Perimeter preservers of nonnegative integer matrices

Seok-Zun Song, Kyung-Tae Kang, Sucheol Yi

Abstract. We investigate the perimeter of nonnegative integer matrices. We also characterize the linear operators which preserve the rank and perimeter of nonnegative integer matrices. That is, a linear operator T preserves the rank and perimeter of rank-1 matrices if and only if it has the form $T(A) = P(A \circ B)Q$, or $T(A) = P(A^t \circ B)Q$ with appropriate permutation matrices P and Q and positive integer matrix B, where $\circ$ denotes Hadamard product.

Keywords: linear operator, rank, perimeter, (P, Q, B)-operator

Classification: 15A04, 15A33, 15A48

1. Introduction and preliminaries

Nonnegative integer matrices are combinatorially interesting matrices. So it has been a subject of many research works (see [5]). In [1], Beasley and Pullman defined the perimeter of a Boolean rank-1 matrix in order to characterize the linear operators that preserve Boolean rank. In this paper, we consider the nonnegative integer matrices of rank-1 and their perimeters. We also characterize the linear operators that preserve the rank and perimeter of the rank-1 matrices over nonnegative integers.

Let $\mathbb{Z}_+$ be a semiring of nonnegative integers and let $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ denote the set of all $m \times n$ matrices with entries in $\mathbb{Z}_+$. The *rank* or *factor rank* [2], $r(A)$, of a nonzero matrix $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is defined as the least integer k for which there exist $m \times k$ and $k \times n$ matrices B and C with $A = BC$. The rank of a zero matrix is zero. If $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ has rank 1, there exist nonzero vectors $\mathbf{u} \in \mathcal{M}_{m,1}(\mathbb{Z}_+)$ and $\mathbf{v} \in \mathcal{M}_{n,1}(\mathbb{Z}_+)$ such that $A = \mathbf{u}\mathbf{v}^t$. The *perimeter* [1] of this rank 1 matrix A, $p(A)$ is defined as $|\mathbf{u}| + |\mathbf{v}|$ for arbitrary factorization $A = \mathbf{u}\mathbf{v}^t$, where $|\mathbf{u}|$ denotes the number of nonzero entries in $\mathbf{u}$. It is clear that the perimeter of a rank 1 matrix is uniquely determined by the given matrix. Let $A \circ B$ denote the Hadamard (or Schur) product, the (i, j) entry of $A \circ B$ is $a_{ij}b_{ij}$.

A matrix in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ is called a *cell* [3] if it has exactly one nonzero entry, that being a 1. We denote the cell whose nonzero entry is in the (i, j)th position by E_{ij}. Let $\mathbb{E}_{m,n} = \{E_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. For $A = [a_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we define $A^* = [{a_{ij}}^*]$ to be the $m \times n$ $(0, 1)$-matrix whose (i, j)th entry is 1 if and only if $a_{ij} > 0$.

It follows from the definition that $p(A) = p(A^*)$ and $(AB)^* = A^*B^*$, $(B + C)^* = B^* +_B C^*$, where $1 +_B 1 = 1$ is Boolean arithmetic, for all $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ and all $B, C \in \mathcal{M}_{n,r}(\mathbb{Z}_+)$.

If A and B are in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we say that A *dominates* B (written $B \le A$ or $A \ge B$) if $a_{ij} = 0$ implies $b_{ij} = 0$ for all i, j ([4]). Then we can obtain the fact that $A \ge B$ if and only if $(A + B)^* = A^*$ for any $m \times n$ matrices A and B.

2. Perimeter preservers

A mapping $T : \mathcal{M}_{m,n}(\mathbb{Z}_+) \to \mathcal{M}_{m,n}(\mathbb{Z}_+)$ is called a *linear operator* if T satisfies

$$T(\alpha A + \beta B) = \alpha T(A) + \beta T(B)$$

for all $A, B \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ and for all $\alpha, \beta \in \mathbb{Z}_+$.

In this section, we will characterize the linear operators that preserve both the rank and the perimeter of every rank-1 matrix in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$.

Suppose T is a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then

(1) T is a (P, Q, B)*-operator* if there exist permutation matrices $P \in \mathcal{M}_{m,m}(\mathbb{Z}_+)$, $Q \in \mathcal{M}_{n,n}(\mathbb{Z}_+)$ and a positive matrix $B \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ with $r(B) = 1$ such that $T(A) = P(A \circ B)Q$ for all A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, or $m = n$ and $T(A) = P(A^t \circ B)Q$ for all A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$;
(2) T *preserves rank* 1 if $r(T(A)) = 1$ whenever $r(A) = 1$ for all $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$;
(3) T *preserves perimeter* k of rank-1 matrices if $p(T(A)) = k$ whenever $p(A) = k$ for all $A \in \mathcal{M}_{m,n}(\mathbb{Z}_+)$ with $r(A) = 1$.

Theorem 2.1. *If T is a (P, Q, B)-operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, then T preserves both rank and perimeter of every rank-1 matrix.*

Proof: Since the operators Hadamard product, transpose and permutational equivalence preserve the rank and perimeter of every rank-1 matrix, the theorem follows. □

We note that an $m \times n$ matrix has perimeter 2 if and only if it is a positive integer multiple of a cell. We say that A is *a row (column) matrix* if A has nonzero entries only in one row (column, respectively). Thus we have the following lemma:

Lemma 2.2. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. If T preserves rank 1 and perimeter 2 of every rank-1 matrix, then the following statements hold:*

(1) *there exist positive integers u_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$, and a mapping $f : \mathbb{E}_{m,n} \to \mathbb{E}_{m,n}$ such that for $A = [a_{ij}]$, $T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} u_{ij} f(E_{ij})$;*
(2) *T maps a row (column) matrix to a row (column) matrix or if $m = n$, a row (column) matrix to a column (row) matrix.*

Proof: (1) Since T preserves perimeter 2, T maps a cell into a positive integer multiple of a cell.

(2) If not, then there exist two distinct cells E_{ij}, E_{ih} in some ith row such that $T(E_{ij})$ and $T(E_{ih})$ lie in two different rows and different columns. Then the rank of $E_{ij}+E_{ih}$ is 1 but that of $T\big(E_{ij}+E_{ih}\big)=T(E_{ij})+T(E_{ih})$ is 2. Therefore T does not preserve rank 1, a contradiction. $\square$

An example follows of a linear operator that preserves rank 1 and perimeter 2 of a rank-1 matrix, but the operator does not preserve perimeter 3 and is not a (P,Q,B)-operator.

Example 2.3. Let $T:\mathcal{M}_{2,2}(\mathbb{Z}_+)\to\mathcal{M}_{2,2}(\mathbb{Z}_+)$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d\end{bmatrix}\right)=(a+b+c+d)\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}.$$

It is easy to verify that T is a linear operator which preserves rank 1 and perimeter 2. But T does not preserve perimeter 3 and hence it is not a (P,Q,B)-operator. $\square$

Let $R_i=\{E_{ij}\,|\,1\le j\le n\}$, $C_j=\{E_{ij}\,|\,1\le i\le m\}$, $\mathcal{R}=\{R_i\,|\,1\le i\le m\}$ and $\mathcal{C}=\{C_j\,|\,1\le j\le n\}$. For a linear operator T on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, define $T^*(A)=[T(A)]^*$ for all A in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Let $T^*(R_i)=\{T^*(E_{ij})\,|\,1\le j\le n\}$ for all $i=1,\dots,m$ and $T^*(C_j)=\{T^*(E_{ij})\,|\,1\le i\le m\}$ for all $j=1,\dots,n$.

Lemma 2.4. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Suppose that T preserves rank 1 and perimeters 2 and p (≥ 3) of every rank-1 matrix. Then*

(1) *T maps two distinct cells in a row (or column) into positive multiples of two distinct cells in a row or in a column;*
(2) *for the case $m=n$, if T maps some R_i into a row (column) matrix then T maps every row matrix into a row (column) matrix, and if T maps some C_j into a row (column) matrix then T maps every column matrix into a row (column) matrix.*

Proof: (1) Suppose $T(E_{ij})=\alpha E_{rl}$ and $T(E_{ih})=\beta E_{rl}$ for some cells $E_{ij}\ne E_{ih}$ and some positive integers $\alpha,\beta\in\mathbb{Z}_+$. Then T maps the ith row of a matrix A into rth row or lth column by Lemma 2.2. Without loss of generality, we assume the former. Thus for any rank-1 matrix A with perimeter p (≥ 3) which dominates $E_{ij}+E_{ih}$, we can show that $T(A)$ has perimeter at most $p-1$, a contradiction. Thus T maps two distinct cells in a row into two distinct cells in a row or in a column.

(2) If not, then there exist rows R_i and R_j such that $T^*(R_i) \subseteq R_r$ and $T^*(R_j) \subseteq C_s$ for some r, s. Consider a rank-1 matrix $D = E_{ip} + E_{iq} + E_{jp} + E_{jq}$ with $p \neq q$. Then we have

$$T(D) = T(E_{ip} + E_{iq}) + T(E_{jp} + E_{jq}) = (\alpha_1 E_{rp'} + \alpha_2 E_{rq'}) + (\beta_1 E_{p''s} + \beta_2 E_{q''s})$$

for some $p' \neq q'$ and $p'' \neq q''$ and some positive integers $\alpha_i, \beta_i \in \mathbb{Z}_+$ by (1). Therefore $r\,(T(D)) \neq 1$ and T does not preserve rank 1, a contradiction. Hence T maps each row of A into a row (or a column) of $T(A)$. Similarly, T maps each column of A into a column (or a row) of $T(A)$. □

Now we have an interesting example:

Example 2.5. Consider a linear operator T on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ with $m \geq 3$ and $n \geq 4$ such that

$$T(A) = B = [b_{ij}]$$

where $A = [a_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, $b_{ij} = 0$ if $i \geq 2$ and $b_{1j} = \sum_{i=1}^{m} a_{ir}$ with $r \equiv i + (j-1) \pmod{n}$ and $1 \leq r \leq n$. Then T maps each row and each column into the first row with some positive integer multiplication. And T preserves both rank and perimeters $2, 3$ and $n+1$ of rank-1 matrices. But T does not preserve perimeters k $(k \geq 4$ and $k \neq n+1)$ of rank-1 matrices: For if $4 \leq k \leq n$, then we can choose a $2 \times (k-2)$ submatrix with perimeter k which is mapped to distinct k positions in the first row of B under T. Then this $1 \times k$ submatrix has perimeter $k+1$. Therefore T does not preserve perimeter k of rank-1 matrices. □

Lemma 2.6. *Let T be a linear operator defined by*

$$T(A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} u_{ij} f(E_{ij})$$

for some function $f : \mathbb{E}_{m,n} \to \mathbb{E}_{m,n}$ and for some positive integers u_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$. If T preserves both rank and perimeters 2 and k $(k \geq 4, k \neq n+1)$ of rank-1 matrices, then the corresponding map f is a bijection on $\mathbb{E}_{m,n}$.

Proof: By Lemma 2.2, $T(E_{ij}) = b_{ij} E_{rl}$ for some $E_{rl} \in \mathbb{E}_{m,n}$ and some positive integer $b_{ij} \in \mathbb{Z}_+$. Without loss of generality, we may assume that T maps the ith row of a matrix into the rth row with positive integer multiplication. Suppose $f(E_{ij}) = f(E_{pq})$ for some distinct pairs $E_{ij}, E_{pq} \in \mathbb{E}_{m,n}$. Then we have $T(E_{ij}) = b_{ij} E_{rl}$ and $T(E_{pq}) = c_{pq} E_{rl}$ for some positive integers $b_{ij}, c_{pq} \in \mathbb{Z}_+$. If $i = p$ or $j = q$, then we have contradictions by Lemma 2.4. So let $i \neq p$ and $j \neq q$.

If $4 \leq k \leq n$, we will show that we can choose a $2 \times (k-2)$ submatrix from the ith and pth row whose image under T has a $1 \times k$ submatrix in the rth row as follows: Since $T(E_{ij}) = b_{ij} E_{rl}$ and $T(E_{pq}) = c_{pq} E_{rl}$, T maps the ith row and the pth row into the rth row. But T maps distinct cells in each row (or column) to distinct cells by Lemma 2.4. Now, choose E_{ij}, E_{pj} but do not choose E_{iq}, E_{pq}. Since there is a cell E_{ph} $(h \neq j, q)$ in the pth row such that $f(E_{ph}) = f(E_{iq})$ but $f(E_{ih}) \neq f(E_{pj})$, we choose the 2×2 submatrix $E_{ij} + E_{ih} + E_{pj} + E_{ph}$ whose image under T is a 1×4 submatrix in the rth row. And we can choose a cell E_{ps} $(s \neq q, j, h)$ such that $f(E_{is}) \neq f(E_{pj}), f(E_{pq}), f(E_{ph})$. Then we have a 2×3 submatrix $E_{ij} + E_{ih} + E_{is} + E_{pj} + E_{ph} + E_{ps}$ whose image under T is a 1×5 submatrix in the rth row. Similarly, we can choose a $2 \times (k-2)$ submatrix whose image under T is a $1 \times k$ submatrix in the rth row. This shows that T does not preserve the perimeter k of a rank-1 matrix, a contradiction.

If $k = n + k' \geq n + 2$, consider the matrix

$$D = \sum_{s=1}^{n} E_{is} + \sum_{t=1}^{n} E_{p\,t} + \sum_{h=1}^{k'-2} \sum_{g=1}^{n} E_{hg}$$

with rank 1 and perimeter $n + k' = k$. Then T maps the ith and pth row of D into the rth row with positive integer multiplication by Lemma 2.4. Thus the perimeter of $T(D)$ is less than $n + k' = k$, a contradiction.

Hence $f(E_{ij}) \neq f(E_{pq})$ for any two distinct cells $E_{ij}, E_{pq} \in \mathbb{E}_{m,n}$. Therefore f is a bijection. $\square$

We obtain the following characterization theorem for linear operators preserving the rank and the perimeter of rank-1 matrices over nonnegative integers.

Theorem 2.7. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then the following are equivalent:*

(1) *T is a (P, Q, B)-operator;*
(2) *T preserves both rank and perimeter of rank-1 matrices;*
(3) *T preserves both rank and perimeters 2 and k $(k \geq 4, k \neq n+1)$ of rank-1 matrices.*

Proof: (1) implies (2) by Theorem 2.1. It is obvious that (2) implies (3). We now show that (3) implies (1). Assume (3). Then T induces a bijection $f : \mathbb{E}_{m,n} \to \mathbb{E}_{m,n}$ by Lemma 2.6. By Lemma 2.4, there are two cases; (a) T^* maps $\mathcal{R}$ onto $\mathcal{R}$ and maps $\mathcal{C}$ onto $\mathcal{C}$ or (b) T^* maps $\mathcal{R}$ onto $\mathcal{C}$ and $\mathcal{C}$ onto $\mathcal{R}$.

Case (a). We note that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all i, j, where σ and τ are permutations of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. Let P and Q be the permutation matrices corresponding to σ and τ, respectively. Then for any $E_{ij} \in \mathbb{E}_{m,n}$, we can write $T(E_{ij}) = b_{ij} E_{\sigma(i)\tau(j)}$ for some positive integer $b_{ij} \in \mathbb{Z}_+$. Now we claim that $B = (b_{ij})$ has rank 1. For, consider an $m \times n$ matrix J, all of whose entries are 1's. Then we have

$$T(J) = T\left(\sum_{i=1}^{m} \sum_{j=1}^{n} E_{ij}\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} T(E_{ij}) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} E_{\sigma(i)\tau(j)} = PBQ.$$

Since J has rank 1, it follows that $r(T(J)) = 1$ and hence $r(B) = 1$ since permutational equivalences preserve rank. Therefore for any $A = [a_{ij}]$ in $\mathcal{M}_{m,n}(\mathbb{Z}_+)$, we have

$$T(A) = T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}T(E_{ij})$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}E_{\sigma(i)\tau(j)} = P(A \circ B)Q.$$

Thus T is a (P, Q, B)-operator.

Case (b). We note that $m = n$, $T^*(R_i) = C_{\sigma(i)}$ and $T^*(C_j) = R_{\tau(j)}$ for all i, j, where σ and τ are permutations of $\{1, \ldots, m\}$. By an argument similar to case (a), we obtain that $T(A)$ is of the form $T(A) = P(A^t \circ B)Q$. Thus T is a (P, Q, B)-operator. $\square$

We say that a linear operator T on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$ *strongly preserves* perimeter k of rank-1 matrices if $p\,(T(A)) = k$ if and only if $p(A) = k$.

Consider a linear operator T on $\mathcal{M}_{2,2}(\mathbb{Z}_+)$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + b + c + d)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then T preserves both rank and perimeter 2 of rank-1 matrices but does not strongly preserve perimeter 2.

Theorem 2.8. *Let T be a linear operator on $\mathcal{M}_{m,n}(\mathbb{Z}_+)$. Then T preserves both rank and perimeter of rank-1 matrices if and only if it preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices.*

Proof: Suppose T preserves perimeter 3 and strongly preserves perimeter 2 of rank-1 matrices. Then T maps each row of a matrix into a row or a column (if $m = n$) with positive integer multiplication. Since T strongly preserves perimeter 2, T maps each cell onto a positive integer multiple of a cell. This means that T induces a bijection f on $\mathbb{E}_{m,n}$. Thus T preserves both rank and perimeter of rank-1 matrices by a method similar to that in the proof of Theorem 2.7.

The converse is immediate. $\square$

Thus we have characterizations of the linear operators that preserve both rank and perimeter of rank-1 matrices over nonnegative integers.

References

[1] Beasley L.B., Pullman N.J., *Boolean rank-preserving operators and Boolean rank-*1 *spaces*, Linear Algebra Appl. **59** (1984), 55–77.

[2] Beasley L.B., Gregory D.A., Pullman N.J., *Nonnegative rank-preserving operators*, Linear Algebra Appl. **65** (1985), 207–223.

[3] Beasley L.B., Pullman N.J., *Term-rank, permanent and rook-polynomial preservers*, Linear Algebra Appl. **90** (1987), 33–46.

[4] Beasley L.B., Song S.Z., Lee S.G., *Zero term rank preservers*, Linear and Multilinear Algebra **48** (2001), 313–318.

[5] Song S.Z., *Linear operators that preserve maximal column ranks of nonnegative integer matrices*, Proc. Amer. Math. Soc. **126** (1998), 2205–2211.

Seok-Zun Song, Kyung-Tae Kang:
Department of Mathematics, Cheju National University, Jeju 690–756,
Republic of Korea

E-mail: szsong@cheju.ac.kr
kangkt@cheju.ac.kr

Sucheol Yi:
Department of Applied Mathematics, Changwon National University,
Changwon 641–773, Republic of Korea

E-mail: scyi@changwon.ac.kr

(*Received* May 16, 2003, *revised* August 13, 2003)