Mittag-Leffler type expansions of $\bar{\partial}$-closed $(0, n-1)$-forms in certain domains in $\mathbb{C}^n$

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Abstract. In this paper we will prove a Mittag-Leffler type theorem for $\bar{\partial}$-closed $(0, n-1)$-forms in $\mathbb{C}^n$ by addressing the question of constructing such differential forms with prescribed periods in certain domains.

Keywords: Mittag-Leffler type expansions, $\bar{\partial}$-closed forms, Bochner-Martinelli kernel

Classification: 32A25

1. Introduction

Let us recall that given a sequence c_k, $k = 0, 1, 2, \ldots$, of complex numbers, there exists a holomorphic function $f(z)$ defined for $z \in \mathbb{C} - \{0\}$ so that

$$\int_{|z|=r} z^k f(z)\, dz = c_k, \quad k = 0, 1, 2, \ldots \ (r > 0),$$

if and only if

$$\sum_{k=0}^{\infty} |c_k| s^k < \infty, \quad \text{for every} \quad s > 0,$$

and that, moreover, such a function is of the form

$$f(z) = \frac{1}{2\pi i} \sum_{k=0}^{\infty} c_k \frac{1}{z^{k+1}} + \quad \text{a holomorphic function in } \ \mathbb{C}.$$

More generally, if $D \subset \mathbb{C}$ is an open set, $A = \{\alpha^l : l = 1, 2, 3, \ldots\}$ is a discrete subset of D and if for each l we are given a sequence c_k^l of complex numbers which satisfies the condition

$$\sum_{k=0}^{\infty} |c_k^l| s^k < \infty, \quad \text{for every} \quad s > 0,$$

then there exists $f \in \mathcal{O}(D - A)$ so that

$$\int\limits_{|z-\alpha^l|=r_l} (z-\alpha^l)^k f(z)\, dz = c_k^l, \quad k = 0, 1, 2, \ldots, \; l = 1, 2, 3, \ldots,$$

where $r_l > 0$ are sufficiently small. And, moreover, such a function f is unique up to a holomorphic function in D.

In $\mathbb{C}^n$, we may consider systems $(f_1, \ldots, f_n)$ of C^∞ functions, which satisfy the differential equation

$$\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial f_j}{\partial \bar{z}_j} = 0.$$

This means that the $(0, n-1)$-form

$$\theta = \sum_{j=1}^{n} f_j d\bar{z}_1 \wedge \ldots (j) \ldots \wedge d\bar{z}_n$$

is $\bar{\partial}$-closed, and therefore

$$d[\theta(z) \wedge \omega(z)] = 0,$$

where $\omega(z) = dz_1 \wedge \ldots \wedge dz_n$.

By Stokes's theorem, this implies that

$$\int\limits_{\mathcal{S}_1} \theta(z) \wedge \omega(z) = \int\limits_{\mathcal{S}_2} \theta(z) \wedge \omega(z),$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are $(2n-1)$-dimensional closed surfaces, homotopic in the domain where θ is defined and $\bar{\partial}$-closed.

Thus such $\bar{\partial}$-closed $(0, n-1)$-forms play, in certain cases, roles of holomorphic functions.

Also, again by Stokes's theorem,

$$\int\limits_{\mathcal{S}} \theta(z) \wedge \omega(z) = 0,$$

if the $(0, n-1)$-form θ is $\bar{\partial}$-exact in a neighborhood of the $(2n-1)$-dimensional closed surface $\mathcal{S}$. Thus the $\bar{\partial}$-exact $(0, n-1)$-forms are, in a sense, negligible, at least as far as their periods are concerned.

As for the notation, we will denote by $Z_{\bar{\partial}}^{(0,q)}(D)$ the set of $\bar{\partial}$-closed $(0, q)$-forms with C^∞ coefficients in an open set D and by $\mathcal{O}(D)$ the set of holomorphic functions in D. Also we will denote by $B_{\bar\partial}^{(0,q)}(D)$ the set of $(0,q)$-forms which are $\bar\partial$-exact in D and $H_{\bar\partial}^{(0,q)}(D) = Z_{\bar\partial}^{(0,q)}(D)/B_{\bar\partial}^{(0,q)}(D)$.

In this paper we will prove a Mittag-Leffler type theorem for $\bar\partial$-closed $(0,n-1)$-forms in $\mathbb{C}^n$ by addressing the question of constructing such differential forms with prescribed periods in certain domains. More precisely we will prove the following theorems.

Theorem 1. *Suppose that for each $k=(k_1,\dots,k_n)$, where k_j are non-negative integers, we are given a complex number $c_k = c_{k_1,\dots,k_n}$. Then there exists $\theta \in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\})$ with*

$$\int_{|z|=r} z_1^{k_1}\cdots z_n^{k_n}\theta(z)\wedge\omega(z) = c_{k_1,\dots,k_n}, \quad \text{for every } \ k$$

(where $r>0$), if and only if the sequence $c_{k_1,\dots,k_n}$ satisfies the condition

$$(*) \qquad \sum_{k_1,\dots,k_n\ge 0} |c_{k_1,\dots,k_n}| s_1^{k_1}\dots s_n^{k_n} < \infty, \quad \text{for every} \quad s_1,\dots,s_n>0.$$

Theorem 2. *Let D be an open set in $\mathbb{C}^n$ and A a discrete subset of D. Suppose that for each $\alpha\in A$, we are given a sequence $c_k^\alpha\in\mathbb{C}$ which satisfies condition $(*)$. Then there exists $\theta\in Z_{\bar\partial}^{(0,n-1)}(D-A)$ so that*

$$(**) \quad \int_{|z-\alpha|=r_\alpha} (z_1-\alpha_1)^{k_1}\dots(z_n-\alpha_n)^{k_n}\theta(z)\wedge\omega(z) = c^\alpha_{k_1,\dots,k_n}, \text{ for every } k \text{ and } \alpha,$$

where $r_\alpha>0$ are sufficiently small.
If, moreover, D is pseudoconvex, the differential form θ which satisfies $(**)$ is unique up to a $\bar\partial$-exact $(0,n-1)$-form in $D-A$.

Before we prove Theorem 1, let us observe that the sequence $c_{k_1,\dots,k_n}$ satisfies condition $(*)$ if and only if

$$(1) \qquad \sum_{k_1,\dots,k_n\ge 0} \frac{n(n+1)\cdots(n+k_1+\cdots+k_n-1)}{k_1!\dots k_n!}|c_{k_1,\dots,k_n}|s_1^{k_1}\dots s_n^{k_n}<\infty \qquad \text{for every} \ \ s_1,\dots,s_n>0.$$

Indeed, first, (1) implies $(*)$ because of the inequalities

$$\frac{n\cdots(n+k_1-1)}{k_1!}\ge 1,\dots,\frac{(n+k_1+\cdots+k_{n-1})\cdots(n+k_1+\cdots+k_n-1)}{k_n!}\ge 1.$$

To prove the other direction, let us set $N = n + k_1 + \cdots + k_n - 1$ and notice that

$$\begin{aligned}(2)\qquad &\frac{n(n+1)\cdots(n+k_1+\cdots+k_n-1)}{k_1!\ldots k_n!}\\ &\le \sum_{0\le p_1,\ldots,p_n\le N} \frac{N!}{p_1!\ldots p_n!(N-p_1-\cdots-p_n)!}\\ &= (n+1)^N = (n+1)^{n-1}(n+1)^{k_1}\ldots(n+1)^{k_n}.\end{aligned}$$

This gives that the sum in (1) is

$$\le (n+1)^{n-1} \sum_{k_1,\ldots,k_n\ge 0} |c_{k_1,\ldots,k_n}| [(n+1)s_1]^{k_1}\ldots[(n+1)s_n]^{k_n}.$$

Therefore $(*)$ implies (1).

2. Proof of Theorem 1

For the one direction, let us consider a sequence c_k of complex numbers which satisfies $(*)$. We will construct a $\theta \in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\})$ so that

$$\int_{|z|=r} z_1^{k_1}\cdots z_n^{k_n}\theta(z)\wedge\omega(z) = c_k, \quad \text{for every } k.$$

For $z \ne w$, set

$$M(z,w) = \frac{\beta_n}{|z-w|^{2n}} \sum_{j=1}^{n} (-1)^{j-1}(\bar z_j - \bar w_j)d\bar z_1 \wedge \ldots (j) \ldots d\bar z_n,$$

where $\beta_n = (-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$, and, as in [1], for each $k = (k_1,\ldots,k_n)$, define

$$\begin{aligned}\eta_k(z) &= \left.\frac{\partial^{k_1+\cdots+k_n} M(z,w)}{\partial w_1^{k_1}\cdots\partial w_n^{k_n}}\right|_{w=0}\\ &= \beta_n n(n+1)\cdots(n+k_1+\cdots+k_n-1)\frac{\bar z_1^{k_1}\cdots\bar z_n^{k_n}}{|z|^{2(n+k_1+\cdots+k_n)}}\times\\ &\qquad\times\sum_{j=1}^{n}(-1)^{j-1}\bar z_j d\bar z_1\wedge\ldots(j)\ldots\wedge d\bar z_n.\end{aligned}$$

Then $\eta_k \in Z_{\bar{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{0\})$. Also, by the Bochner-Martinelli formula, for $f \in \mathcal{O}(\mathbb{C}^n)$,

(3)
$$\int\limits_{|z|=r} f(z)\eta_{k_1,\dots,k_n}(z) \wedge \omega(z) = \left.\frac{\partial^{k_1+\dots+k_n} f(w)}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}}\right|_{w=0}.$$

But the sequence c_k satisfies (1), since as we pointed out, $(*)$ implies (1). Writing the factor $\dfrac{\bar{z}_1^{k_1} \cdots \bar{z}_n^{k_n}}{|z|^{2(n+k_1+\dots+k_n)}}$ of $\eta_{k_1,\dots,k_n}(z)$ in the form

$$\frac{1}{|z|^{2n}} \left(\frac{\bar{z}_1}{|z|^2}\right)^{k_1} \cdots \left(\frac{\bar{z}_n}{|z|^2}\right)^{k_n},$$

we see that (1) implies that the series

$$\theta(z) = \sum_{k_1,\dots,k_n \geq 0} \frac{c_{k_1,\dots,k_n}}{k_1! \dots k_n!} \eta_{k_1,\dots,k_n}(z)$$

converges and defines a $\bar{\partial}$-closed $(0, n-1)$-form with C^∞ coefficients in $\mathbb{C}^n - \{0\}$. Also (1) implies that

$$\int\limits_{|z|=r} z_1^{p_1} \cdots z_n^{p_n} \theta(z) \wedge \omega(z)$$
$$= \sum_{k_1,\dots,k_n \geq 0} \frac{c_{k_1,\dots,k_n}}{k_1! \dots k_n!} \int\limits_{|z|=r} z_1^{p_1} \cdots z_n^{p_n} \eta_{k_1,\dots,k_n}(z) \wedge \omega(z).$$

Applying (3) with $f(z) = z_1^{p_1} \cdots z_n^{p_n}$, we find that

(4)
$$\int\limits_{|z|=r} z_1^{p_1} \cdots z_n^{p_n} \eta_{k_1,\dots,k_n}(z) \wedge \omega(z) = \begin{cases} p_1! \dots p_n! & \text{if } (k_1, \dots, k_n) = (p_1, \dots, p_n) \\ 0 & \text{otherwise.} \end{cases}$$

This gives that

$$\int\limits_{|z|=r} z_1^{p_1} \cdots z_n^{p_n} \theta(z) \wedge \omega(z) = c_{p_1,\dots,p_n}$$

and completes the proof of the theorem in this direction.

To prove the other direction of the theorem, we consider (as in [3]) the function $F(\zeta)$ defined by the integral

$$F(\zeta) = \int\limits_{|z|=r} e^{\langle \zeta, z \rangle} \theta(z) \wedge \omega(z), \quad \zeta \in \mathbb{C}^n,$$

where $\langle \zeta, z\rangle = \sum \zeta_j z_j$. It is easy to see that F is an entire holomorphic function and that

$$c_k = \int_{|z|=r} z_1^{k_1} \cdots z_n^{k_n} \theta(z) \wedge \omega(z) = \left. \frac{\partial^{k_1+\cdots+k_n} F(\zeta)}{\partial \zeta_1^{k_1} \cdots \partial \zeta_n^{k_n}} \right|_{\zeta=0}.$$

Since r (in the definition of F) can be arbitrarily small, it follows that the function F satisfies the following: For every $\delta > 0$ there exists a constant $C_\delta > 0$ so that

$$|F(\zeta)| \leq C_\delta \, \mathrm{e}^{\delta|\zeta|} \quad \text{for every} \ \ \zeta \in \mathbb{C}^n.$$

Therefore, by Cauchy's inequalities, applied to the entire function $F(\zeta)$, the coefficients c_k satisfy the inequality: For $\delta > 0$,

$$\frac{|c_{k_1,\dots,k_n}|}{k_1! \dots k_n!} \leq C_\delta \frac{\mathrm{e}^{\delta(R_1+\cdots+R_n)}}{R_1^{k_1} \dots R_n^{k_n}}, \quad \text{for every } R_1, \dots, R_n > 0.$$

Applying this inequality with $R_1 = k_1/\delta, \dots, R_n = k_n/\delta$ we obtain that for every $\delta > 0$,

$$\frac{|c_{k_1,\dots,k_n}|}{k_1! \dots k_n!} \leq C_\delta \frac{(\delta\, \mathrm{e})^{k_1+\cdots+k_n}}{k_1^{k_1} \dots k_n^{k_n}}, \quad \text{for all} \ \ k_1, \dots, k_n.$$

(In case some $k_j = 0$, the above inequality also holds with the convention $k_j^{k_j} = 1$.) Thus

$$|c_{k_1,\dots,k_n}| \leq C_\delta (\delta\, \mathrm{e})^{k_1+\cdots+k_n}, \quad \text{for all} \ \ k_1, \dots, k_n, \quad \text{and for all} \ \ \delta > 0.$$

Therefore

$$\sum_{k_1,\dots,k_n \geq 0} |c_{k_1,\dots,k_n}| s_1^{k_1} \dots s_n^{k_n} \leq C_\delta \sum_{k_1,\dots,k_n \geq 0} (\delta\, \mathrm{e}\, s_1)^{k_1} \dots (\delta\, \mathrm{e}\, s_n)^{k_n} < \infty,$$

provided that $\delta < \min\{1/(\mathrm{e}\, s_j) : \ j = 1, \dots, n\}$. Thus the sequence c_k satisfies $(*)$. The proof of the theorem is now complete.

3. Remark

According to Theorem 1, to each $\theta \in Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\})$, we may associate an entire function T_θ defined by the formula:

$$T_\theta(\zeta) = \sum_{k_1,\dots,k_n \geq 0} c_{k_1,\dots,k_n} \zeta_1^{k_1} \dots \zeta_n^{k_n}, \quad \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n,$$

where

$$c_{k_1,\dots,k_n} = \int\limits_{|z|=r} z_1^{k_1}\cdots z_n^{k_n}\theta(z)\wedge\omega(z).$$

Then the transform $T : Z_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\}) \to \mathcal{O}(\mathbb{C}^n)$, $\theta \to T_\theta$, is linear and its kernel is the set of $\bar\partial$-exact forms, i.e.,

$$\ker T = B_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\}).$$

(This follows from Lemma 2, below).
Thus T induces an isomorphism of linear spaces:

$$\tilde T : H_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\}) \to \mathcal{O}(\mathbb{C}^n), \quad \text{defined by} \quad \tilde T([\theta]) = T(\theta),$$

for $[\theta] \in H_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\})$.
In particular we may transfer, in a natural way, the multiplication structure from $\mathcal{O}(\mathbb{C}^n)$ to $H_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\})$:

$$[\theta_1]\cdot[\theta_2] = \tilde T^{-1}(T(\theta_1)\cdot T(\theta_2)), \quad \text{for} \quad [\theta_1],[\theta_2] \in H_{\bar\partial}^{(0,n-1)}(\mathbb{C}^n - \{0\}).$$

According to this multiplication,

$$[\eta_{k_1,\dots,k_n}]\cdot[\eta_{p_1,\dots,p_n}] = \frac{k_1!\dots k_n! p_1!\dots p_n!}{(k_1+p_1)!\dots(p_n+k_n)!}[\eta_{k_1+p_1,\dots,k_n+p_n}].$$

This follows from (4) and the fact that

$$(\zeta_1^{k_1}\dots\zeta_n^{k_n})\cdot(\zeta_1^{p_1}\dots\zeta_n^{p_n}) = \zeta_1^{k_1+p_1}\dots\zeta_n^{k_n+p_n}.$$

For the proof of Theorem 2, we will need the following lemmas. The proof of Lemma 1 is based on a classical idea of a “patching” process, using a partition of unity, and a “correction” process, using a solution of an appropriate differential equation. (The case $n=1$ is in [4, p. 13]). Lemma 2 is a generalization of a case of [2, Lemma 5].

Lemma 1. *Let D be an open set in $\mathbb{C}^n$ and V_l, $l = 1,2,3,\dots$, a sequence of open subsets of D with $D = \bigcup_l V_l$. Suppose that, for each pair $(l,m) \in \mathbb{N}\times\mathbb{N}$, we are given a differential form $\theta_{lm} \in Z_{\bar\partial}^{(0,n-1)}(V_l\cap V_m)$ (here we assume that $Z_{\bar\partial}^{(0,n-1)}(\emptyset) = \{0\}$) in such a way that*

$$\theta_{lm} + \theta_{mp} + \theta_{pl} = 0 \;\; \textit{in} \;\; V_l\cap V_m\cap V_p, \quad \textit{for every} \;\; l,m,p \in \mathbb{N}.$$

Then there exist $\theta_l \in Z_{\bar\partial}^{(0,n-1)}(V_l)$, $l \in \mathbb{N}$, *so that*

$$\theta_l - \theta_m = \theta_{lm} \quad \textit{in} \quad V_l \cap V_m, \quad \textit{for every} \quad l, m \in \mathbb{N}.$$

Proof: First, let us notice that the assumptions, imposed on θ_{lm}, imply that

$$\theta_{ll} = 0 \ \text{ in } \ V_l \ \text{ and } \ \theta_{lm} + \theta_{ml} = 0 \ \text{ in } \ V_l \cap V_m, \ \text{ for every } \ l, m \in \mathbb{N}.$$

Then, let us consider a partition of unity subordinate to the cover $\{V_l\}$, i.e., we consider functions $\chi_l \in C^\infty(D)$, with the following properties: $0 \le \chi_l \le 1$, $\mathrm{supp}(\chi_l) \subset V_l$, the family $\{\mathrm{supp}(\chi_l) : l \in \mathbb{N}\}$ should be locally finite, and $\sum \chi_l = 1$ in D.

For $l \in \mathbb{N}$, we define the $(0, n-1)$-forms

$$\Theta_l = \sum_{m\in\mathbb{N}} \chi_m \theta_{lm}, \quad \text{with} \quad C^\infty \quad \text{coefficients in} \quad V_l.$$

Here, the differential form $\chi_m\theta_{lm}$ is defined to be 0 in $V_l - V_m$. Writing the set V_l as the union of the open sets $V_l \cap V_m$ and $V_l - \mathrm{supp}(\chi_m)$, and observing that, according to the above definition of the differential form $\chi_m\theta_{lm}$, $\chi_m\theta_{lm} = 0$ in $V_l - \mathrm{supp}(\chi_m)$, we see that, indeed, the sum $\sum_m \chi_m\theta_{lm}$ has C^∞ coefficients in V_l. A computation shows that

$$\begin{aligned}\Theta_l - \Theta_m = \sum_{p\in\mathbb{N}} \chi_p\theta_{lp} - \sum_{p\in\mathbb{N}} \chi_p\theta_{mp} &= \sum_{p\in\mathbb{N}} \chi_p(\theta_{lp} - \theta_{mp}) \\ &= \sum_{p\in\mathbb{N}} \chi_p\theta_{lm} = \theta_{lm}, \ \text{ in } \ V_l \cap V_m.\end{aligned}$$

But $\theta_{lm} \in Z_{\bar\partial}^{(0,n-1)}(V_l \cap V_m)$, i.e., $\bar\partial\theta_{lm} = 0$, and therefore

$$\bar\partial\Theta_l = \bar\partial\Theta_m, \quad \text{in} \quad V_l \cap V_m.$$

Since $H_{\bar\partial}^{(0,n)}(D) = 0$, it follows that there exists a $(0, n-1)$-form Θ with C^∞ coefficients in D, so that

$$\bar\partial\Theta = \bar\partial\Theta_l, \quad \text{in} \quad V_l.$$

Thus if we set $\theta_l = \Theta_l - \Theta$, we obtain differential forms $\theta_l \in Z_{\bar\partial}^{(0,n-1)}(V_l)$ which satisfy the equations

$$\theta_l - \theta_m = (\Theta_l - \Theta) - (\Theta_m - \Theta) = \theta_{lm} \ \text{ in } \ V_l \cap V_m, \ \text{ for every } \ l, m \in \mathbb{N}.$$

This completes the construction of the lemma. □

Lemma 2. *Let D be an open pseudoconvex set in $\mathbb{C}^n$, A a discrete subset of D and $\theta \in Z^{(0,n-1)}_{\bar\partial}(D-A)$. Then the following are equivalent:*

(I) *θ is $\bar\partial$-exact in $D-A$;*

(II) *$\int_{|z-\alpha|=r_\alpha} \mathrm{e}^{\langle\zeta,z\rangle}\,\theta(z)\wedge\omega(z)=0$, for every $\alpha\in A$ and $\zeta\in\mathbb{C}^n$ (where $r_\alpha>0$ are sufficiently small);*

(III) *$\int_{|z-\alpha|=r_\alpha} f(z)\theta(z)\wedge\omega(z)=0$, for every $\alpha\in A$ and for every $f\in\mathcal{O}(\mathbb{C}^n)$;*

(IV) *$\int_{|z-\alpha|=r_\alpha}(z_1-\alpha_1)^{k_1}\cdots(z_n-\alpha_n)^{k_n}\theta(z)\wedge\omega(z)=0$, for every k and $\alpha\in A$.*

Proof: Since the set of linear combinations of the functions $\mathrm{e}^{\langle\zeta,z\rangle}$, $\zeta\in\mathbb{C}^n$, is dense in $\mathcal{O}(\mathbb{C}^n)$, it is clear that (II)$\Leftrightarrow$(III).
Also, since every entire function can be expanded in a power series with center α, it follows that (III)$\Leftrightarrow$(IV).
The implication (I)$\Rightarrow$(III) follows from Stokes's theorem.
Thus it remains to show that (III)$\Rightarrow$(I). The proof of this is based on the Cauchy-Leray formula and it is similar to the proof of [2, Lemma 5].
First, let us notice that we may find a sequence $G_\nu \subset\subset D$, $\nu=1,2,3,\dots$, of strictly pseudoconvex sets with smooth boundary, which exhaust the pseudoconvex set D, and in such a way that $(\partial G_\nu)\cap A=\emptyset$, for every ν. This is possible, since A is discrete in D. Then each set G_ν will contain finitely many points from the set A. It follows that the set $D-A$ can be exhausted by a sequence of compact sets of the form

$$K=\bar G-[B(\alpha^1,\varepsilon^1)\cup\ldots\cup B(\alpha^N,\varepsilon^N)],$$

where $G\subset\subset D$ is strictly pseudoconvex with smooth boundary, $\alpha^l\in A$, $\varepsilon^l>0$, $l=1,2,\dots,N$, and the closures of the balls

$$B(\alpha^l,\varepsilon^l)=\{z\in\mathbb{C}^n:|z-\alpha^l|<\varepsilon^l\}$$

are pair-wise disjoint.
Fixing such a set K, we consider the map $\gamma:(\partial K)\times\mathrm{int}(K)\to\mathbb{C}^n$ as follows: For $(\zeta,z)\in(\partial K)\times\mathrm{int}(K)$, $\{\gamma_j(\zeta,z)\}_{j=1}^n$ is defined to be a Henkin-Ramirez map of the strictly pseudoconvex set G, if $\zeta\in\partial G$, and

$$\gamma_j(\zeta,z)=\frac{\partial\rho_l}{\partial\zeta_j}(z)\quad\text{if}\quad\zeta\in\{\rho_l=0\},$$

where $\rho_l(\zeta)=|\zeta-\alpha^l|^2-(\varepsilon^l)^2$.
(For exhaustions of pseudoconvex sets by strictly pseudoconvex domains and constructions of Henkin-Ramirez maps, see [5] and [6].)
Then

$$\sum_{j=1}^n(\zeta_j-z_j)\gamma_j(\zeta,z)\neq 0,\quad\text{for}\quad(\zeta,z)\in(\partial K)\times\mathrm{int}(K),$$

and therefore we may write down the Cauchy-Leray formula:

$$(5) \qquad u = \bar{\partial}_z(T_{q-1}u) + T_q(\bar{\partial}u) + L_q^{\gamma}(u), \quad \text{for } (0,q)\text{-forms } u \text{ in a neighborhood of } K$$

(notation is as in [2, p. 912]).
Now if $\theta \in Z_{\bar{\partial}}^{(0,n-1)}(D-A)$ satisfies (III), it follows as in the proof of [2, Lemma 5] that $L_{n-1}^{\gamma}(\theta) = 0$, and therefore (5) gives

$$\theta = \bar{\partial}_z(T_{n-2}\theta), \quad \text{in} \quad \text{int}(K).$$

Now the conclusion that θ is $\bar{\partial}$-exact in $D - A$, follows from [2, Lemma 4], and this completes the proof of the implication (III)⇒(I).
The proof of the lemma is complete. □

4. Proof of Theorem 2

First, with D being an arbitrary open set in $\mathbb{C}^n$, we will use Lemma 1 in order to construct a $\theta \in Z_{\bar{\partial}}^{(0,n-1)}(D - A)$ which satisfies $(**)$.
Let α^l, $l = 1, 2, 3, \dots$, be an enumeration of the set A. By Theorem 1, for each $l = 1, 2, 3, \dots$, there exists $\theta_l \in Z_{\bar{\partial}}^{(0,n-1)}(\mathbb{C}^n - \{\alpha^l\})$ so that

$$\int_{|z-\alpha^l|=r_{\alpha^l}} (z_1 - \alpha_1^l)^{k_1} \cdots (z_n - \alpha_n^l)^{k_n} \theta_l(z) \wedge \omega(z) = c_{k_1,\dots,k_n}^{\alpha^l}, \quad \text{for every} \quad k.$$

Next we consider an open cover $\{V_0, V_1, V_2, \dots\}$ of D, which is of the form: $V_0 = D - A$, and, for $l \geq 1$, V_l is a small ball centered at the point α^l, so that $V_l \cap V_m = \emptyset$ for $l \neq m$, $l, m \geq 1$.
For each pair (l, m) with $l, m \geq 0$, we define $\theta_{lm} \in Z_{\bar{\partial}}^{(0,n-1)}(V_l \cap V_m)$ in the following way:

$$\theta_{00} = 0, \ \theta_{lm} = 0 \ \text{ if } \ l, m \geq 1, \ \text{ and } \ \theta_{0l} = -\theta_{l0} = \theta_l \quad \text{in } \ V_0 \cap V_l = V_l - \{\alpha^l\} \ \text{ for } \ l \geq 1.$$

Then

$$\theta_{lm} + \theta_{mp} + \theta_{pl} = 0 \quad \text{in} \quad V_l \cap V_m \cap V_p, \quad \text{for every} \quad l, m, p \geq 0.$$

Therefore, from Lemma 1, there exist $\tilde{\theta}_l \in Z_{\bar{\partial}}^{(0,n-1)}(V_l)$, $l \geq 0$, so that

$$\tilde{\theta}_l - \tilde{\theta}_m = \theta_{lm} \ \text{ in } \ V_l \cap V_m, \quad \text{for every} \quad l, m \geq 0.$$

In particular,

$$\tilde\theta_0 - \tilde\theta_l = \theta_{0l} = \theta_l, \quad \text{in} \quad V_0 \cap V_l = V_l - \{\alpha^l\}, \quad \text{for} \quad l \geq 1.$$

Define $\theta = \tilde\theta_0 \in Z^{(0,n-1)}_{\bar\partial}(D - A)$. Since $\tilde\theta_l \in Z^{(0,n-1)}_{\bar\partial}(V_l)$, the $(0, n-1)$-form

$$(z_1 - \alpha_1^l)^{k_1} \cdots (z_n - \alpha_n^l)^{k_n} \tilde\theta_l(z)$$

is also $\bar\partial$-closed in V_l.
It follows that the differential form $(z_1-\alpha_1^l)^{k_1} \cdots (z_n-\alpha_n^l)^{k_n} \tilde\theta_l(z) \wedge \omega(z)$ is d-closed in V_l, and therefore, by Stokes's theorem,

$$\int\limits_{|z-\alpha^l|=r_{\alpha^l}} (z_1 - \alpha_1^l)^{k_1} \cdots (z_n - \alpha_n^l)^{k_n} \tilde\theta_l(z) \wedge \omega(z) = 0, \quad \text{for every} \quad k \quad \text{and} \quad l \geq 1.$$

(Recall that the r_α's are assumed sufficiently small.) Since for each $l \geq 1$,

$$\theta = \theta_l + \tilde\theta_l, \quad \text{in} \quad V_l - \{\alpha^l\},$$

it follows that θ satisfies $(**)$.
This completes the proof of the first part of the theorem.

Finally, assume that D is pseudoconvex and that two differential forms $\theta, \hat\theta \in Z^{(0,n-1)}_{\bar\partial}(D - A)$ satisfy $(**)$. Then their difference $\theta - \hat\theta$ satisfies the following equations

$$\int\limits_{|z-\alpha^l|=r_{\alpha^l}} (z_1 - \alpha_1)^{k_1} \cdots (z_n - \alpha_n)^{k_n} [\theta(z) - \hat\theta(z)] \wedge \omega(z) = 0,$$

$$\text{for every} \quad k \quad \text{and every} \quad \alpha \in A.$$

It follows from Lemma 2, that $\theta - \hat\theta$ is $\bar\partial$-exact in $D - A$. This completes the proof of the theorem. □

We close with the following remark. In the case D is pseudoconvex, Theorem 2 establishes a bijection from the $\bar\partial$-cohomology group $H^{(0,n-1)}_{\bar\partial}(D - A)$ to the set $[\mathcal{O}(\mathbb{C}^n)]^A$ of all maps $A \to \mathcal{O}(\mathbb{C}^n)$. Thus we have, in a natural way, an isomorphism of linear spaces:

$$H^{(0,n-1)}_{\bar\partial}(D - A) \approx [\mathcal{O}(\mathbb{C}^n)]^A.$$

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(*Received* March 8, 2002, *revised* November 19, 2002)