[0,1]} \to S_{C[0,1]}$ as follows: $\left(T_{[a,b]} x\right)(t) = (Tx)(t)$ for $t \in [c, d]$ and $\left(T_{[a,b]} x\right)(t) = 0$ for every $t \in [0, a] \cup [b, 1]$. On two intervals:

$[a,c]$ and $[d,b]$ we define $T_{[a,b]}x$ as affine functions such that $\left(T_{[a,b]}x\right)(c) = -1$, $\left(T_{[a,b]}x\right)(a) = \left(T_{[a,b]}x\right)(b) = 0$ and $\left(T_{[a,b]}x\right)(d) = 1$ for any $x \in B_{C[c,d]}$. The map $T_{[a,b]} \in L(k)$ and $\left\|x - T_{[a,b]}x\right\| \geq \max_{t\in[c,d]} |x(t) - (Tx)(t)| > 1 - \frac{1}{k}$ according to the previous lemma. □

Now we can proceed to the example.

Example. Let $\{t_i\}_{i=1}^{\infty}$ be a dense sequence in the interval $[0,1]$. It can be shown that the space of continuous functions $X = C[0,1]$ equipped with the norm

$$\|x\|_X = \|x\|_{C[0,1]} + \left[\sum_{i=1}^{\infty}\left(2^{-i}x(t_i)\right)^2\right]^{1/2}$$

is URED (see [9]). Fix an arbitrary $\epsilon \in (0,1)$. Find then an interval $[a,b] \subset [0,1]$ such that

$$\sum_{i, t_i\in[a,b]} 2^{-2i} \leq \epsilon^2.$$

Let $T_{[0,1]} : B_{C[0,1]} \to S_{C[0,1]}$ be the mapping from Lemma 2. Then let us consider the mapping $T_\epsilon x = (1-\epsilon)T_{[a,b]}x$. Observe that

$$\begin{aligned}
\|T_\epsilon x\|_X &= \max_{t\in[0,1]} |(T_\epsilon x)(t)| + \left[\sum_{i=1}^{\infty}\left(2^{-i}(T_\epsilon x)(t_i)\right)^2\right]^{1/2} \\
&= \max_{t\in[a,b]} |(T_\epsilon x)(t)| + \left[\sum_{t_i\in[a,b]}\left(2^{-i}(T_\epsilon x)(t_i)\right)^2\right]^{1/2} \\
&\leq 1 - \epsilon + \epsilon = 1,
\end{aligned}$$

which shows that $T_\epsilon : B_X \to B_X$. We prove that T_ϵ is Lipschitzian. We have

$$\begin{aligned}
\|T_\epsilon x - T_\epsilon y\|_X &= \max_{t\in[0,1]} |(T_\epsilon x)(t) - (T_\epsilon y)(t)| \\
&\quad + \left[\sum_{i=1}^{\infty}\left(2^{-i}\left[(T_\epsilon x)(t_i) - (T_\epsilon y)(t_i)\right]\right)^2\right]^{1/2} \\
&= \max_{t\in[a,b]} |(T_\epsilon x)(t) - (T_\epsilon y)(t)| \\
&\quad + \left[\sum_{t_i\in[a,b]}\left(2^{-i}\left[(T_\epsilon x)(t_i) - (T_\epsilon y)(t_i)\right]\right)^2\right]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\epsilon)k \max_{t\in[a,b]} |x(t)-y(t)| \\
&\quad + (1-\epsilon)k \max_{t\in[a,b]} |x(t)-y(t)| \left[\sum_{t_i\in[a,b]} 2^{-2i}\right]^{1/2} \\
&\leq \left(1-\epsilon^2\right) k\|x-y\|.
\end{aligned}
$$

This implies that $T_\epsilon \in L\left(\left(1-\epsilon^2\right)k\right)$. The minimal displacement of T_ϵ can be evaluated in the following way

$$
\begin{aligned}
\|x-T_\epsilon x\|_X &= \max_{t\in[0,1]} |x(t)-(T_\epsilon x)(t)| + \left[\sum_{i=1}^{\infty}\left(2^{-i}\left[x(t_i)-(T_\epsilon x)(t_i)\right]\right)^2\right]^{1/2} \\
&\geq \max_{t\in[a,b]} |x(t)-(T_\epsilon x)(t)| \\
&\geq 1-\frac{1}{k}-\epsilon.
\end{aligned}
$$

From the definition of the minimal displacement characteristic we have

$$\psi_X\left(\left(1-\epsilon^2\right)k\right) \geq 1-\frac{1}{k}-\epsilon.$$

This holds for every $k \geq 1$ and for every $\epsilon > 0$. Since ϵ can be arbitrarily small and the function ψ_X is continuous (see [6]) we deduce that

$$\psi_X(k) = 1-\frac{1}{k}\,.$$

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