$C^{1,\alpha}$ local regularity for the solutions of the p-Laplacian on the Heisenberg group. The case $1+\frac{1}{\sqrt{5}} < p \leq 2$

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Abstract. We prove the Hölder continuity of the homogeneous gradient of the weak solutions $u \in W^{1,p}_{\rm loc}$ of the p-Laplacian on the Heisenberg group $\mathcal{H}^n$, for $1+\frac{1}{\sqrt{5}} < p \leq 2$.

Keywords: degenerate elliptic equations, weak solutions, regularity, higher differentiability

Classification: 35D10, 35J60, 35J70

1. Introduction

In this paper we deal with the regularity of the weak solutions $u \in W^{1,p}_{\rm loc}(\Omega, X)$, $1+\frac{1}{\sqrt{5}} < p \leq 2$, of the equation

$$\mathrm{div}_{\mathcal{H}}\, \vec{a}(Xu) = 0, \tag{1}$$

where $\mathrm{div}_{\mathcal{H}}\, \vec{a}(Xu) = \sum_{k=1}^{2n} X_k a^k(Xu)$ and $a^k(q) = |q|^{p-2} q_k$, $k = 1, \dots, 2n$. Here Ω is an open subset of the Heisenberg group $\mathcal{H}^n$, the vector fields X_k, $k = 1, \dots, 2n$, are the generators of the corresponding Lie algebra with their commutators up to the first order and $Xu = (X_1 u, \dots, X_{2n} u)$.

Our main object is the local Hölder continuity of the homogeneous gradient Xu. To this aim we consider approximate equations and we prove the property uniformly for their solutions. Then we gain the result for the solutions u of the equation (1) via a limit argument.

Let us recall the definitions of the functional spaces needed (see [7]). For any positive integer j, let us set $s = (s_1, \dots, s_j)$, where $s_i \in \{1, \dots, 2n\}$ for any $i = 1, \dots, j$, and set $|s| = j$.

Let us denote by X_s the operator $X_{s_1} X_{s_2} \dots X_{s_j}$. For any $q \geq 1$ and any positive integer h, $W^{h,q}(\mathcal{H}^n, X)$ denotes the set of functions $f \in L^q(\mathcal{H}^n)$ such that $X_s f \in L^q(\mathcal{H}^n)$ for $|s| \leq h$, with norm $\|f\|_{h,q} = \|f\|_{L^q(\mathcal{H}^n)} + \sum_{|s| \leq h} \|X_s f\|_{L^q(\mathcal{H}^n)}$.

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$W^{h,q}_{\mathrm{loc}}(\Omega, X)$ is the set of functions f such that $\varphi f \in W^{h,q}(\mathcal{H}^n, X)$ for any $\varphi \in C_0^\infty(\Omega)$.

We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega, X)$ is a local weak solution of (1) if

$$\int_\Omega a^k(Xu)\, X_k(\varphi)\, dx = 0 \tag{2}$$

for all $\varphi \in W^{1,p}(\Omega, X)$ with $\operatorname{supp}\varphi \subset \Omega$. Here and in the following repeated indices denote summation.

We can now state the main results of this paper.

From now on Ω' will denote an arbitrary open subset of Ω such that $\Omega' \subset\subset \Omega$ and $B(\rho)$, $\rho > 0$, will denote any homogeneous ball of radius ρ (see Section 3).

Theorem 1.1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, X)$, $1 + \frac{1}{\sqrt{5}} < p \le 2$, be a local weak solution of (1). Then for any $\sigma \in (0,1)$ there exists a positive constant $\gamma(\sigma)$ depending only on σ and the data such that, for any homogeneous ball $B(R) \subset\subset \Omega'$,*

$$\|Xu\|_{\infty, B(R-\sigma R)} \le \gamma(\sigma)\left(\frac{1}{|B(R)|}\int_{B(R)} |Xu|^p\, dx\right)^{1/p}. \tag{3}$$

In particular $|Xu| \in L^\infty_{\mathrm{loc}}(\Omega')$ and for every compact $K \subset \Omega'$, there exists a constant $C_0 > 0$ depending only on the data and on $\operatorname{dist}(K, \partial\Omega')$ *such that*

$$\|Xu\|_{\infty,K} \le C_0.$$

Theorem 1.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega, X)$, $1 + \frac{1}{\sqrt{5}} < p \le 2$, be a local weak solution of (1). Then for any homogeneous ball $B(R) \subset\subset \Omega'$ there exist positive constants ν and $\eta \in (0,1)$ depending only on the data and on* $\operatorname{dist}(B(R), \partial\Omega')$ *such that*

$$\max_{i=1,\dots,2n} \operatorname{osc}_{B(\rho)} X_i u \le \nu \left(\frac{\rho}{R}\right)^\eta \sup_{B(R/2)} |Xu| \tag{4}$$

for all $\rho < R/2$. In particular Xu is locally Hölder continuous in Ω', i.e. for every compact $K \subset \Omega'$ there exist $C_1 > 0$ and $\alpha \in (0,1)$ depending only on data and on $\operatorname{dist}(K, \partial\Omega')$ *such that*

$$|Xu(x) - Xu(y)| \le C_1 d(x,y)^\alpha, \quad x, y \in K,$$

where d denotes the homogeneous distance associated to $\mathcal{H}^n$ (see Section 3).

Our results extend to the Heisenberg group setting some properties which hold, in the Euclidean context, for the solutions of the p-Laplacian, but even of more general nonlinear elliptic equations.

Let us recall in particular on this subject the papers of K. Uhlenbeck [27], N.N. Ural'tzeva [28], L. Evans [6] for $p \geq 2$, and P. Tolksdorf [26], E. Di Benedetto [5] and J.L. Lewis [14] for $1 < p < +\infty$.

In general these methods consist in differentiating the equation and proving that the derivatives of the solutions solve another partial differential equation.

But this procedure does not fit the Heisenberg context due to the lack of commutativity of the vector fields. In fact, even difference quotiens along any left-invariant vector field produce derivatives in the second commutator's direction.

If $p = 2$ L. Capogna [1] solved this problem for sub-elliptic equations having the p-Laplacian as a prototype, establishing at first a control on the L^2 norm of the derivatives in the commutator's direction. This is the key point in the matter. Thanks to this result he could prove the differentiability of the equation and gradient's Hölder continuity.

In [18] we proved the same result for the p-Laplacian when $2 \leq p < 1+\sqrt{5}$. Here we extend it to $1 + \frac{1}{\sqrt{5}} < p < 2$. Because of the worsening of the degeneracy, in both cases we are forced to smooth the problem introducing regularized equations

$$\operatorname{div}_{\mathcal{H}} \vec{a}_\epsilon(Xu_\epsilon) = 0 \tag{5}$$

for small $\epsilon > 0$, where $\vec{a}_\epsilon(q) = [(\epsilon + |q|^2)^{(p-2)/2} q]$. Following an adaptation of Di Benedetto's method [5] we attempt to obtain "uniform" Hölder continuity for Xu_ϵ. However this method requires differentiability of equations (5) too, that in turn requires a control in L^p of some derivatives of u_ϵ.

If $p > 2$ we could limit ourselves to establish an L^p estimate for Tu_ϵ (T is the second commutator of the vector fields). But the case $p < 2$ is much more tricky. In fact, besides the L^p estimate for Tu_ϵ we proved in [19], here we further need an L^p-control of the derivatives of Tu_ϵ along the vector fields. We prove this result via an iterated application of fractional difference quotiens and repeated inclusions between functional spaces. As this is a crucial step we treat it apart in Section 7. The limit $p > 1 + \frac{1}{\sqrt{5}}$ comes from [19]. A different technique could improve the result.

Finally we would give a brief description of the content of each section. Sections 2 and 3 are devoted to recall basic knowledge and preliminary results. In Section 4 we prove the differentiability of equations (5). We multiply them by a particular test function, defined by double difference quotiens and apply a Lemma of Cutrí-Garroni [4] which enables us to commute the vector fields with the double difference quotiens. Thanks to this tool and the L^p estimate of Tu_ϵ from Section 7, we can then apply Giusti's method [Giusti] and conclude about the $W^{2,p}$ local regularity for u_ϵ (see Theorem 4.1). This is enough to differentiate equations (5).

Thanks to this tool in Sections 5 and 6 we prove boundedness and local Hölder continuity of $X_i u_\epsilon$ by the methods of Di Benedetto [5]. As these estimates are uniform in ϵ, this enables us to establish Theorems 1.1 and 1.2 about u by standard arguments [13], [14], possibly up to subsequences.

The general plan of this paper is the same as of [19]. The principal differences concern the crucial Sections 4 and 7 and part of Section 6. They are outlined in detail.

We limit ourselves to sketch the remainder, referring the reader to [5], [18], [19] for a closer examination.

2. Basic knowledge

The Heisenberg group $\mathcal{H}^n$ is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ with the following group law: for all $x = (x', t) = (x_1, \dots, x_{2n}, t)$, $y = (y', s) = (y_1, \dots, y_{2n}, s)$,

$$x \circ y = (x' + y', t + s + 2[x', y'])$$

where $[x', y'] := \sum_{i=1}^{n} (y_i x_{i+n} - x_i y_{i+n})$.

$\mathcal{H}^n$ is a homogeneous group, that is a group with dilations. A norm for $\mathcal{H}^n$ which is homogeneous of degree 1 with respect to the dilations is

$$|x|^4 = |(x', t)|^4 = |x'|^4 + t^2 \quad \text{for any} \quad x = (x', t) \in \mathcal{H}^n$$

and the associated distance is

$$d(x, y) := |y^{-1} \circ x|, \quad x, y \in \mathcal{H}^n, \quad \text{where} \quad y^{-1} = -y.$$

$B(x, r)$ will denote the homogeneous ball centered in $x \in \mathcal{H}^n$ with radius $r > 0$.

The Lie algebra $\mathcal{L}(X)$ of left-invariant vector fields corresponding to $\mathcal{H}^n$ is generated by

$$\begin{aligned} X_i &= \partial_{x_i} + 2x_{i+n}\partial_t \\ X_{i+n} &= \partial_{x_{i+n}} - 2x_i\partial_t \\ T &= -4\partial_t \end{aligned}$$

for $i = 1, \dots, n$, where $[X_i, X_{i+n}] = -[X_{i+n}, X_i] = T$, $i = 1, \dots, n$, and $[X_j, X_k] = 0$ in any other case.

The vector fields X_i do not commute with right translations. In particular we cannot interchange them with difference quotiens operators

$$D_h w(x) = \frac{w(x \circ h) - w(x)}{|h|}, \quad D_{-h} w(x) = \frac{w(x \circ h^{-1}) - w(x)}{-|h|}$$

for any $x \in \mathcal{H}^n$, $h = (h', 0)$, $h_i \geq 0$ for any $i = 1, \dots, 2n$.

For any $i = 1, \dots, 2n$ let h^i, $(h^i)^{-1}$ be the elements of the group whose j-th component is h_i, or resp. $-h_i$, if $j = i$ and 0 otherwise. We have

$$X_i D_{\pm h^i} = D_{\pm h^i} X_i \tag{6}$$

for every $i = 1, \dots, 2n$, but $X_k D_{\pm h^i} \neq D_{\pm h^i} X_k$ if $k \neq i$.

For any $s > 0$ let h^*_s, $(h^*_s)^{-1}$ be the elements of the group whose $(2n+1)$-th component is s or $-s$ respectively and 0 otherwise. For any $s > 0$ and any $\alpha \in (0, 1)$ let

$$D_{h^*_{s,\alpha}} w(x) = \frac{w(x \circ h^*_s) - w(x)}{s^\alpha}, \quad D_{-h^*_{s,\alpha}} w(x) = \frac{w(x \circ (h^*_s)^{-1}) - w(x)}{-s^\alpha}.$$

For every $i = 1, \dots, 2n$ we have

$$X_i D_{\pm h^*_{s,\alpha}} = D_{\pm h^*_{s,\alpha}} X_i. \tag{7}$$

3. Difference quotiens and a priori bounds

For more details on this argument see also [4], [1]. Let us consider any $w \in C^\infty_0(\Omega)$ and any $h = (h', 0) = (h_1, \dots, h_{2n}, 0)$ with $h_i \geq 0$ for $i = 1, \dots, 2n$.

Remark 3.1. It is easy to show that

$$D_{-h} D_h w(x) = \frac{2w(x) - w(x \circ h) - w(x \circ h^{-1})}{-|h|^2} = D_h D_{-h} w(x). \tag{8}$$

Remark 3.2. For any function $w \in L^p(\Omega)$ with compact support $\omega \subset \Omega$, for any $f \in L^{p/(p-1)}_{\mathrm{loc}}(\Omega)$ and for any h such that $|h| < d(\omega, \partial\Omega)$ we have

$$\int f D_{\pm h} w \, dx = -\int w D_{\mp h} f \, dx. \tag{9}$$

Lemma 3.3 (see [4, Lemma 2.7], [18, Lemma 3.3])**.** *For any $w \in C^\infty_0(\Omega)$ and for any $i = 1, \dots, n$,*

$$X_i(D_{-h} D_h w(x)) = D_{-h} D_h (X_i w(x)) - \frac{h_{i+n}}{2|h|^2}[(Tw)(x \circ h) - (Tw)(x \circ h^{-1})], \tag{10}$$

$$\begin{aligned} &X_{i+n}(D_{-h} D_h w(x)) \\ &\quad = D_{-h} D_h (X_{i+n} w(x)) + \frac{h_i}{2|h|^2}[(Tw)(x \circ h) - (Tw)(x \circ h^{-1})]. \end{aligned} \tag{11}$$

Lemma 3.4 (see [4, Lemma 2.9], [18, Lemma 3.4]). *For any $w \in C_0^\infty(\Omega)$ and for any $i = 1, \ldots, 2n$,*

$$\lim_{h_i \to 0} D_{\pm h^i} w = X_i w. \tag{12}$$

Lemma 3.5 ([1, Proposition 2.3]). *Let $p > 1$ and let $\psi \in L^p_{\mathrm{loc}}(\Omega)$ and $g \in C_0^\infty(\Omega)$ with $\omega = \operatorname{supp} g \subset\subset \Omega$. Let $i \in \{1, \ldots, 2n\}$. If there are some constants $\epsilon_0 > 0$ and $C > 0$ such that*

$$\sup_{0<h_i<\epsilon_0} \int_\omega |D_{\pm h^i} \psi|^p \, dx \le C^p \tag{13}$$

then $X_i\psi \in L^p(\omega)$ and $\|X_i\psi\|_{L^p(\omega)} \le C$. Conversely, if $X_i\psi \in L^p_{\mathrm{loc}}(\Omega)$, then (13) *holds for any $\omega = \operatorname{supp} g \subset\subset \Omega$, $g \in C_0^\infty(\Omega)$ and $C = 2\|X_i\psi\|_{L^p(\omega)}$. The same result holds if we substitute $D_{\pm h^i}$ and X_i by $D_{\pm h^*_{s,1}}$ and ∂_t, respectively.*

Lemma 3.6 (see [1, Theorem 2.6], [19, Lemma 3.6]). *Let $\psi \in C^\infty(\Omega)$ and let $g \in C_0^\infty(\omega)$, with $\operatorname{supp} \omega \subset\subset \Omega$. Then there exists a positive constant C such that, for any small $\epsilon_0 > 0$ and any $p > 1$*

$$\begin{aligned} &\sup_{0<s<\epsilon_0} \int_\Omega |D_{\pm h^*_{s,1/2}}(\psi g)|^p \, dx \\ &\le C \sum_{i=1}^{2n} \left\{ \sup_{0<h_i<\epsilon_0} \int_\Omega |D_{h^i}(\psi g)|^p \, dx + \sup_{0<h_i<\epsilon_0} \int_\Omega |D_{-h^i}(\psi g)|^p \, dx \right\}. \end{aligned} \tag{14}$$

From Lemmas 3.5 and 3.6 we easily deduce

Corollary 3.7. *Let the assumptions of Lemma* 3.6 *hold true. Then there exists a constant $C > 0$ such that, for any small $\epsilon_0 > 0$ and any $p > 1$*

$$\sup_{0<s<\epsilon_0} \int_\Omega |D_{\pm h^*_{s,1/2}}(\psi g)|^p \, dx \le C \int_\Omega |X(\psi g)|^p \, dx. \tag{15}$$

4. $W^{2,p}_{\mathrm{loc}}$ regularity for the solutions of the approximate equation

This section is devoted to prove the $W^{2,p}$ local regularity of the solutions of equations (5) and, as a by-product, to differentiate equations (5). This will be a basic tool in order to apply the Di Benedetto's machinery [5] to obtain uniform boundedness and Hölder continuity of ∇u_ϵ (see Sections 5, 6).

Here we will exploit a local $W^{1,p}$ estimate of Tu_ϵ whose proof can be found in Section 7.

Theorem 4.1. *Let* $1 + \frac{1}{\sqrt{5}} < p \leq 2$ *and, for any* $\epsilon \in (0,1)$*, let* $u_\epsilon \in W^{1,p}_{\mathrm{loc}}(\Omega, X)$ *be a local weak solution of* (5).

Then $u_\epsilon \in W^{2,p}_{\mathrm{loc}}(\Omega, X)$ *and, for any* $\Omega'' \subset\subset \Omega'$

$$\int_{\Omega''} V_\epsilon^{p-2} |X^2 u_\epsilon|^2 \, dx \leq C(\Omega'', \Omega', \epsilon, H_\epsilon, p),$$

where $H_\epsilon = \int_{\Omega'} (V_\epsilon^p + |u_\epsilon|^p) \, dx$ *and* $V_\epsilon^2 = \epsilon + |X u_\epsilon|^2$.

Proof: For notational simplicity we will drop the subscript ϵ and denote the solution of (5) by u. We briefly recall some piece of notation used in the previous sections; for any $\epsilon > 0$ and for any $z \in \mathbb{R}^{2n}$ we will denote

$$\begin{aligned}
V^2(z) &= \epsilon + |z|^2,\\
W^2_{h^i}(x) &= \epsilon + |Xu(x)|^2 + |Xu(x \circ h^i)|^2,\\
z^{h^i}(\theta) &= Xu + \theta h_i D_{h^i} X u,\\
z^{h^i}_k(\theta) &= X_k u + \theta h_i D_{h^i} X_k u.
\end{aligned}$$

Let $B(3R)$ be a homogeneous ball of radius $3R$ such that $B(3R) \subset \Omega'$. For an arbitrary $i = 1, \dots, n$, let $\varphi = -(D_{-h^i} D_{h^i} + D_{-h^{i+n}} D_{h^{i+n}} + D_{h^i} D_{-h^i} + D_{h^{i+n}} D_{-h^{i+n}}) w$, where $w = g^{12} u$ and g is a cut-off function between $B(R)$ and $B(2R)$. Let us observe that the existence of cut-off functions in the Heisenberg group follows from standard methods whenever one observes that the horizontal gradient of the gauge distance has length less or equal than 1 (this is a trivial computation from the definition in Section 2). Let us recall that h_i is always assumed to be nonnegative.

In [19] we proved $Tw \in L^p_{\mathrm{loc}}(\Omega')$ (see Theorem 7.1 in Section 7 of the present paper). Thanks to this fact and to Lemma 3.3 we obtain $\varphi \in W^{1,p}_0(\Omega, X)$; this makes φ a right test function for equation (5).

Let us multiply equation (5) for the test function φ. On account of Remark 3.2 and Lemma 3.3 we obtain

$$\begin{aligned}
0 &= \sum_{k=1}^{2n} \int_\Omega D_{\pm h^i} a^k \, D_{\pm h^i} X_k w \, dx + \sum_{k=1}^{2n} \int_\Omega D_{\pm h^{i+n}} a^k \, D_{\pm h^{i+n}} X_k w \, dx\\
&\quad + \int_\Omega \left[D_{\pm h^i} a^{i+n} - D_{\pm h^{i+n}} a^i \right] Tw \, dx\\
&= I_1 + I_2 + I_3,
\end{aligned} \tag{16}$$

where $\pm$ in I_1, I_2 and I_3 means the sum of the terms corresponding to both the signs. Let us observe that $D_{\pm h^i} a^k \, D_{\pm h^i} X_k w$ denotes the product of the functions $D_{\pm h^i} a^k$ and $D_{\pm h^i} X_k w$. Here and in the following we omit the parentheses for sake of simplicity.

Estimates of I_1 and I_2. Let us observe that, for any $i,k=1,\dots,2n$

$$
\begin{aligned}
D_{h^i}a^k &= \frac{1}{h_i}\int_0^1 \frac{d}{d\theta}a^k(Xu+\theta h_i D_{h^i}Xu)\,d\theta \\
&= \int_0^1 a^k_j(Xu+\theta h_i D_{h^i}Xu)D_{h^i}X_j u\,d\theta \\
&= \alpha^{kj}_{h^i}D_{h^i}X_j u
\end{aligned}
\tag{17}
$$

where $\alpha^{kj}_{h^i} := \int_0^1 a^k_j(Xu+\theta h_i D_{h^i}Xu)\,d\theta$ and the sum over j is understood even if not explicitly written. Here a^k_j denotes the derivative of a^k with respect to its j-th variable.

Using the previous notation we have

$$
a^k_j(z^{h^i}) = (p-2)V^{p-4}(z^{h^i})z^{h^i}_k z^{h^i}_j + V^{p-2}(z^{h^i})\delta_{kj}
\tag{18}
$$

where $\delta_{kj}=1$ if $k=1$ and $\delta_{kj}=0$ if $k\neq j$. An easy calculation gives

$$
\sum_{k,j=1}^{2n} a^k_j(z^{h^i})D_{h^i}X_k u\,D_{h^i}X_j u \geq (p-1)V^{p-2}(z^{h^i})\,|D_{h^i}Xu|^2.
\tag{19}
$$

In virtue of (17) and (19) we easily obtain

$$
\begin{aligned}
\sum_{k=1}^{2n} D_{h^i}a^k\,D_{h^i}X_k u &= \sum_{k,j=1}^{2n}\alpha^{kj}_{h^i}D_{h^i}\,X_k u\,D_{h^i}X_j u \\
&\geq c\int_0^1 V^{p-2}(z^{h^i})d\theta\,|D_{h^i}Xu|^2.
\end{aligned}
\tag{20}
$$

By [9, Lemma 8.3] we have

$$
\int_0^1 V^{p-2}(z^{h^i})\,d\theta \geq c\,W^{p-2}_{h^i}.
\tag{21}
$$

Hence, from (20) and (21) we have

$$
\sum_{k,j=1}^{2n} D_{h^i}a^k\,D_{h^i}X_k u \geq cW^{p-2}_{h^i}\,|D_{h^i}Xu|^2.
\tag{22}
$$

Let us observe that

$$
\begin{aligned}
(23)\quad D_{h^i} X_k w = g^{12} D_{h^i} X_k u + 12\, g^{11}\, X_k u D_{h^i} g \\
+ 12 g^{11}\, D_{h^i} u X_k g + 12 g^{11}\, u D_{h^i} X_k g + 132 u g^{10}\, D_{h^i} g X_k g.
\end{aligned}
$$

Then, from (22) and (23) we obtain

$$
\begin{aligned}
&\sum_{k=1}^{2n} \int_{\Omega} D_{h^i} a^k D_{h^i} X_k w \, dx \geq c \int_{\Omega'} g^{12} W_{h^i}^{p-2} \, |D_{h^i} X u|^2 \, dx \\
(24)\qquad &+ 12 \int_{\Omega'} g^{11} D_{h^i} a^k X_k u \, D_{h^i} g \, dx + 12 \int_{\Omega'} g^{11} D_{h^i} a^k D_{h^i} u \, X_k g \, dx \\
&+ 12 \int_{\Omega'} g^{11}\, u \, D_{h^i} a^k \, D_{h^i} X_k g \, dx \; + 132 \int_{\Omega'} g^{10}\, u \, D_{h^i} a^k \, D_{h^i} X_k g \, dx \\
&= J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}
$$

Estimate of $J_2, \ldots, J_5$. Let us observe that, for any $k, j = 1, \ldots, 2n$,

$$
(25)\qquad |\alpha_{h^i}^{kj}| \leq c\, W_{h^i}^{p-2}.
$$

On account of (17), (25), Hölder inequality and the decomposition $p-1 = \frac{p+(p-2)}{2}$, we have

$$
\begin{aligned}
|J_2| &= 12 \, | \int_{\Omega'} g^{11} \alpha_{h^i}^{kj} \, D_{h^i} X_j u \, X_k u \, D_{h^i} g \, dx | \\
(26)\qquad &\leq c \int_{\Omega'} g^{11} W_{h^i}^{p-1} |D_{h^i} X_j u| |D_{h^i} g| \, dx \\
&\leq \delta \int_{\Omega'} g^{12} W_{h^i}^{p-2} |D_{h^i} X u|^2 \, dx + c\delta^{-1} \int_{\Omega'} g^{10} W_{h^i}^{p} |D_{h^i} g|^2 \, dx.
\end{aligned}
$$

As for $h_i < R$

$$
(27)\qquad \int_{B_{2R}} W_{h^i}^p \, dx \leq \int_{B_{3R}} V^p \, dx,
$$

it follows from (26) and (27) that

$$
(28)\qquad |J_2| \leq \delta \int_{\Omega'} g^{12} W_{h^i}^{p-2} |D_{h^i} X u|^2 \, dx + c\delta^{-1} R^{-2} \int_{\Omega'} V^p \, dx.
$$

We choose another suitable approach to estimate $|J_3|$, $|J_4|$ and $|J_5|$.

To this end, let us observe that for any $i, k = 1, \ldots, 2n$

$$(29)\qquad D_{h^i} a^k = \frac{1}{h_i} \int_0^1 \nabla_H a^k(Xu(x \circ \delta_\theta h^i)) \cdot h^i \, d\theta = \int_0^1 X_i a^k(Xu(x \circ \delta_\theta h^i)) \, d\theta = X_i \alpha^k_{h^i}$$

where the functions $\alpha^k_{h^i} := \int_0^1 a^k(Xu(x \circ \delta_\theta h^i)) \, d\theta$ can be estimated as

$$(30)\qquad |\alpha^k_{h^i}| \le Y_{h^i} := \int_0^1 (\epsilon + |Xu(x \circ \delta_\theta h^i)|^2)^{\frac{p-1}{2}} \, d\theta.$$

On account of (6), (29) and Remark 3.2 we have

$$(31)\qquad \begin{aligned} J_3 = & -12 \int_{\Omega'} \alpha^k_{h^i} X_i [g^{11} D_{h^i} u X_k g] \, dx \\ = & -12 \int_{\Omega'} \alpha^k_{h^i} D_{h^i} X_i u \, g^{11} X_k g \, dx \\ & -12 \int_{\Omega'} \alpha^k_{h^i} D_{h^i} u \, [11 \, g^{10} X_i g X_k g + g^{11} X_i X_k g] \, dx \end{aligned}$$

and then, by (30)

$$(32)\qquad |J_3| \le c R^{-1} \int_{\Omega'} g^{11} Y_{h^i} |D_{h^i} Xu| \, dx + c R^{-2} \int_{B_{2R}} g^{10} Y_{h^i} |D_{h^i} u| \, dx.$$

The first integral on the right-hand side of (32) can be estimated taking into account the following decomposition

$$(33)\qquad \begin{aligned} & R^{-1} g^{11} Y_{h^i} |D_{h^i} Xu| = R^{-1} g^{11} Y_{h^i} W_{h^i}^{\frac{2-p}{2}} W_{h^i}^{\frac{p-2}{2}} |D_{h^i} Xu| \\ & \le \delta g^{12} W_{h^i}^{p-2} |D_{h^i} Xu|^2 + c \delta^{-1} R^{-2} g^{10} Y_{h^i}^2 W_{h^i}^{2-p} \\ & \le \delta g^{12} W_{h^i}^{p-2} |D_{h^i} Xu|^2 + c \delta^{-1} R^{-2} g^{10} (W_{h^i}^p + Y_{h^i}^{\frac{p}{p-1}}). \end{aligned}$$

To estimate the second integral on the right-hand side of (32) let us observe at first that

$$(34)\qquad Y_{h^i} |D_{h^i} u| \le c (|D_{h^i} u|^p + Y_{h^i}^{\frac{p}{p-1}}).$$

Moreover

$$(35)\qquad \int_{B(2R)} g^{10} \, Y_{h^i}^{\frac{p}{p-1}} \, dx \le \int_0^1 \{ \int_{B(2R)} g^{10} \, (1 + |Xu(x \circ \delta_\theta h^i)|^2)^{\frac{p}{2}} \, dx \} \, dt \le \int_{B(3R)} V^p \, dx.$$

From (27), Lemma 3.5 and (32), ... , (35) we finally obtain

$$|J_3| \leq \delta \int_{\Omega'} g^{12} W_{h^i}^{p-2} |D_{h^i} X u|^2 \, dx + cR^{-2} \int_{\Omega'} V^p \, dx. \tag{36}$$

Analogously we have

$$\begin{aligned} J_4 = & -12 \int_{\Omega'} \alpha_{h^i}^k \, X_i [g^{11} \, u \, D_{h^i} X_k g] \, dx \\ = & -12 \int_{\Omega'} \alpha_{h^i}^k \, X_i u \, g^{11} \, D_{h^i} X_k g \, dx \\ & -12 \int_{\Omega'} \alpha_{h^i}^k \, u \, [11 g^{10} X_i g \, D_{h^i} X_k g + g^{11} \, X_i D_{h^i} X_k g \,] \, dx \end{aligned} \tag{37}$$

and then, by (30)

$$|J_4| \leq 12 R^{-2} \int_{\Omega'} g^{11} Y_{h^i} \, |X u| \, dx + 144 R^{-3} \int_{\Omega'} g^{10} Y_{h^i} \, u \, dx. \tag{38}$$

Estimating as in (34), (35) we finally obtain

$$|J_4| \leq c R^{-3} \int_{\Omega'} (V^p + |u|^p) \, dx. \tag{39}$$

The same holds for $|J_5|$:

$$|J_5| \leq c R^{-3} \int_{\Omega'} (V^p + |u|^p) \, dx. \tag{40}$$

From (22), (28), (36), (39), (40) and choosing δ small, we obtain that there exist some positive constants c and $c^{'}$ such that

$$\begin{aligned} & \sum_{k=1}^{2n} \int_{\Omega'} D_{h^i} a^k D_{h^i} X_k w \, dx \\ & \qquad \geq c \int_{\Omega'} g^{12} W_{h^i}^{p-2} |D_{h^i} X u|^2 \, dx - c^{'} R^{-3} \int_{\Omega'} (V^p + |u|^p) \, dx. \end{aligned} \tag{41}$$

An analogous result can be obtained switching between h^i and $-h^i$. In conclusion there are some positive constants c and $c^{'}$, possibly different from those in (41), such that

$$\begin{aligned} I_1 = & \sum_{k=1}^{2n} \int_{\Omega'} D_{\pm h^i} a^k D_{\pm h^i} X_k w \, dx \\ & \qquad \geq c \int_{\Omega'} g^{12} W_{\pm h^i}^{p-2} |D_{\pm h^i} X u|^2 \, dx - c^{'} R^{-3} \int_{\Omega'} (V^p + |u|^p) \, dx. \end{aligned} \tag{42}$$

The estimate of I_2 proceeds exactly in the same way.

Estimate of I_3. By Theorem 7.2, we have $Tw \in W^{1,p}_{\mathrm{loc}}(\Omega', X)$ and

$$\int_{\Omega'} |XTw|^p \, dx \le C(\epsilon, R, H, p)$$

for a certain positive constant C depending on ϵ, R, H, p. This inequality and the methods applied to $J_3, \ldots, J_5$ give now

$$\left| \int_{\Omega'} D_{h^i} a^{i+n} Tw \, dx \right| \le c \int_{\Omega'} |\alpha^{i+n}_{h^i}| \, |XTw| \, dx \tag{43}$$

$$\le c \int_{\Omega'} Y_{h^i} |XTw| \, dx \le C(\epsilon, R, H, p)$$

for some other constant $C(\epsilon, R, H, p) > 0$. The other three terms of I_3, that is $\int_{\Omega'} D_{-h^i} a^{i+n} Tw \, dx$, $\int_{\Omega'} D_{+h^{i+n}} a^i Tw \, dx$, $\int_{\Omega'} D_{-h^{i+n}} a^i Tw \, dx$ can be estimated in the same way. So we obtain

$$I_3 \ge -C(\epsilon, R, H, p). \tag{44}$$

From (42), the analogous estimate of I_2 and (44) we finally obtain, for any $i = 1, \ldots, 2n$,

$$\int_{\Omega'} g^6 W^{2-p}_{h^i} |D_{h^i} Xu|^2 \, dx \le C(\epsilon, R, H, p). \tag{45}$$

If $2\alpha = p(p-2)$ then, for any $i = 1, \ldots, 2n$

$$|D_{h^i} Xu|^p = W^{\alpha}_{h^i} W^{-\alpha}_{h^i} |D_{h^i} Xu|^p \le W^p_{h^i} + W^{p-2}_{h^i} |D_{h^i} Xu|^2. \tag{46}$$

Inequalities (46), together with (27) and (45) enable us to affirm that, for any $i = 1, \ldots, 2n$

$$D_{h^i} Xu \text{ is bounded in } L^p(B(R)).$$

By Lemma 3.4, possibly up to a subsequence, $D_{h^i} Xu$ converges in $L^p_{\mathrm{loc}}(B(R))$ to $X_i Xu$ for $h^i \to 0$ and then $u \in W^{2,p}_{\mathrm{loc}}(B(R), X)$.

Moreover we can extract from it a subsequence converging a.e. $x \in B(R)$. By Lemma 3.4

$$W_{h^i} \to (\epsilon + 2|Xu|^2)^{1/2} \text{ a.e. } x \in B(R) \text{ as } h^i \to 0.$$

The proof of Theorem 4.1 is then finished passing to the limit $h^i \to 0$ in (45) on account that Ω'' can be covered by a finite number of balls $B(R)$ for R small enough. □

Remark 4.2. We would like to point out that, thanks to Theorem 4.1, we can now differentiate formally equations $\int a^k_\epsilon(Xu_\epsilon) \, X_k \varphi = 0$ along X_i, $i = 1, \ldots, 2n$, obtaining

$$\int_{B(R)} a^k_{\epsilon,j}(Xu_\epsilon) X_i X_j u_\epsilon X_k \varphi \, dx = 0 \tag{47}$$

for any $\varphi \in W^{1,p}_0(B(R), X)$, $\overline{B(R)} \subset \Omega'$.

5. Local boundedness of the gradient

Here we are concerned with the uniform, local boundedness of u_ϵ and ∇u_ϵ (see Propositions 5.1, 5.2, for the proofs we refer to [3] and [18] respectively).

We point out that the proof of Theorem 5.2 insists on the differentiability of equations (5), proved in Theorem 4.1, so its validity is limited to the range $1+\frac{1}{\sqrt{5}} < p \le 2$. However Proposition 5.1 holds for any $p > 1$.

Let us observe that Theorem 1.1 is an easy consequence of Theorem 5.2, standing its uniform validity, via a standard limit argument ([13], [14]).

Proposition 5.1 ([3, Theorem 3.4]). *Let $p > 1$. For any compact $K \subset \Omega'$ there exists a constant $C > 0$ depending only on the structural constants and on* $\operatorname{dist}(K, \partial\Omega')$ *such that*

$$\|u_\epsilon\|_{\infty,K} \le C$$

for all $\epsilon > 0$.

Let $x_0 \in \Omega'$ be arbitrary fixed and, for any $\rho > 0$, let $B(\rho)$ be the ball centered at x_0 of radius ρ. Let $B(R) \subset\subset \Omega'$.

Theorem 5.2 ([18, Theorem 5.2]). *Let $1+\frac{1}{\sqrt{5}} < p \le 2$. For any $\sigma \in (0,1)$ there exists a constant $\gamma(\sigma)$ depending only on the structural constants and σ such that*

$$\|[\epsilon + |Xu_\epsilon|^2]\|_{\infty, B(R-\sigma R)} \le \gamma(\sigma) \frac{1}{|B(R)|} \int_{B(R)} [\epsilon + |Xu_\epsilon|^2]^{p/2}\, dx$$

for all $\epsilon > 0$.

Although the proof of Theorem 5.2 is referred to [18, Theorem 5.2], we want to underline its dependence on the differentiability of equation (5). In fact it is accomplished substituting in (47) the test function $\varphi = X_i u_\epsilon V_\epsilon^\alpha g^2$, $\alpha > 0$, where g is a cut-off function between $B(R-\sigma R)$ and $B(R)$, $\sigma \in (0,1)$, and applying standard methods.

6. Local Hölder continuity of the gradient

Our purpose is to establish the Hölder continuity of Xu_ϵ at x_0, uniformly in $\epsilon > 0$. The technique is due to [6], [5], with few adaptations due to [17].

We will not deal with all the proofs in depth. We will mostly refer to [5], even if we will discuss all needed modifications in details. We outline that Proposition 6.1 holds true for any $p > 1$, while the validity of Propositions 6.2, 6.4 and Theorem 6.5, which depend on the results of the former section, is limited to the range $1+\frac{1}{\sqrt{5}} < p \le 2$.

Let us observe that Theorem 1.2 easily follows from Theorem 6.5 via a standard limit technique (see [13], [14]).

The following result can be found in [3, Theorem 3.35].

Proposition 6.1 (Local Hölder continuity of u_ϵ)**.** *For any compact $K \subset \Omega'$ there exist constants $C, \beta \in (0,1)$ depending only on the structural constants and* $\operatorname{dist}(K, \partial\Omega')$ *such that*

$$|u_\epsilon(x) - u_\epsilon(y)| \le C|x-y|^{\beta}, \quad x, y \in K,$$

for all $\epsilon > 0$.

As before let x_0 be an arbitrary point of Ω' and, for any $\rho > 0$, $B(\rho)$ be the ball centered at x_0 of radius ρ. We will choose $R > 0$ in such a way that $\overline{B(2R)} \subset \Omega'$.

Let us now set $\varphi = \pm(X_i u_\epsilon - k)^{\pm}\xi^2$ in (47), for $k \in \mathbb{R}$ and $i = 1, \dots, 2n$, where ξ is a cut-off function with support in $B(R)$. We easily obtain

$$\int_{B(R)} V_\epsilon^{p-2} |X(X_i u_\epsilon - k)^{\pm}|^2 \, \xi^2 \, dx \le \gamma \int_{B(R)} V_\epsilon^{p-2} |(X_i u_\epsilon - k)^{\pm}|^2 |X\xi|^2 \, dx \tag{48}$$

for all $\epsilon > 0$, where $V_\epsilon^2 = \epsilon + |Xu_\epsilon|^2$ and γ is a structural constant independent on ϵ, R.

Let us observe that, due to Theorem 5.1 and the results of [1], the solutions u_ϵ are now smooth. Therefore, for any $\rho \le R$, $\epsilon > 0$, we can set

$$\mu_\epsilon(\rho) = \max_i \sup_{B(\rho)} |X_i u_\epsilon|$$

$$\omega_\epsilon(\rho) = \max_i \operatorname{osc}_{B(\rho)} X_i u_\epsilon.$$

Proposition 6.2. *Let $2\rho < R$. Set*

$$\lambda = \frac{\mu_\epsilon(2\rho)}{2}\,.$$

Then there exists a positive constant C_0 depending only on the data but independent of ϵ, R, λ, such that, if for some $1 \le i \le 2n$

$$|\{x \in B(2\rho) \,|\, X_i u_\epsilon < \lambda\}| \le C_0 \, |B(2\rho)|$$

then

$$X_i u_\epsilon \ge \frac{\lambda}{4}\,, \quad \forall\, x \in B(\rho).$$

Analogously if

$$|\{x \in B(2\rho) \,|\, X_i u_\epsilon > -\lambda\}| \le C_0 \, |B(2\rho)|$$

then

$$X_i u_\epsilon \le -\frac{\lambda}{4}\,, \quad \forall\, x \in B(\rho).$$

Proof: We will drop the subscript ϵ. As in [5, Proposition 4.1] we distinguish between $\epsilon \ge \lambda^2$ and $\epsilon < \lambda^2$. In the first simpler case the proof is easily accomplished using (48) as in [5, Proposition 4.1]. Let now $\epsilon < \lambda^2$.

Lemma 6.3. *Let* $v = |X_i u|^{p/2} \operatorname{sign} X_i u$. *Then*

$$\int_{B(r-\sigma r)} |X(v-h)^-|^2\, dx \le \gamma h_0^2 (\sigma r)^{-2} |A^-_{h,r}| \tag{49}$$

for any $\sigma \in (0,1)$, $r \le 2\rho$, $h \le h_0 = \lambda^{p/2}$ *and for a suitable positive structural constant* γ, *independent on* ϵ, r, σ, h, *where* $A^-_{h,r} =: \{x \in B(r) \,|\, v(x) < h\}$.

Proof of Lemma 6.3: We refer the reader to [5, Section 4] for the details of the proof. Here we recall only the main steps of it for convenience of the reader.

If we set in (47) the test function $\varphi_\eta = -[(|X_i u| + \eta)^{p-2} X_i u - k]^- \xi^2$, $k \in \mathbb{R}^+$, where η is a small positive number which will be let tend to 0 and ξ is a cut-off function with support in $B(\rho)$, then we obtain

$$\begin{aligned}\int_{B(\rho)} V^{p-2} |X(v - k^{\frac{p}{2(p-1)}})^-|^2 \xi^2\, dx \\ \le \gamma \int_{B(\rho)} |((|X_i u|^{p-2} X_i u - k)^-|^2 |X\xi|^2\, dx\end{aligned} \tag{50}$$

where $v := |X_i u|^{p/2} \operatorname{sign} X_i u$, for a structural constant γ independent on ϵ. From (50), recalling the definition of λ we deduce for any $r < 2\rho$,

$$\begin{aligned}\lambda^{p-2} \int_{B(r)} |X(v - k^{\frac{p}{2(p-1)}})^-|^2 \xi^2\, dx \\ \le \gamma \int_{B(r)} |((|X_i u|^{p-2} X_i u - k)^-|^2 |X\xi|^2\, dx\end{aligned} \tag{51}$$

for a new constant γ independent on ϵ, r, σ. If we choose $k \le \lambda^{p-1}$ in (51) and denote by h any number such that $h \le h_0 = \lambda^{p/2}$, then we obtain (49).

Let us now continue the **proof of Proposition 6.2**.

Let $H = \sup_{B(2r)} (v - h_0)^-$. Let us observe that if $H < \frac{h_0}{2}$, then $X_i u > \frac{\lambda}{4}$, for any $x \in B(2\rho)$. Therefore we may assume $H \ge \frac{h_0}{2}$. For any integer $j \ge 0$ let

$$\begin{aligned} r_j &= \rho + \frac{\rho}{2^j}, \quad h_j = h_0 - \frac{H}{4}\Big(1 - \frac{1}{2^j}\Big), \\ B_j &= B(r_j), \qquad A_j = A^-_{h_j,r_j}, \qquad \mu_j = |A^-_{h_j,r_j}|. \end{aligned} \tag{52}$$

If we set in (49) $h = h_j$, $r = r_j$, $r - \sigma r = r_{j+1}$, for an arbitrary $j \ge 0$, then we obtain

$$\int_{B_{j+1}} |X(v-h_j)^-|^2\, dx \le C\, 2^{2j} \frac{h_0^2}{\rho^2} |A_j|. \tag{53}$$

Let $s \in (p, \frac{pQ}{Q-p})$. Applying Poincaré's inequality [15] to the function $(v-h_j)_{-}\xi$, where ξ is a cut-off function between B_{j+2} and B_{j+1} we have, on account of the doubling property,

$$\begin{aligned}(54)\quad &\left(\int_{A_{j+1}} |(v-h_j)^{-}\,\xi|^s\,dx\right)^{1/s}\\ &\le c\,\rho\left(\int_{A_{j+1}} |X(v-h_j)^{-}|^p\,dx + \rho^{-p}2^{pj}\int_{A_{j+1}} |(v-h_j)^{-}|^p\,dx\right)^{1/p} |B(\rho)|^{1/s-1/p}.\end{aligned}$$

By Hölder inequality, (53) and (54) we obtain
(55)

$$\begin{aligned}\frac{H}{2^{j+1}}|A_{j+2}| &\le \int_{A_{j+2}} |(v-h_j)^{-}|\,dx\\ &\le (\int_{A_{j+1}} |(v-h_j)^{-}\,\xi|^s\,dx)^{1/s}\,|A_{j+1}|^{1-1/s}\\ &\le c\rho\{(\int_{A_{j+1}} |X(v-h_j)^{-}|^2\,dx\,)^{p/2}\,|A_{j+1}|^{\frac{2-p}{2}}\\ &\quad + \rho^{-p}2^{pj}\int_{A_{j+1}} |(v-h_j)^{-}|^p\,dx\}^{1/p}\,|B(\rho)|^{1/s-1/p}\,|A_{j+1}|^{1-1/s}\\ &\le c\,\{\rho(\int_{A_{j+1}} |X(v-h_j)^{-}|^2\,dx\,)^{1/2}\,|A_{j+1}|^{\frac{2-p}{2p}}\\ &\quad + 2^j\,(\int_{A_{j+1}} |(v-h_j)^{-}|^p\,dx)^{1/p}\}|B(\rho)|^{1/s-1/p}\,|A_{j+1}|^{1-1/s}\\ &\le c2^jH|A_j|^{1/p}\,|B(\rho)|^{1/s-1/p}|A_{j+1}|^{1-1/s}\end{aligned}$$

from which we obtain for any $j \ge 0$

$$(56)\qquad \frac{|A_{j+2}|}{|B(\rho)|} \le c\,2^{4j}\left(\frac{|A_j|}{|B(\rho)|}\right)^{1+\chi}$$

where $\chi = \frac{1}{p} - \frac{1}{s} > 0$. In particular (56) gives for any $l \ge 1$

$$(57)\qquad \frac{|A_{2l}|}{|B(\rho)|} \le c(2^8)^{(l-1)}\left(\frac{|A_{2(l-1)}|}{|B(\rho)|}\right)^{1+\chi}.$$

It follows from (57) and [12, Lemma 4.7, p. 66] that there exists a positive constant C_0 depending only on c and $b = 2^8$ such that, if $|A_0| \le C_0|B_0|$, then $\lim_{l\to+\infty} A_{2l} = 0$, which implies $|\{x \in B(\rho)\,|\,X_i u < \frac{\lambda}{2^{2/p}}\}| = 0$, and then $X_i u \ge \frac{\lambda}{4}$ for any $x \in B(\rho)$, so Proposition 6.2 is proved. □

Proposition 6.4 ([5, Proposition 4.2], [18, Proposition 6.4]). *Let $2\rho < R$. If the assumptions of Proposition 6.4 fail, then there exists a positive structural constant $\sigma_0 \in (0,1)$ independent on ϵ, ρ, such that*

$$\mu_\epsilon(\rho/2) \le \sigma_0 \mu_\epsilon(2\rho).$$

Theorem 6.5. *There exist positive constants γ and $\eta \in (0,1)$ depending only on the data and $\operatorname{dist}(B(R), \partial\Omega')$ such that*

$$\operatorname{osc}_{B(\rho)} X_i u_\epsilon \le \gamma \Big(\frac{\rho}{R}\Big)^\eta \sup_{B(R/2)} |Xu_\epsilon|, \quad i = 1, \dots, 2n$$

for every $2\rho < R$ and every $\epsilon > 0$.

Proof: The proof is the same as that of [5, Proposition 4.3] using a result of [17]. Here we limit ourselves to describe the general idea of the proof and we refer the reader to [18] for any details.

We prove the existence of positive structural constants $\alpha \in (0,1)$, δ_0 and σ_0 independent of ϵ such that, for all small $\rho > 0$, if the subset of $B(\rho)$ where Xu_ϵ degenerates is "small", then the equation behaves in $B(\rho)$ as a nondegenerate elliptic equation (see Proposition 6.2). In this case, by [17, Theorem 2.1], we obtain $\omega_\epsilon(\rho/2) \le \delta_0 \rho^\alpha$.

On the other hand if Xu_ϵ degenerates in a "thick" portion of $B(\rho)$, then we have $\mu_\epsilon(\rho/2) \le \mu_\epsilon(2\rho)$ (see Proposition 6.4).

The Hölder continuity follows from both cases by a standard iteration technique [12].

□

7. Estimate of Tu_ϵ

In this section we prove that, for any $1 + \frac{1}{\sqrt{5}} < p \le 2$, the local weak solutions u_ϵ of equation (5) satisfy Tu_ϵ, $XTu_\epsilon \in L^p_{\mathrm{loc}}(\Omega')$. Just as before, Ω' will denote an arbitrary open bounded subset of Ω such that $\Omega' \subset\subset \Omega$.

Theorem 7.1 ([19, Theorem 1.1]). *Let $1 + \frac{1}{\sqrt{5}} < p \le 2$ and, for any $\epsilon \in (0,1)$, let $u_\epsilon \in W^{1,p}_{\mathrm{loc}}(\Omega, X)$ be a local weak solution of (5). Let $B(3R)$ be an arbitrary homogeneous ball of radius $3R$ such that $B(3R) \subset \Omega'$ and let g be a cut-off function between $B(R)$ and $B(2R)$. Then $T(g^4 u_\epsilon) \in L^p(\Omega')$ and*

$$\int_{\Omega'} |T(g^4 u_\epsilon)|^p \, dx \le C R^{-4p} \int_{\Omega'} (V_\epsilon^p + |u_\epsilon|^p) \, dx \tag{58}$$

where $V_\epsilon^2 = \epsilon + |Xu_\epsilon|^2$.

Theorem 7.2. *Let the assumptions of Theorem 7.1 hold. Then* $T(g^{12}u_\epsilon) \in W^{1,p}_{\mathrm{loc}}(\Omega', X)$ *and*

$$\int_{\Omega'} |XT(g^{12}u_\epsilon)|^p \, dx \le C(R, \epsilon, H_\epsilon, p) \tag{59}$$

where $H_\epsilon = \int_{\Omega'} (V_\epsilon^p + |u_\epsilon|^p)\, dx$ *and* $V_\epsilon^2 = \epsilon + |Xu_\epsilon|^2$.

Proof: From Lemma 3.5 and Theorem 7.1 we easily deduce for any small $s > 0$

$$\int_{\Omega'} |D_{h^*_{s,1/2}}(g^4 u_\epsilon)|^p \, dx \le c s^{p/2} C(R, H_\epsilon, p). \tag{60}$$

Let us multiply the equation (5) by the test function $\varphi = D_{-h^*_{s,1/2}}(g^{10} D_{h^*_{s,1/2}} u_\epsilon)$. Let us observe that $\varphi \in W^{1,p}_0(\Omega, X)$. In the following we will drop the subscript ϵ for the sake of simplicity. On account of (7) and Remark 3.2 we obtain

$$\int_{\Omega'} D_{h^*_{s,1/2}} a^k \, g^{10} \, X_k D_{h^*_{s,1/2}} u \, dx + 10 \int_{\Omega'} D_{h^*_{s,1/2}} a^k \, D_{h^*_{s,1/2}} u \, g^9 X_k g \, dx \; = 0. \tag{61}$$

For any $p > 1$ the first integral on the left-hand side of (61) can be estimated by the same argument we applied to J_2 in Section 4: as

$$D_{h^*_{s,1/2}} a^k = \alpha^{kj}_{h^*_{s,1/2}} D_{h^*_{s,1/2}} X_j u \tag{62}$$

where $\alpha^{kj}_{h^*_{s,1/2}} := \int_0^1 a^k_j(Xu + \theta s^{1/2} D_{h^*_{s,1/2}} Xu)\, d\theta$, then we have

$$\int_{\Omega'} D_{h^*_{s,1/2}} a^k \, g^{10} \, X_k D_{h^*_{s,1/2}} u \, dx \ge c \int_{\Omega'} g^{10} \, W^{p-2}_{h^*_s} |D_{h^*_{s,1/2}} Xu|^2 \, dx \tag{63}$$

where $W^2_{h^*_s}(x) = \epsilon + |Xu(x)|^2 + |Xu(x \circ h^*_s)|^2$.

Let us now estimate the second integral on the left-hand side of (61). As

$$|s^{1/2} D_{h^*_{s,1/2}} Xu| \le 2 W_{h^*_s},$$

we obtain, for $\gamma = \frac{2-p}{p}$,

$$\begin{aligned} &\Big| \int_{\Omega'} D_{h^*_{s,1/2}} a^k \, D_{h^*_{s,1/2}} u \, g^9 X_k g \, dx \Big| \\ &\le \int_{\Omega'} W^{p-2}_{h^*_s} |D_{h^*_{s,1/2}} Xu| \, |D_{h^*_{s,1/2}} u| \, g^9 |X_k g| \, dx \end{aligned} \tag{64}$$

$$\leq c\int_{\Omega'} W_{h_s^*}^{p-2+\gamma} s^{-\gamma/2} |D_{h^*_{s,1/2}} Xu|^{1-\gamma} |D_{h^*_{s,1/2}} u|\, g^9 |X_k g|\, dx$$

$$\text{(by Young's inequality with exponents } \frac{p}{p-1} \text{ and } p\text{)}$$

$$\leq \delta \int_{\Omega'} g^{10}\, W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2\, dx$$

$$+ cR^{-p}\delta^{-1} \int_{\Omega'} (|D_{h^*_{s,1/2}} (g^4 u)|^p + |u|^p |D_{h^*_{s,1/2}} g^4|^p) s^{-\gamma p/2}\, dx$$

$$\text{(on account of (60))}$$

$$\leq \delta \int_{\Omega'} g^{10}\, W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2\, dx$$

$$+ \delta^{-1} s^{p-1} C(R,H,p).$$

Hence, on account of (63) and (64) with small δ, (61) gives

$$\int_{\Omega'} g^{10} W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^2\, dx \leq s^{p-1} C(R,H,p). \tag{65}$$

If $2\sigma = p(2-p)$ and $q = \frac{p(p-1)(2-p)}{4}$ then, by Young's inequality with exponents $\frac{2}{2-p}$ and $\frac{2}{p}$, (65) gives

$$\int_{\Omega'} g^{10} |D_{h^*_{s,1/2}} Xu|^p\, dx = \int_{\Omega'} g^{10} W_{h_s^*}^{\sigma} W_{h_s^*}^{-\sigma} |D_{h^*_{s,1/2}} Xu|^p\, dx$$

$$\leq c s^{\frac{2q}{2-p}} \int_{\Omega'} g^{10} W_{h_s^*}^{p}\, dx + s^{\frac{-2q}{p}} \int_{\Omega'} g^{10} W_{h_s^*}^{p-2} |D_{h^*_{s,1/2}} Xu|^p\, dx$$

$$\leq s^{\frac{p(p-1)}{2}} C(R,H,p)$$

from which we deduce

$$\int_{\Omega'} |D_{h^*_{s,1/2}} X(g^{10}u)|^p\, dx \leq s^{\frac{p(p-1)}{2}} C(R,H,p)$$

and hence

$$\int_{\Omega'} |D_{h^*_{s,p/2}} X(g^{10}u)|^p\, dx \leq C(R,H,p). \tag{66}$$

Let us now recall some basic definitions and relative properties we will need in the sequel.

For $p \geq 1$ and fractional $l > 0$, let $w^{l,p}$ denote the completion of $\mathcal{D}(\mathbb{R})$ with respect to the norm

$$\|\varphi\|_{w^{l,p}} = \left(\int \|\Delta_s \varphi\|^p_{w^{[l],p}}\, |s|^{-1-p\{l\}}\, ds \right)^{1/p}.$$

Here $\Delta_s\varphi(t) := \varphi(t+s) - \varphi(s)$, $[l]$ and $\{l\}$ are the integer and fractional parts of l, respectively. Further $W^{l,p}$ denotes the completion of $\mathcal{D}(\mathbb{R})$ with respect to the norm $\|\varphi\|_{w^{l,p}} + \|u\|_{L^p}$.

If $\mathcal{F}$ denotes the Fourier transform in the real variable t, then for $l \in \mathbb{R}$ we define the fractional derivative of φ as

$$\partial_t^\alpha \varphi = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}\varphi$$

and we denote $\mathcal{L}_t^l\varphi := \mathcal{F}^{-1}(1+|\xi|^2)^{l/2}\mathcal{F}\varphi$. For $p > 1$ and $l > 0$, let $h^{l,p}$ and $H^{l,p}$ denote the completion of $\mathcal{D}(\mathbb{R})$ with respect to the norms

$$\|\varphi\|_{h^{l,p}} = \|\mathcal{F}^{-1}|\xi|^l\mathcal{F}\varphi\|_{L^p} \ , \ \|\varphi\|_{H^{l,p}} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{l/2}\mathcal{F}\varphi\|_{L^p}$$

respectively. Let us observe that

$$(L^p, W^{1,p})_{l+\tau,\infty} \subset W^{l,p} \tag{67}$$

for $0 < l < 1$, $p > 1$ and small $\tau > 0$ (see [24, Theorem 1, p. 64; (1), (4), p. 25; (11), p. 189]) where $(L^p, W^{1,p})_{\theta,\infty} = \{\varphi \in L^p : \sup_{0<|s|<\sigma} \frac{\|\varphi(\cdot+s)-\varphi(\cdot)\|_{L^p}}{|s|^\theta} < \infty\}$, and

$$w^{l,p} \subset h^{l,p} \tag{68}$$

for $l > 0$, $1 < p < 2$ (see [16, Theorem 7.1.3-1]).

Now we can **continue the proof of Theorem 7.2**. The following lemma holds true.

Lemma 7.3. *Let η, $w \in C_0^\infty(\Omega)$ and let $p > 1$, $0 < l < 1$. Then*

$$\|\eta X(\partial_t^l w)\|_{L^p(\Omega')} \leq c\,\|\partial_t^l(\eta X w)\|_{L^p(\Omega')}. \tag{69}$$

We omit the proof of Lemma 7.3 because it obviously follows from an important result due to Capogna [2, Theorem 2.12] in the more general Carnot group setting. Let us observe that even if the proof of Capogna is accomplished for $p = 2$, still it works alike for $p \neq 2$.

From (66), (67), (68) and for $1 + \frac{1}{\sqrt{5}} < p \leq 2$ and small $\tau > 0$ we obtain

$$\|\partial_t^{p/2-\tau} X(g^{10}u)\|^p_{L^p(\Omega')} \leq C(R, H, p) \tag{70}$$

and then, by Lemma 7.3

$$\|X(\partial_t^{p/2-\tau}(g^{10}u))\|^p_{L^p(\Omega')} \leq C(R, H, p). \tag{71}$$

From Corollary 3.7, (67) and (68) we deduce

$$\|\partial_t^{1/2-\tau}(\eta w)\|^p_{L^p(\Omega')} \leq \|X(\eta w)\|^p_{L^p(\Omega')} \tag{72}$$

for any $\eta,\, w \in C_0^\infty(\Omega')$ and small $\tau > 0$. From (71) and (72) we deduce

$$\|\partial_t^{1/2-\tau}(\partial_t^{p/2-\tau}(g^{10}u))\|_{L^p(\Omega')} \leq C(R,H,p). \tag{73}$$

As in virtue of Theorem 7.1

$$\|\partial_t^{p/2-\tau}(g^{10}u)\|_{L^p(\Omega')} \leq C(R,H,p) \tag{74}$$

then from (73) and (74) we obtain (see [23, Lemma V-3.2.2, p. 133])

$$\|\mathcal{L}_t^{1/2-\tau}(\partial_t^{p/2-\tau}(g^{10}u))\|_{L^p(\Omega')} \leq C(R,H,p)$$

and then

$$\|\partial_t^{p/2-\tau}(\mathcal{L}_t^{1/2-\tau}(g^{10}u))\|_{L^p(\Omega')} \leq C(R,H,p).$$

As in virtue of Theorem 7.1, $\|\mathcal{L}_t^{1/2-\tau}(g^{10}u)\|_{L^p(\Omega')} \leq C(R,H,p)$, then

$$\|\mathcal{L}_t^{(p+1)/2-2\tau}(g^{10}u)\|_{L^p(\Omega')} \leq C(R,H,p). \tag{75}$$

As for $q \geq r > 1$, $l - \frac{1}{r} \geq m - \frac{1}{q}$

$$H^{l,r}(\mathbb{R}) \subset H^{m,q}(\mathbb{R})$$

(see for example [24, (15), p. 206]), we deduce from (75)

$$\|T(g^{10}u)\|_{L^q(\Omega')} \leq C(R,H,p)$$

whenever $\frac{p-1}{2} - 2\tau - \frac{1}{p} \geq -\frac{1}{q}$. In particular for $1 + \frac{1}{\sqrt{5}} < p < 2$ and small τ we obtain

$$\|T(g^{10}u)\|_{L^2(\Omega')} \leq C(R,H,p)$$

and then, by Lemma 3.5 also

$$\int_{\Omega'} |D_{h^*_{s,1}}(g^{10}u)|^2\,dx \leq C(R,H,p) \tag{76}$$

for any small $s > 0$.

At this point let us set $\varphi = D_{-h^*_{s,1}}(g^{12} D_{h^*_{s,1}} u)$ in equation (5) and repeat the machinery from (61) to (63) obtaining

$$(77) \qquad \int_{\Omega'} D_{h^*_{s,1}} a^k \, g^{12} \, X_k D_{h^*_{s,1}} u \, dx \geq c \int_{\Omega'} g^{12} W^{p-2}_{h^*_s} |D_{h^*_{s,1}} X u|^2 \, dx.$$

In virtue of (76), the second term on the left-hand side of (61) can now be estimated as follows

$$(78) \qquad \begin{aligned} &| \int_{\Omega'} D_{h^*_{s,1}} a^k \, D_{h^*_{s,1}} u \, g^{11} \, X_k g \, dx| \\ &\leq \int_{\Omega'} W^{p-2}_{h^*_s} |D_{h^*_{s,1}} X u| \, |D_{h^*_{s,1}} u| \, g^{11} |X g| \, dx \\ &\leq \delta \int_{\Omega'} g^{12} \, W^{p-2}_{h^*_s} |D_{h^*_{s,1}} X u|^2 + C R^{-2} \delta^{-1} \int_{\Omega'} g^{10} \, W^{p-2}_{h^*_s} |D_{h^*_{s,1}} X u|^2 \, dx \\ &\leq \delta \int_{\Omega'} g^{12} W^{p-2}_{h^*_s} |D_{h^*_{s,1}} X u|^2 + \delta^{-1} \epsilon^{p-2} C(R, H, p). \end{aligned}$$

Hence (76) and (77) give now for small $\delta > 0$

$$(79) \qquad \int_{\Omega'} g^{12} W^{p-2}_{h^*_s} |D_{h^*_{s,1}} X u|^2 \, dx \leq C(\epsilon, R, H, p).$$

If $2\alpha = p(p-2)$, then

$$(80) \qquad |D_{h^*_{s,1}} X u|^p = W^{\alpha}_{h^*_s} W^{-\alpha}_{h^*_s} |D_{h^*_{s,1}} X u|^p \leq W^p_{h^*_s} + W^{p-2}_{h^*_s} |D_{h^*_{s,1}} X u|^2.$$

It follows from (79) and (80) that for any $i = 1, \ldots, 2n$, $D_{h^*_{s,1}} X_i u$ is bounded in $L^p(B(R))$ and thus, possibly up to a subsequence, it converges in $L^p(B(R))$ to $T X_i u$ as $s \to 0$ and a.e. in $B(R)$. So, since T commutes with X_i, $Tu \in W^{1,p}_{\text{loc}}(B(R), X)$; moreover the limit $s \to 0$ on (79), (80) gives

$$\int_{\Omega'} g^{12} |D_{h^*_{s,1}} X u|^p \, dx \leq C(\epsilon, R, H, p)$$

and then, by Lemma 3.5

$$\int_{\Omega'} g^{12} \, |T X u|^p \, dx \leq C(\epsilon, R, H, p)$$

from which we easily deduce

$$\int_{\Omega'} |X T(g^{12} u)|^p \, dx \leq C(\epsilon, R, H, p)$$

and the proof is concluded. □

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