Totally non-remote points in $\beta\mathbb{Q}$

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Abstract. Totally nonremote points in $\beta\mathbb{Q}$ are constructed. The number of these points is $2^{\mathfrak{c}}$.

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1. Introduction

All spaces considered are normal. We follow [2] in our terminology and notations. For a space X we identify a point a of the Čech-Stone remainder $X^* = \beta X - X$ with

$$\{A \subseteq X : A \ \text{ closed in } \ X \ \text{ and } \ a \in \mathrm{Cl}_{\beta X} A\}.$$

We call $a \in X^*$

far: if no element of a is discrete;
crowded: if every element of a is dense-in-itself;
remote: if no element of a is nowhere dense; and
totally nonremote: if for every $A \in a$ there is $B \in a$ that is nowhere dense in A.

After the introduction of remote points in [1] and [3], they became one of the most intriguing in the theory of Čech-Stone compactifications. What about the existence of points with antipodial properties? E. van Douwen set the following question:

Does $\mathbb{Q}$ have a crowded totally nonremote point ?

and showed, in particular, that CMA (Martin's Axiom for countable posets) implies yes [2]. Every totally nonremote point is, obviously, far.

We prove naively the following

Theorem 1.1. *There are totally nonremote points in $\beta\mathbb{Q}$. The number of these points is $2^{\mathfrak{c}}$.*

The question above remains open.

2. Proofs

In this paper $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ is the set of rational numbers. We will use $S = \bigcup_{n=0}^{\infty} \mathbb{N}^{2n}$ as a set of indexes.

By recursion on $s \in S$ we will choose p_s and O_s as follows:

Let $O_\emptyset = \mathbb{Q}$ and $p_\emptyset = q_1$.

If O_s and p_s are found we let $\{O_{s,i} : i \in \mathbb{N}\}$ be a strictly decreasing local base at p_s, consisting of clopen sets and with $O_{s,1} = O_s$. For every $U_{s,i} = O_{s,i} - O_{s,i+1}$ we choose an infinite pairwise disjoint clopen cover $\mathcal{U}_{s,i} = \{O_{s,i,j} : j \in \mathbb{N}\}$ of $U_{s,i}$. We let $p_{s,i,j} \in O_{s,i,j}$ be the q_n with minimal index (in this way we get $\{p_s : s \in S\} = \mathbb{Q}$). Finally, $\mathcal{O} = \{O_s : s \in S\}$.

We use $\mathcal{F}$ to denote the set of maps f from S to $\operatorname{Exp}\mathcal{O}$ with the properties that $f(s) \subset \bigcup_i \mathcal{U}_{s,i}$ and $|f(s) \cap \mathcal{U}_{s,i}| \leq 1$ for all s and i. For every $f \in \mathcal{F}$ we put $\mathcal{O}(f) = \bigcup\{f(s) : s \in S\}$.

We denote $\mathcal{D}$ all closed and discrete subsets of $\mathbb{Q}$. For every $s \in \mathbb{N}^{2n}$ and $D \in \mathcal{D}$ we set

$$i(s, D) = \min\{i : O_{s,i} \cap D \subseteq \{p_s\}\} = \min\{i : (\forall j \geq i)(U_{s,j} \cap D = \emptyset)\}.$$

We say that s is D-good if $s_{2k+1} < i(s \upharpoonright 2k, D)$ for all $k < n$, and we put

$$\mathcal{O}(D) = \{U_{s,i(s,D)} : s \in S \text{ is } D\text{-good}\}.$$

Claim 1. *For any $D \in \mathcal{D}$, $\mathcal{O}(D)$ is locally finite in $\mathbb{Q}$.*

Proof: Let $p_s \in \mathbb{Q}$. If the index s is D-good, then the neighborhood $Op_s = O_{s,i(s,D)+1}$ does not intersect any member of $\mathcal{O}(D)$. Otherwise, we choose the maximal k such that $t = s \upharpoonright 2k$ is D-good. If $s_{2k+1} > i(t, D)$ then $Op_s = O_s$ meets no member of $\mathcal{O}(D)$ and if $s_{2k+1} = i(t, D)$ then $U_{t,i(t,D)}$ is the unique member of $\mathcal{O}(D)$ that O_s intersects. □

Claim 2.

$$\bigcap_{k=1}^{n} \big(\bigcup \mathcal{O}(D_k)\big) - \bigcup_{j=1}^{n} \bigcup \mathcal{O}(f_j) \neq \emptyset$$

for any $n \in \mathbb{N}$, $D_k \in \mathcal{D}$ and $f_j \in \mathcal{F}$.

Proof: We shall construct an $s \in S$ such that p_s belongs to the set in question. To begin, let $s \upharpoonright 0 = \emptyset$ and $F_0 = \{1, 2, \ldots, n\}$.

Assume $a \upharpoonright 2m$ and F_m have been found with $F_m \neq \emptyset$ and $s \upharpoonright 2m$ a D_k-good sequence for $k \in F_m$. Let $s_{2m+1} = \min\{(s \upharpoonright 2m, D_k) : k \in F_m\}$ and choose s_{2m+2} so large that $O_{s,s_{2m+1},s_{2m+2}} \in \mathcal{U}_{s,s_{2m+1}} - \bigcup_{k \leq n} f_k(s)$. Let $F_{m+1} = \{k \in F_m : i(s \upharpoonright 2m, D_k) > s_{2m+1}\}$; observe that F_{m+1} is a proper subset of F_m and that $s \upharpoonright (2m+2)$ is D_k-good for $k \in F_{m+1}$.

There will be an m with $F_m = \emptyset$; then $s = s \upharpoonright 2m$ is as required. □

It follows that the point a in Claim 3 does really exist.

Claim 3. *Every point $a \in \mathbb{Q}^*$ such that*

$$a \in \bigcap\{\mathrm{Cl}_{\beta\mathbb{Q}} \bigcup \mathcal{O}(D) : D \in \mathcal{D}\} - \bigcup\{\mathrm{Cl}_{\beta\mathbb{Q}} \bigcup \mathcal{O}(f) : f \in \mathcal{F}\}$$

is totally nonremote.

Proof: Let $A \in a$. If $\mathrm{Cl}_X D \in a$ for $D = \{q \in A : q \text{ is isolated in } A\}$, then $\mathrm{Cl}_X D - D \in a$, because a is a far point. Otherwise, if $Oa \cap D = \emptyset$ for a clopen neighborhood of a, then $G = Oa \cap A$ has no isolated points. Define $f_G \in \mathcal{F}$ so that for any $s \in S$, $\bigcup f_G(s)$ meets every nonempty intersection $U_{s,i} \cap G$. Then $G - \bigcup \mathcal{O}(f_G) \in a$ is nowhere dense in A. □

Claim 4. *The number of totally nonremote points in $\beta\mathbb{Q}$ is $2^{\mathfrak{c}}$.*

Proof: For every $Q_j = (\sqrt{2}j, \sqrt{2}j + 1) \cap \mathbb{Q}$ we fix a totally nonremote point $a_j \in Q_j^*$ and put $A = \{a_j : j \in \mathbb{N}\}$. Then $Y = \mathbb{Q} \cup A$ is normal and $\mathrm{Cl}_{\beta\mathbb{Q}} Y$ is equivalent to βY, because $\mathbb{Q} \subset Y \subset \beta\mathbb{Q}$. Hence $\mathrm{Cl}_{\beta\mathbb{Q}} A \subset \mathbb{Q}^*$ has cardinality $2^{\mathfrak{c}}$. Let $a \in \mathrm{Cl}_{\beta\mathbb{Q}} A$ and $B \in a$. For each $a_j \in A$ there is $G_j \subset Q_j$, which belongs to a_j and has nowhere dense intersection (possibly empty) with B. Then $G = \bigcup_{j \in \mathbb{N}} G_j$ belongs to a and has nowhere dense intersection with B. Our proof is complete. □

References

[1] van Douwen E.K., *Remote points*, Dissertations Math. **188** (1981), 1–45.
[2] van Douwen E.K., *Better closed ultrafilters on Q*, Topology Appl. **47** (1992), 173–177.
[3] Fine N.J., Gillman L., *Remote points in βR*, Proc. Amer. Math. Soc. **13** (1962), 29–36.

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