The structure of the σ-ideal of σ-porous sets

Miroslav Zelený, Jan Pelant

Abstract. We show a general method of construction of non-σ-porous sets in complete metric spaces. This method enables us to answer several open questions. We prove that each non-σ-porous Suslin subset of a topologically complete metric space contains a non-σ-porous closed subset. We show also a sufficient condition, which gives that a certain system of compact sets contains a non-σ-porous element. Namely, if we denote the space of all compact subsets of a compact metric space E with the Vietoris topology by $\mathcal{K}(E)$, then it is shown that each analytic subset of $\mathcal{K}(E)$ containing all countable compact subsets of E contains necessarily an element, which is a non-σ-porous subset of E. We show several applications of this result to problems from real and harmonic analysis (e.g. the existence of a closed non-σ-porous set of uniqueness for trigonometric series). Finally we investigate also descriptive properties of the σ-ideal of compact σ-porous sets.

Keywords: σ-porosity, descriptive set theory, σ-ideal, trigonometric series, sets of uniqueness

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1. Introduction

Let (P, ρ) be a metric space, $x \in P$ and $r > 0$. We denote

$$B(x,r) = \{y \in P;\ \rho(x,y) < r\} \text{ (the open ball with radius } r \text{ and center } x),$$
$$\overline{B}(x,r) = \{y \in P;\ \rho(x,y) \le r\} \text{ (the closed ball with radius } r \text{ and center } x).$$

Let $M \subset P$, $x \in P$ and $R > 0$. Then we define

$$\begin{aligned} \theta(x,R,M) = {} & \sup\{r > 0;\ \text{there exists an open ball } B(z,r) \\ & \qquad \text{such that } \rho(x,z) < R \text{ and } B(z,r) \cap M = \emptyset\}, \\ p(x,M) = {} & \limsup_{R \to 0+} \frac{\theta(x,R,M)}{R}\,. \end{aligned}$$

We say that $M \subset P$ is *porous at* $x \in P$ *(in the space* P*)* if $p(x,M) > 0$. We say that $M \subset P$ is *porous* (*in* P), if $p(x,M) > 0$ whenever $x \in M$. A set $M \subset P$ is said to be σ*-porous* (*in* P), if it is a countable union of porous sets (in P).

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The notion of σ-porosity was firstly defined by Dolzhenko in [Do] to describe certain sets of exceptional points in the theory of boundary behavior of functions. This notion appears naturally in many problems from cluster sets theory and differentiation theory. The reader can consult [Za$_2$] for more information. The notion of σ-porosity was used to obtain stronger versions of some results on exceptional sets replacing some kind of smallness (for example meagerness) by σ-porosity.

Let us explain how the paper is organized. In §2 we introduce auxiliary notions and notation. Then we prove a series of technical lemmas, which provide a basic tool for our constructions of non-σ-porous sets. The most complicated result is contained in §3. We prove:

Each Suslin non-σ-porous subset of a topologically complete metric space contains a closed non-σ-porous subset.

This result was firstly obtained by J. Pelant. M. Zelený proved independently the same result in compact metric spaces using a completely different method. The proof presented combines Pelant's original idea and techniques from §2 developed by Zelený. The other results of the paper are due to Zelený.

In §4 we prove theorems, which provide a general method of construction of small but non-σ-porous subsets of a compact metric space. We obtain in particular the following result:

Let E be a compact metric space. Suppose that A is an analytic subset of the space of all compact subsets of E (with the Vietoris topology) and contains all countable compact subsets of E. Then there exists $L \in A$ such that the set L is not a σ-porous subset of E.

We show that several results from different fields can be obtained using this theorem. Section 5 is devoted to applications of the preceding results. We show the existence of closed non-σ-porous sets of Hausdorff dimension zero. We answer negatively Laczkovich's question, whether each proper analytic subgroup of $\mathbb{R}$ is necessarily σ-porous. We answer also the question posed in [Za$_2$] and [BKR] showing that there exists a closed non-σ-porous set of uniqueness. Such a set of uniqueness is in some sense big.

The last section is devoted to an investigation of descriptive properties of the σ-ideal $I_{\sigma\text{-}p}(E)$ of compact σ-porous subsets of a nonempty compact metric space E. It is shown that if E has no isolated point, then $I_{\sigma\text{-}p}(E)$ has no Borel basis. In this case we reprove the unpublished result of Debs and Preiss that $I_{\sigma\text{-}p}(E)$ is $\mathbf{\Pi^1_1}$-complete and also the result of Reclaw ([Re]) that $I_{\sigma\text{-}p}(E)$ is not thin.

2. Several lemmas

Lemma 2.1. *Let P be a metric space. Let $A \subset T \subset P$ and let T be porous at no point of A. Then $C \subset A$ is σ-porous in T if and only if C is σ-porous in P.*

Proof: If C is σ-porous in T, then C is obviously σ-porous in P. Now assume that $C \subset A$ is porous in P. Suppose that $x \in C$. There exist $\eta > 0$ and a sequence of open balls $(B(y_n, r_n))_{n=1}^{\infty}$ such that $\lim_{n\to\infty} y_n = x$, $r_n/\rho(x, y_n) > \eta$ and $B(y_n, r_n) \cap C = \emptyset$. The set T is not porous at x. Thus there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $B(y_n, r_n/2) \cap T \neq \emptyset$. This means that for every $n \in \mathbb{N}$, $n \geq n_0$, there exists a ball $B(w_n, r_n/2) \subset B(y_n, r_n)$ with $w_n \in T$. The sequence $\{B(w_n, r_n/2)\}_{n=n_0}^{\infty}$ shows that C is porous at x in the space T. Thus C is porous in T. It gives that each set $C \subset A$, which is σ-porous in P, is σ-porous in T, too. □

Setting 2.2. Throughout the rest of this section we will work in a nonempty complete metric space (P, ρ) without isolated points.

Thus each ball (open or closed) in P has a positive diameter. A ball (open or closed), considered as a set, does not uniquely determine its center and its radius, therefore a ball will be identified with a pair (center, radius). From this point of view, two distinct balls need not be geometrically different.

We will use the convention that $\operatorname{dist}(A, \emptyset) = +\infty$, whenever $A \subset P$. The symbol $\mathbb{N}$ ($\mathbb{N}_0$, respectively) stands for the set of positive (non-negative, respectively) integers.

Some ideas of this section can be found in [Ze$_2$].

Definition 2.3. (i) Let $H \subset P$, $\eta > 0$. We say that $A \subset P$ is an *η-net of H* if for every $h \in H$ there exists $a \in A$ with $\rho(a, h) < \eta$.

(ii) Let $\mathcal{V}$ be a system of closed balls. Then the symbol $\operatorname{ap}(\mathcal{V})$ stands for the set of all points $x \in P$ such that for every $\varepsilon > 0$ there exist infinitely many $B \in \mathcal{V}$ with $B \cap B(x, \varepsilon) \neq \emptyset$.

(iii) Let B be a ball. Then $c(B)$ denotes the center of B.

(iv) Let $\mathcal{V}$ be a system of closed balls. Then $c(\mathcal{V})$ denotes the set of centers of balls from $\mathcal{V}$.

(v) Let $\mathcal{V}$ be a nonempty system of closed balls satisfying

(a) $\mathcal{V}$ is point finite, i.e. each $x \in P$ is contained at most in finitely many balls from $\mathcal{V}$,

(b) $\operatorname{ap}(\mathcal{V}) \subset c(\mathcal{V})$.

Then we say that $\mathcal{V}$ is a *B-system.*

Lemma 2.4. (i) *Let $\mathcal{V}$ be a B-system and for every $B \in \mathcal{V}$ let $\mathcal{V}(B)$ be a B-system such that $\bigcup \mathcal{V}(B) \subset B$ and $c(B) \in c(\mathcal{V}(B))$. Then $\mathcal{U} = \bigcup\{\mathcal{V}(B);\ B \in \mathcal{V}\}$ is a B-system.*

(ii) *Let $\mathcal{V}$ be a B-system and for every $B \in \mathcal{V}$ let $F(B)$ be a closed subset of B such that $c(B) \in F(B)$. Then the set $F = \bigcup\{F(B);\ B \in \mathcal{V}\}$ is closed.*

(iii) *Let $\mathcal{V}$ be a B-system. Then $\bigcup \mathcal{V}$ is a closed set.*

(iv) *A union of finitely many B-systems is a B-system.*

Proof: (i) The system $\mathcal{U}$ clearly contains only closed balls and is point finite. Let $x \in \operatorname{ap}(\mathcal{U})$. Then we distinguish the following two possibilities.

1) There exist a sequence $(B_n)_{n=1}^{\infty}$ of pairwise distinct closed balls and $D \in \mathcal{V}$ such that $B_n \in \mathcal{V}(D)$ and $B(x, 1/n) \cap B_n \neq \emptyset$. Then we have $x \in \operatorname{ap}(\mathcal{V}(D)) \subset c(\mathcal{V}(D)) \subset c(\mathcal{U})$.

2) There exist sequences $(B_n)_{n=1}^{\infty}$, $(D_n)_{n=1}^{\infty}$ of closed balls such that $B_n \in \mathcal{V}(D_n)$, $D_n \in \mathcal{V}$, the D_n's are pairwise distinct and $B(x, 1/n) \cap B_n \neq \emptyset$. We have $x \in \operatorname{ap}(\mathcal{V}) \subset c(\mathcal{V})$. Thus there exists a ball $H \in \mathcal{V}$ with $c(H) = x$. We obtain $x \in c(\mathcal{V}(H)) \subset c(\mathcal{U})$ and assertion (i) is proved.

The proofs of (ii) and (iv) are straightforward and will be omitted. Assertion (iii) follows immediately from (ii). $\square$

Definition 2.5. (i) Let $M \subset P$, $x \in P$ and B_1, B_2 be two closed balls with $x \in B_2 \subset B_1$. Then we denote

$$\Gamma(x, B_1, B_2, M) = \sup\{r/\rho(x,z);\ z \in B_1 \setminus B_2,\ B(z,r) \subset B_1 \setminus M\}.$$

(ii) Let $M \subset P$, B be a closed ball and $x \in B$. Then we denote

$$\Gamma^{\star}(x, B, M) = \sup\{r/\rho(x,z);\ B(z,r) \subset B \setminus M,\ z \neq x\}.$$

(iii) Let $S \subset P$. The set of all accumulating points of S is denoted by S'.
(iv) Let $M \subset P$, $x \in P$ and $\mu > 0$. We say that x is a point of *μ-porosity* (*non-μ-porosity*, respectively) *of* M, if $p(x, M) \geq \mu$ ($p(x, M) < \mu$, respectively). We say that x is a point of *non-porosity of* M, if $p(x, M) \leq 0$.
(v) Let $\mathcal{S}$ be a system of subsets of P and $A \subset P$. We say that $\mathcal{S}$ is *discrete in* $P \setminus A$, if for every $x \in P \setminus A$ there is $r > 0$ such that $B(x, r)$ intersects at most one element of $\mathcal{S}$.

Roughly speaking the quantity $\Gamma(x, B_1, B_2, M)$ measures porosity of M at x with respect to $B_1 \setminus B_2$ and $\Gamma^{\star}(x, B, M)$ does the same job with respect to B. Now we introduce inductively two notions, which will play a key role in the sequel.

Definition 2.6. Let B be a closed ball, S be a closed nonempty subset of B and $n \in \mathbb{N}$, $\delta, \kappa, \alpha \in (0,1)$. We say that S has the *$\mathcal{C}(0, \delta, \kappa, \alpha)$-property in B* if $S = \{c(B)\}$. We say that S has the *$\mathcal{C}(n, \delta, \kappa, \alpha)$-property in B* if

$(\mathrm{C1})_n$ $\forall\, x \in S:\ \operatorname{dist}(x, B^c) > \delta^n \operatorname{diam} B$,
$(\mathrm{C2})_n$ $\Gamma^{\star}(y, B, S) \leq \kappa$ whenever $y \in S'$,
$(\mathrm{C3})_n$ each point $x \in S'$ is a point of non-$\alpha\kappa$-porosity of the set S,
$(\mathrm{C4})_n$ S' has the $\mathcal{C}(n-1, \delta, \kappa, \alpha)$-property in B.

Definition 2.7. Let B be a closed ball, $\mathcal{V}$ be a B-system, $n \in \mathbb{N}$, $\delta, \beta, \varepsilon \in (0,1)$. We say that $\mathcal{V}$ has the *$\mathcal{P}(0,\delta,\beta,\varepsilon)$-property in B* if $\mathcal{V} = \{B_0\}$, $c(B_0) = c(B)$ and $B_0 \subset B$. We say that $\mathcal{V}$ has the *$\mathcal{P}(n,\delta,\beta,\varepsilon)$-property in B* if

- $(\mathrm{P1})_n$ $\forall V \in \mathcal{V}:\ \operatorname{dist}(V, B^c) > \operatorname{diam} V$,
- $(\mathrm{P2})_n$ $\forall V \in \mathcal{V}:\ \operatorname{dist}(V, B^c) > \delta^n \operatorname{diam} B$,
- $(\mathrm{P3})_n$ $\forall V \in \mathcal{V}:\ \operatorname{diam} V \le \frac{1}{2} \operatorname{diam} B$,
- $(\mathrm{P4})_n$ there exists a B-system $\mathcal{R} \subset \mathcal{V}$ with the $\mathcal{P}(n-1,\delta,\beta,\varepsilon)$-property in B such that, for an arbitrary set J intersecting each ball from $\mathcal{V}$, we have

$$\forall R \in \mathcal{R}\ \forall x \in R:\ \operatorname{dist}(x, R^c) > \beta \operatorname{diam} R \Rightarrow \Gamma(x, B, R, J) < \varepsilon.$$

We will need the following two easy observations later.

Observation 2.8. *If P is separable, $B \subset P$ is a closed ball and $S \subset P$ is a set with the $\mathcal{C}(n,\delta,\kappa,\alpha)$-property in B for some $n \in \mathbb{N}$, $\delta, \kappa, \alpha \in (0,1)$, then S is countable.*

Observation 2.9. *Let $n \in \mathbb{N}$, $\delta, \beta, \varepsilon \in (0,1)$. If $\mathcal{V}$ has the $\mathcal{P}(n,\delta,\beta,\varepsilon)$-property in a closed ball B and $k \in \mathbb{N}_0$, $k < n$, then there exists a B-system $\mathcal{R} \subset \mathcal{V}$ with the $\mathcal{P}(k,\delta,\beta,\varepsilon)$-property in B such that, for an arbitrary set J intersecting each ball from $\mathcal{V}$, we have*

$$\forall R \in \mathcal{R}\ \forall x \in R:\ \operatorname{dist}(x, R^c) > \beta \operatorname{diam} R \Rightarrow \Gamma(x, B, R, J) < \varepsilon.$$

We show how to construct B-systems with the $\mathcal{P}(n,\delta,\beta,\varepsilon)$-property (under some conditions on $\delta, \beta, \varepsilon$) for an arbitrary $n \in \mathbb{N}$. The following picture indicates how these B-systems will look like for $n = 0, 1, 2$ and some $\delta, \beta, \varepsilon \in (0,1)$. The B-system $\mathcal{V}_0$ has the $\mathcal{P}(0,\delta,\beta,\varepsilon)$-property in B, the B-system $\mathcal{V}_1$ has the $\mathcal{P}(1,\delta,\beta,\varepsilon)$-property in B and the B-system $\mathcal{V}_2$ has the $\mathcal{P}(2,\delta,\beta,\varepsilon)$-property in B.

$\mathcal{V}_0$ $\mathcal{V}_1$ $\mathcal{V}_2$

The next lemma will enable us to deal with simpler sets during our construction.

Lemma 2.10. *Let B be a closed ball, $\xi > 0$. Let A and T be nonempty subsets of B such that A is closed, $A \subset T$ and $\Gamma^\star(y, B, T) < \xi$ whenever $y \in A$. Then there exists a closed set D such that $D \subset T$, $D' = A$ and $\Gamma^\star(y, B, D) < 5\xi$ whenever $y \in A$.*

Proof: Put

$$F_k = \{z \in T;\ \operatorname{dist}(z, A) \le (1+\xi)\operatorname{diam} B/k\}, \qquad k \in \mathbb{N}.$$

Let $W_k \subset F_k$, $k \in \mathbb{N}$, be a discrete $\xi \operatorname{diam} B/k$-net of F_k. Put $D = A \cup \bigcup_{k=1}^{\infty} W_k$. It is easy to see that D is closed, $D \subset T$ and $D' = A$.

Take $y \in A$ and a ball $B(z, s) \subset B \setminus D$. We distinguish two cases.

1) If $s \ge \xi\rho(y, z)$, then $B(z, \xi\rho(y, z)) \subset B(z, s) \subset B \setminus D$. Since $\Gamma^\star(y, B, T) < \xi$, we have $B(z, \xi\rho(y, z)) \cap T \neq \emptyset$ and we can find $t \in T$ with $\rho(t, z) < \xi\rho(y, z)$. There exists $k_0 \in \mathbb{N}$ such that $\operatorname{diam} B/(k_0+1) \le \rho(y, z) \le \operatorname{diam} B/k_0$. Then we have

$$\rho(t, y) \le \rho(t, z) + \rho(z, y) < (\xi+1)\rho(z, y) \le (1+\xi)\operatorname{diam} B/k_0$$

and therefore $t \in F_{k_0}$. Thus we can find $w \in W_{k_0}$ with $\rho(t, w) < \xi \operatorname{diam} B/k_0$. We estimate

$$s \le \rho(z, w) \le \rho(z, t) + \rho(t, w) < \xi\rho(y, z) + \xi \operatorname{diam} B/k_0$$

and

$$\frac{s}{\rho(y, z)} < \xi + \frac{\xi \operatorname{diam} B/k_0}{\operatorname{diam} B/(k_0+1)} = \xi + \xi\frac{k_0+1}{k_0} \le 3\xi < 5\xi.$$

2) If $s < \xi\rho(y, z)$, then obviously $s/\rho(y, z) < \xi < 5\xi$.

The above considerations show that $\Gamma^\star(y, B, D) \le 3\xi < 5\xi$. $\square$

Lemma 2.11. *Let $M \subset P$, $x \in M$ and $\xi > 0$. If $p(x, M) < \xi$, then there exists a closed set D such that $D \subset M$, $D' = \{x\}$ and $p(x, D) < 5\xi$.*

Proof: Since $p(x, M) < \xi$, there exists $r_0 > 0$ such that $\Gamma^\star(x, \overline{B}(x, r_0), M \cap \overline{B}(x, r_0)) < \xi$. Putting $A := \{x\}$, $T := M \cap \overline{B}(x, r_0)$, $B := \overline{B}(x, r_0)$ and applying Lemma 2.10 we obtain the desired set D. $\square$

The aim of Lemma 2.12 (Lemma 2.13, respectively) is to provide some conditions under which one can construct sets with the property $\mathcal{C}(n, \delta, \kappa, \alpha)$ (systems with the property $\mathcal{P}(m, \delta, \alpha, \varepsilon)$, respectively).

Lemma 2.12. *Let B be a closed ball, $m \in \mathbb{N}_0$, $\delta, \kappa, \alpha \in (0, 1)$, $40\delta < \kappa$ and $P_0 \subset P_1 \subset \dots \subset P_m$ be subsets of P such that $c(B) \in P_0$ and $\Gamma^\star(y, B, P_{j+1}) < \alpha\kappa/10$ whenever $j \in \{0, 1, \dots, m-1\}$, $y \in P_j \cap B$.*

Then there exist sets $S_0, \dots, S_m$ such that

(i) *$S_j \subset P_j$, $j = 0, \dots, m$,*
(ii) *$S'_{j+1} = S_j$, $j = 0, \dots, m-1$,*
(iii) *S_j has the $\mathcal{C}(j, \delta, \kappa, \alpha)$-property in B, $j = 0, \dots, m$.*

PROOF: We will proceed by induction. If $m = 0$, then we put $S_0 = \{c(B)\}$. The set S_0 has clearly the $\mathcal{C}(0, \delta, \kappa, \alpha)$-property in B. Observe that $2 \operatorname{dist}(c(B), B^c) \geq \operatorname{diam} B$ and $\delta < 1/40$. Thus $\operatorname{dist}(c(B), B^c) > \delta \operatorname{diam} B$.

Now assume that we have proved the case "$m = k$". We will deal with the case "$m = k + 1$". Suppose that sets $P_0, \dots, P_{k+1}$ are given. According to the induction hypothesis there exist sets $S_0, \dots, S_k$ satisfying (i)–(iii) for $m = k$.

Define

$$\tilde{B} = \{x \in B;\ \operatorname{dist}(x, B^c) > \delta^{k+1} \operatorname{diam} B\}, \qquad T = P_{k+1} \cap \tilde{B}.$$

Choose $y \in S_k$ and a ball $B(z, s)$ with $B(z, s) \subset B \setminus T$. We have two possibilities:

1) If $B(z, s) \subset \tilde{B}$, then $s/\rho(y, z) < \kappa/10$ since $B(z, s) \subset B \setminus P_{k+1}$ and $\Gamma^\star(y, B, P_{k+1}) < \alpha\kappa/10 < \kappa/10$.

2) If $B(z, s) \cap \tilde{B}^c \neq \emptyset$, then $s \leq \operatorname{dist}(z, \tilde{B}^c) + \delta^{k+1} \operatorname{diam} B$. We have also $\operatorname{dist}(z, \tilde{B}^c) < \kappa\rho(y, z)/10$ since $B(z, \operatorname{dist}(z, \tilde{B}^c)) \subset B \setminus P_{k+1}$ and $\Gamma^\star(y, B, P_{k+1}) < \kappa/10$. We have

$$\begin{aligned}
\rho(y, z) &\geq \operatorname{dist}(y, B^c) - \operatorname{dist}(z, B^c) \geq \delta^k \operatorname{diam} B - (\operatorname{dist}(z, \tilde{B}^c) + \delta^{k+1} \operatorname{diam} B) \\
&\geq \delta^k (1 - \delta) \operatorname{diam} B - \kappa\rho(y, z)/10, \\
\rho(y, z) &\geq \frac{1}{1 + \kappa/10} \delta^k (1 - \delta) \operatorname{diam} B > \frac{1}{2} \delta^k (1 - \delta) \operatorname{diam} B.
\end{aligned}$$

We estimate

$$\begin{aligned}
\frac{s}{\rho(y, z)} &= \frac{\operatorname{dist}(z, \tilde{B}^c)}{\rho(y, z)} + \frac{s - \operatorname{dist}(z, \tilde{B}^c)}{\rho(y, z)} < \kappa/10 + \frac{\delta^{k+1} \operatorname{diam} B}{\frac{1}{2} \delta^k (1 - \delta) \operatorname{diam} B} \\
&= \kappa/10 + \frac{2\delta}{1 - \delta} < \kappa/10 + 4\delta < \kappa/10 + \kappa/10 = \kappa/5.
\end{aligned}$$

The above discussion gives that $\Gamma^\star(y, B, T) < \kappa/5$ whenever $y \in S_k$.

According to Lemma 2.10 there exists a closed set $D \subset T$ such that $D' = S_k$ and $\Gamma^\star(y, B, D) < \kappa$ whenever $y \in S_k$. We associate with each point $x \in S_k \setminus S'_k$ a closed ball B_x centered at x in such a way that

- the system $\mathcal{S} := \{B_x;\ x \in S_k \setminus S'_k\}$ is discrete in $P \setminus S'_k$,
- $\forall\, x \in S_k \setminus S'_k :\ \operatorname{dist}(B_x, B^c) > \delta^k \operatorname{diam} B$.

For every $x \in S_k \setminus S'_k$ we have $p(x, B_x \cap P_{k+1}) < \alpha\kappa/10$ since $\Gamma^\star(x, B, P_{k+1}) < \alpha\kappa/10$. Then there exists a closed set D_x (according to Lemma 2.11) such that

- $D_x \subset P_{k+1} \cap B_x$,
- $D'_x = \{x\}$,
- $p(x, D_x) < \alpha\kappa$.

We put $S_{k+1} = D \cup \bigcup\{D_x;\ x \in S_k \setminus S'_k\}$. We show that $S_0, \dots, S_{k+1}$ have the desired properties.

(i) This condition is obviously satisfied.

(ii) We have $S'_{j+1} = S_j$ for $j = 0, \dots, k-1$ by the induction hypothesis. Since $\mathcal{S}$ is discrete in $P \setminus S'_k$ we have that S_{k+1} is closed. Since $D' = S_k$ and $\mathcal{S}$ is discrete in $P \setminus S'_k$ we have $S'_{k+1} = S_k$.

(iii) The induction hypothesis gives that S_j, $j = 0, \dots, k$, has the $\mathcal{C}(j, \delta, \kappa, \alpha)$-property in B. It remains to show that S_{k+1} has the $\mathcal{C}(k+1, \delta, \kappa, \alpha)$-property in B:

$(\mathrm{C1})_{k+1}$ This property follows from the definition of T and from the fact that $\operatorname{dist}(B_x, B^c) > \delta^k \operatorname{diam} B$ for every $x \in S_k \setminus S'_k$.

$(\mathrm{C2})_{k+1}$ We have $\Gamma^\star(y, B, D) < \kappa$ for every $y \in S_k$. Since $S'_{k+1} = S_k$ and $D \subset S_{k+1}$ we conclude that $\Gamma^\star(y, B, S_{k+1}) < \kappa$ for every $y \in S'_{k+1}$.

$(\mathrm{C3})_{k+1}$ Let $x \in S'_{k+1} = S_k$. If $x \in S'_k$, then $p(x, S_k) < \alpha\kappa$ by the induction hypothesis. Thus we also have $p(x, S_{k+1}) < \alpha\kappa$. If $x \in S_k \setminus S'_k$, then x is a point of non-$\alpha\kappa$-porosity of D_x and therefore x is also a point of non-$\alpha\kappa$-porosity of S_{k+1}.

$(\mathrm{C4})_{k+1}$ The set S'_{k+1} has the $\mathcal{C}(k, \delta, \kappa, \alpha)$-property in B by the induction hypothesis. □

Lemma 2.13. *Let B be a closed ball, $m \in \mathbb{N}$, $\delta, \kappa, \alpha, \varepsilon \in (0,1)$, $10\kappa < \varepsilon$, $S_m \subset B$ be a set with the $\mathcal{C}(m, \delta, \kappa, \alpha)$-property in B. Then there exists a function $s : S_m \to (0, +\infty)$ such that, for every function $r : S_m \to (0, +\infty)$ with $r \le s$, we have that $\mathcal{V}_m = \{\overline{B}(x, r(x));\ x \in S_m\}$ forms a B-system with the $\mathcal{P}(m, \delta, \alpha, \varepsilon)$-property in B.*

Proof: We put $S_{m-1} := S'_m$, $S_{m-2} := S'_{m-1}$, $\dots$, $S_0 := S'_1$ and $S_{-1} := \emptyset$. We know that S_j, $j \in \{0, \dots, m\}$, has the $\mathcal{C}(j, \delta, \kappa, \alpha)$-property in B.

Fix $k \in \{0, \dots, m\}$ and $x \in S_k \setminus S_{k-1}$. Observe that

- if $k \ge 1$, then $\operatorname{dist}(x, B^c) > \delta^k \operatorname{diam} B$; if $k = 0$, then $\operatorname{dist}(x, B^c) > \delta \operatorname{diam} B$,
- $\operatorname{dist}(x, S_{k-1}) > 0$ since S_{k-1} is closed and $x \notin S_{k-1}$,
- $\operatorname{dist}(x, S_k \setminus \{x\}) > 0$ since $S_k \setminus \{x\}$ is closed,
- if $k < m$, then $p(x, S_{k+1}) < \alpha\kappa$.

Using these observations we choose $s(x) > 0$ so that $s(x)$ satisfies the following conditions:

(1) $s(x) < \frac{1}{3} \operatorname{dist}(x, B^c)$,

(2) if $k \ge 1$, then $s(x) < \operatorname{dist}(x, B^c) - \delta^k \operatorname{diam} B$; if $k = 0$, then $s(x) < \operatorname{dist}(x, B^c) - \delta \operatorname{diam} B$,

(3) $s(x) < \frac{1}{4} \operatorname{diam} B$,

(4) $s(x) < \frac{1}{4} \alpha\kappa \operatorname{dist}(x, S_{k-1})$,

(5) $s(x) < \frac{1}{4}\operatorname{dist}(x, S_k \setminus \{x\})$,
(6) if $k < m$, then $\theta(x, R, S_{k+1}) < \alpha\kappa R$ whenever $R \in (0, 4s(x))$.

This finishes the construction of the function s.

Let $r : S_m \to (0, +\infty)$ be a function with $r \leq s$. We define

$$\begin{aligned}\mathcal{A}_j &= \{\overline{B}(x, r(x));\ x \in S_j \setminus S_{j-1}\},\\ \mathcal{V}_j &= \{\overline{B}(x, r(x));\ x \in S_j\}, \qquad j \in \{0, \dots, m\}.\end{aligned}$$

Claim. The system $\mathcal{A}_j$ is discrete in $P \setminus S_{j-1}$, $j \in \{0, \dots, m\}$.

Proof of Claim: Let $z \in P \setminus S_{j-1}$. Then we have that $S_j \setminus \{z\}$ is a closed set. Let $d \in (0, \frac{1}{4}\operatorname{dist}(z, S_j \setminus \{z\}))$. Suppose that there are $x_1, x_2 \in S_j \setminus S_{j-1}$, $x_1 \neq x_2$, with $\overline{B}(x_i, r(x_i)) \cap B(z, d) \neq \emptyset$, $i = 1, 2$. This implies $s(x_i) \geq r(x_i) > \frac{3}{4}\rho(x_i, z)$. According to (5) we have $s(x_i) < \frac{1}{4}\rho(x_1, x_2)$. Then we have

$$\frac{3}{4}\rho(x_1, x_2) \leq \frac{3}{4}(\rho(x_1, z) + \rho(z, x_2)) < s(x_1) + s(x_2) < \frac{1}{2}\rho(x_1, x_2),$$

a contradiction. This shows that $\mathcal{A}_j$ is discrete in $P \setminus S_{j-1}$. □

The system $\mathcal{V}_0$ is clearly a B-system with the $\mathcal{P}(0, \delta, \alpha, \varepsilon)$-property in B. Assume that we have proved that $\mathcal{V}_j$, $0 \leq j < m$, is a B-system with the $\mathcal{P}(j, \delta, \alpha, \varepsilon)$-property in B. We shall deal with $\mathcal{V}_{j+1}$. According to (5) we have that $\mathcal{A}_{j+1}$ is a disjoint system. The system $\mathcal{V}_j$ is point finite by the induction hypothesis and therefore $\mathcal{V}_{j+1}$ is also point finite.

Since $\mathcal{A}_{j+1}$ is discrete in $P \setminus S_j$ by Claim, we have $\operatorname{ap}(\mathcal{A}_{j+1}) \subset S_j$. Then we have

$$\operatorname{ap}(\mathcal{V}_{j+1}) = \operatorname{ap}(\mathcal{V}_j) \cup \operatorname{ap}(\mathcal{A}_{j+1}) \subset c(\mathcal{V}_j) \cup S_j = S_j \subset S_{j+1} = c(\mathcal{V}_{j+1}).$$

Thus $\mathcal{V}_{j+1}$ is a B-system. It remains to verify properties $(\mathrm{P1})_{j+1}$–$(\mathrm{P4})_{j+1}$.

$(\mathrm{P1})_{j+1}$: Take $x \in S_{j+1}$. Using (1) we have

$$\operatorname{dist}(\overline{B}(x, r(x)), B^c) \geq \operatorname{dist}(x, B^c) - r(x) > 3s(x) - s(x) = 2s(x).$$

Thus we have

$$\operatorname{dist}(\overline{B}(x, r(x)), B^c) > 2s(x) \geq \operatorname{diam}(\overline{B}(x, r(x))).$$

$(\mathrm{P2})_{j+1}$: Take $x \in S_{j+1}$. According to (2) we have

$$\operatorname{dist}(\overline{B}(x, r(x)), B^c) \geq \operatorname{dist}(x, B^c) - r(x) > s(x) + \delta^{j+1}\operatorname{diam} B - s(x) = \delta^{j+1}\operatorname{diam} B.$$

$(\mathrm{P3})_{j+1}$: This property follows immediately from (3).

$(\mathrm{P4})_{j+1}$: Since $\mathcal{V}_j$ has the $\mathcal{P}(j,\delta,\alpha,\varepsilon)$-property in B, it is sufficient to prove that, for an arbitrary set J intersecting each element of $\mathcal{V}_{j+1}$, we have

$$\forall V \in \mathcal{V}_j \, \forall x \in V: \ \operatorname{dist}(x, V^c) > \alpha \operatorname{diam} V \Rightarrow \Gamma(x, B, V, J) < \varepsilon.$$

Let J be a set intersecting each element of $\mathcal{V}_{j+1}$. Fix $y \in S_j$ and denote $V = \overline{B}(y, r(y))$. Take $x \in V$ with $\operatorname{dist}(x, V^c) > \alpha \operatorname{diam} V$. Consider a ball $B(z,d) \subset B \setminus J$ with $z \notin V$. Denote $d_1 = \operatorname{dist}(z, S_{j+1})$. Since $S'_{j+1} = S_j$ we can find $w \in S_{j+1} \setminus S_j$ with

$$\rho(w,z) < d_1 + \frac{1}{5}d\,. \tag{2.1}$$

Denote $W = \overline{B}(w, r(w))$. Since J intersects W we have

$$d \le \rho(z,w) + \operatorname{diam} W < d_1 + \frac{1}{5}d + \operatorname{diam} W.$$

This gives

$$d \le \frac{5}{4}(d_1 + \operatorname{diam} W) < 2d_1 + 2\operatorname{diam} W. \tag{2.2}$$

Using (4) we obtain

$$\operatorname{diam} W \le 2r(w) \le 2s(w) < \frac{1}{2}\alpha\kappa \operatorname{dist}(w, S_j) \le \frac{1}{2}\alpha\kappa\rho(w,y). \tag{2.3}$$

Inequalities (2.2) and (2.3) imply

$$d < 2d_1 + \alpha\kappa\rho(w,y). \tag{2.4}$$

Now we distinguish two cases.

1) First suppose that $\rho(z,y) < 2\operatorname{diam} V$. Then $\rho(z,y) < 4s(y)$ and according to (6) we have $d_1 \le \alpha\kappa\rho(z,y)$. Using this, (2.1) and (2.4) we obtain

$$\begin{aligned} d &< 2\alpha\kappa\rho(z,y) + \alpha\kappa\rho(w,y) \le 2\alpha\kappa\rho(z,y) + \alpha\kappa(\rho(w,z) + \rho(z,y)) \\ &< 3\alpha\kappa\rho(z,y) + \alpha\kappa(d_1 + \frac{1}{5}d) \le 3\alpha\kappa\rho(z,y) + \alpha^2\kappa^2\rho(z,y) + \frac{1}{5}\alpha\kappa d \\ &< 4\alpha\kappa\rho(z,y) + \frac{1}{5}d. \end{aligned}$$

This implies $d < 5\alpha\kappa\rho(z,y)$. We estimate

$$\frac{d}{\rho(x,z)} = \frac{d}{\rho(z,y)} \cdot \frac{\rho(z,y)}{\rho(x,z)} \le 5\alpha\kappa \cdot \frac{2\operatorname{diam} V}{\alpha \operatorname{diam} V} = 10\kappa.$$

2) Suppose now that $\rho(z,y) \geq 2\operatorname{diam} V$. Moreover suppose that $B(z,d_1) \subset B$. Then $d_1 \leq \kappa\rho(z,y)$ since $\Gamma^\star(y,B,S_{j+1}) < \kappa$. Using this, (2.1) and (2.4) we obtain

$$\begin{aligned} d &< 2d_1 + \alpha\kappa\rho(w,y) \leq 2\kappa\rho(z,y) + \kappa\rho(w,y) \leq 2\kappa\rho(z,y) + \kappa(\rho(w,z) + \rho(z,y)) \\ &< 3\kappa\rho(z,y) + \kappa(d_1 + \frac{1}{5}d) \leq 3\kappa\rho(z,y) + \kappa^2\rho(z,y) + \kappa\frac{1}{5}d < 4\kappa\rho(z,y) + \frac{1}{5}d. \end{aligned}$$

This implies $d < 5\kappa\rho(z,y)$.

If $B(z,d_1) \cap B^c \neq \emptyset$, then $d_1 > d$ and $B(z,d) \cap S_{j+1} = \emptyset$. Using $\Gamma^\star(y,B,S_{j+1}) < \kappa$ we obtain $d \leq \kappa\rho(z,y) < 5\kappa\rho(z,y)$.

In both cases we have $d < 5\kappa\rho(z,y)$. We have also

$$\rho(x,z) \geq \rho(z,y) - \rho(x,y) \geq 2\operatorname{diam} V - \operatorname{diam} V = \operatorname{diam} V.$$

We estimate

$$\frac{d}{\rho(x,z)} = \frac{d}{\rho(z,y)} \cdot \frac{\rho(z,y)}{\rho(x,z)} < 5\kappa \cdot \frac{\rho(y,x) + \rho(x,z)}{\rho(x,z)} \leq 5\kappa\left(\frac{\operatorname{diam} V}{\operatorname{diam} V} + 1\right) = 10\kappa.$$

This shows that $\Gamma(x,B,V,J) \leq 10\kappa < \varepsilon$. $\square$

The next lemma follows easily from the previous one.

Lemma 2.14. *Let B be a closed ball, $m \in \mathbb{N}$, $\delta,\kappa,\alpha,\varepsilon \in (0,1)$, $10\kappa < \varepsilon$, $S_m \subset B$ be a set with the $\mathcal{C}(m,\delta,\kappa,\alpha)$-property in B, let $r : S_m \to (0,+\infty)$ be a function. Then there exists a B-system $\mathcal{V}_m$ such that*

- *$\mathcal{V}_m$ has the $\mathcal{P}(m,\delta,\alpha,\varepsilon)$-property in B,*
- *$c(\mathcal{V}_m) = S_m$,*
- *for every $V \in \mathcal{V}_m$ we have $V \subset B(c(V), r(c(V)))$.*

Lemma 2.15 establishes a relationship between the quantity Γ (defined in Definition 2.5) and the index of porosity p.

Lemma 2.15. *Let $\varepsilon \in (0,1)$, $M \subset P$, $x \in M$ and $(B_n)_{n=1}^\infty$ be a sequence of closed balls such that for every $n \in \mathbb{N}$ we have*

(i) $x \in B_n$,
(ii) $\operatorname{dist}(B_{n+1}, B_n^c) \geq \operatorname{diam} B_{n+1}$,
(iii) $\Gamma(x, B_n, B_{n+1}, M) < \varepsilon$,
(iv) $\operatorname{diam} B_{n+1} \leq \frac{1}{2}\operatorname{diam} B_n$.

Then $p(x,M) < 4\varepsilon$.

Proof: Denote $\psi_n = \Gamma(x, B_n, B_{n+1}, M)$. Suppose that $B(z,s)$ is an open ball with $B(z,s) \subset B_2 \setminus M$. Since $z \neq x$ and condition (iv) holds, there exists $n \in \mathbb{N}$, $n \geq 2$, such that $z \in B_n \setminus B_{n+1}$. Suppose that $B(z,s) \cap B_{n-1}^c \neq \emptyset$. Then we have

$\operatorname{dist}(B_n, B_{n-1}^c) < s$. Since $x \in B_n \setminus B(z,s)$ we have $s \le \rho(x,z) \le \operatorname{diam} B_n$. Thus $\operatorname{dist}(B_n, B_{n-1}^c) < \operatorname{diam} B_n$, a contradiction with (ii). Thus $B(z,s) \subset B_{n-1}$.

Suppose that $B(z,s) \subset B_n$; then

$$\frac{s}{\rho(x,z)} \le \psi_n < \varepsilon.$$

Now suppose that $B(z,s) \cap (B_{n-1} \setminus B_n) \ne \emptyset$. Then we have

$$\frac{\operatorname{dist}(z, B_n^c)}{\rho(x,z)} \le \psi_n. \tag{2.5}$$

There exists $w \in B_{n-1} \setminus B_n$ such that

$$\rho(z,w) < \operatorname{dist}(z, B_n^c) + s/5 \tag{2.6}$$

and $\rho(z,w) < s$. Put $d = \rho(z,w)$ and $r = s - d$. We have $B(w,r) \subset B(z,s) \subset B_{n-1}$ and so

$$\frac{r}{\rho(w,x)} \le \psi_{n-1}. \tag{2.7}$$

Using (2.5) and (2.6) we have

$$d < \operatorname{dist}(z, B_n^c) + s/5 \le \psi_n \rho(x,z) + s/5. \tag{2.8}$$

We have also $\rho(z,w) < s \le \rho(x,z)$. Using this, (2.7) and (2.8) we obtain

$$\begin{aligned} \frac{s}{\rho(x,z)} = \frac{r+d}{\rho(x,z)} &\le \psi_{n-1}\frac{\rho(w,x)}{\rho(x,z)} + \psi_n + \frac{s/5}{\rho(x,z)} \\ &\le \psi_{n-1}\frac{\rho(x,z)+\rho(z,w)}{\rho(x,z)} + \psi_n + \frac{s/5}{\rho(x,z)} \\ &\le \psi_{n-1}\cdot 2 + \psi_n + \frac{s/5}{\rho(x,z)}\,. \end{aligned}$$

This implies

$$\frac{s}{\rho(x,z)} \le \frac{5}{4}(2\psi_{n-1} + \psi_n) < \frac{15}{4}\varepsilon.$$

Thus we have proved that each open ball $B(z,s)$ with $B(z,s) \subset B_2 \setminus M$ satisfies $s/\rho(x,z) < 15\varepsilon/4$. We have also $x \in \operatorname{Int} B_2$ according to condition (ii). We conclude that $p(x,M) < 4\varepsilon$. □

Lemma 2.16. *Let B_1 be a closed ball, $\varepsilon \in (0,1)$, $\alpha_n, \delta_n \in (0,1)$ for every $n \in \mathbb{N}$ and let $(\mathcal{V}_n)_{n=0}^{\infty}$ be a sequence of B-systems satisfying*

(V1) $\mathcal{V}_0 = \{B_1\}$,

(V2) $\mathcal{V}_{n+1} = \bigcup\{\mathcal{V}_{n+1}(C);\ C \in \mathcal{V}_n\}$, *where* $\mathcal{V}_{n+1}(C)$ *has the* $\mathcal{P}(n+1, \delta_{n+1}, \alpha_{n+1}, \varepsilon)$*-property in* C, $n \in \mathbb{N}_0$,

(V3) *for every* $n \in \mathbb{N}$ *we have* $\alpha_n < (\delta_{n+1})^{n+1}$.

Let B_2 be an open set intersecting $\bigcap_{n=0}^{\infty} \bigcup \mathcal{V}_n$. Then there exist $m \in \mathbb{N}$, $B_3 \in \mathcal{V}_m$ with $B_3 \subset B_2$ and a sequence $(\mathcal{R}_l)_{l=0}^{\infty}$ of B-systems such that for every $l \in \mathbb{N}_0$ we have

(R1) $\mathcal{R}_0 = \{B_3\}$,

$(\mathrm{R2})_l$ $\mathcal{R}_{l+1} = \bigcup\{\mathcal{R}_{l+1}(C);\ C \in \mathcal{R}_l\}$, *where* $\mathcal{R}_{l+1}(C)$ *has the* $\mathcal{P}(l+1, \delta_{m+l+1}, \alpha_{m+l+1}, \varepsilon)$*-property in* C,

$(\mathrm{R3})_l$ $\mathcal{R}_l \subset \mathcal{V}_{m+l}$,

(R4) *each point of* $\bigcap_{q=0}^{\infty} \bigcup \mathcal{R}_q$ *is a point of non-*4ε*-porosity of the set* $\bigcap_{n=0}^{\infty} \bigcup \mathcal{V}_n$.

Proof: Choose $x_0 \in B_2 \cap \bigcap_{n=0}^{\infty} \bigcup \mathcal{V}_n$. According to conditions $(\mathrm{P3})_1$–$(\mathrm{P3})_n$ and (V2) we have $\sup\{\operatorname{diam} V;\ V \in \mathcal{V}_n\} \leq 2^{-n} \operatorname{diam} B_1$ for every $n \in \mathbb{N}_0$. Thus there exist $m \in \mathbb{N}$ and $B_3 \in \mathcal{V}_m$ such that $x_0 \in B_3 \subset B_2$. Put $\mathcal{R}_0 = \{B_3\}$.

Now suppose that we have defined B-systems $\mathcal{R}_0, \mathcal{R}_1, \dots, \mathcal{R}_l$ satisfying $(\mathrm{R2})_j$ for $j \in \{0, \dots, l-1\}$, $(\mathrm{R3})_j$ for $j \in \{0, \dots, l\}$, and, for every $j \in \{0, \dots, l-1\}$ and for each set J intersecting each element of $\mathcal{V}_{m+j+1}$, we have

$$\forall\, C \in \mathcal{R}_j\ \forall\, R \in \mathcal{R}_{j+1}(C)\ \forall\, x \in R : \quad \operatorname{dist}(x, R^c) > \alpha_{m+j+1} \operatorname{diam} R \Rightarrow \Gamma(x, C, R, J) < \varepsilon.$$

We will define $\mathcal{R}_{l+1}$. Take $C \in \mathcal{R}_l$. Since the B-system $\mathcal{V}_{m+l+1}(C)$ has the $\mathcal{P}(m+l+1, \delta_{m+l+1}, \alpha_{m+l+1}, \varepsilon)$-property in C, there exists a B-system $\mathcal{R}_{l+1}(C)$ such that (Observation 2.9)

- $\mathcal{R}_{l+1}(C) \subset \mathcal{V}_{m+l+1}(C)$,
- $\mathcal{R}_{l+1}(C)$ has the $\mathcal{P}(l+1, \delta_{m+l+1}, \alpha_{m+l+1}, \varepsilon)$-property in C,
- for each set J intersecting each element of $\mathcal{V}_{m+l+1}(C)$ we have

$$\forall\, R \in \mathcal{R}_{l+1}(C)\ \forall\, x \in R : \ \operatorname{dist}(x, R^c) > \alpha_{m+l+1} \operatorname{diam} R \Rightarrow \Gamma(x, C, R, J) < \varepsilon.$$

Put $\mathcal{R}_{l+1} = \bigcup\{\mathcal{R}_{l+1}(C);\ C \in \mathcal{R}_l\}$. Thus we have defined a sequence $(\mathcal{R}_l)_{l=0}^{\infty}$. These B-systems satisfy $(\mathrm{R2})_l$ and $(\mathrm{R3})_l$ for every $l \in \mathbb{N}_0$. We show that the $\mathcal{R}_l$'s satisfy also condition (R4).

Observe that the set $\bigcap_{n=0}^{\infty} \bigcup \mathcal{V}_n$ intersects each element of $\mathcal{V}_p$, $p \in \mathbb{N}_0$. This follows from (V2), from condition $(\mathrm{P3})_n$ of $\mathcal{V}_n(C)$, where $n \in \mathbb{N}$, $C \in \mathcal{V}_{n-1}$, and from the completeness of the considered metric space P.

Let $x \in \bigcap_{q=0}^{\infty} \bigcup \mathcal{R}_q$. Denote $C_0 = B_3$ and define a tree of sequences of closed balls

$$\mathcal{T} = \{\emptyset\} \cup \{(C_1, C_2, \dots, C_q);\ C_i \in \mathcal{R}_i(C_{i-1}),\ x \in C_i,\ i = 1, \dots, q,\ q \in \mathbb{N}\}.$$

The tree $\mathcal{T}$ is finite splitting since $\mathcal{R}_l$ is point finite for every $l \in \mathbb{N}$. The tree $\mathcal{T}$ is infinite. Thus by König's Lemma (cf. [Ke, Exercise 4.12]) there exists a sequence $(C_l)_{l=1}^{\infty}$ of closed balls such that $x \in C_l \in \mathcal{R}_l(C_{l-1})$. For every $l \in \mathbb{N}$ we have:

- $\operatorname{dist}(C_{l+1}, C_l^c) > \operatorname{diam} C_{l+1}$,
- $\Gamma(x, C_l, C_{l+1}, \bigcap_{n=0}^{\infty} \bigcup \mathcal{V}_n) < \varepsilon$ (since the set $\bigcap_{n=0}^{\infty} \bigcup \mathcal{V}_n$ intersects each element of $\mathcal{V}_{m+l+1}$ and

$$\begin{aligned}\operatorname{dist}(x, C_{l+1}^c) \geq \operatorname{dist}(C_{l+2}, C_{l+1}^c) &> (\delta_{m+l+2})^{l+2} \operatorname{diam} C_{l+1}\\ &> (\delta_{m+l+2})^{m+l+2} \operatorname{diam} C_{l+1} > \alpha_{m+l+1} \operatorname{diam} C_{l+1}),\end{aligned}$$

- $\operatorname{diam} C_{l+1} \leq \frac{1}{2} \operatorname{diam} C_l$.

Now Lemma 2.15 gives (R4). □

The next definition introduces a version of Foran system. It will be a crucial tool when proving non-σ-porosity of a set in Lemma 2.22.

Definition 2.17. Let (X, τ) be a metric space and $\varepsilon > 0$. We say that a nonempty system $\mathcal{F}$ of nonempty closed subsets of X is an *ε-Foran system*, if for every $F \in \mathcal{F}$ and for every open set B intersecting F there exists $F^\star \in \mathcal{F}$ such that $F^\star \subset B \cap F$ and each point of $F^\star$ is a point of non-ε-porosity of F.

Lemma 2.18. *Let (X, τ) be a complete metric space, $\varepsilon \in (0, 1/2)$ and $\mathcal{F}$ be an ε-Foran system. Then each $F \in \mathcal{F}$ is not σ-porous.*

We need the following definition and theorem to prove Lemma 2.18.

Definition 2.19. Let (X, τ) be a metric space, $M \subset X$ and $\varepsilon > 0$. We say that M is *ε-porous* if $p(x, M) \geq \varepsilon$ for every $x \in M$.

Theorem 2.20 (Zajíček [Za$_1$])**.** *Let (X, τ) be a metric space, $\varepsilon \in (0, 1/2)$ and $A \subset X$. The set A is σ-porous if and only if A can be covered by countably many ε-porous sets.*

Remark 2.21. Because of purely technical reasons our definition of ε-porosity is different from Zajíček's one in [Za$_1$]. If a set is ε-porous in Zajíček's sense, then it is ε-porous also in our sense. Thus we can state his theorem as above.

Proof of Lemma 2.18: It is easy to see that the notion of ε-porosity at a point is a special case of the abstract notion of V-porosity (see [Za$_2$, Definition 4.1]). Lemma 4.3 from [Za$_2$] gives that the system $\mathcal{F}$ contains only elements, which cannot be covered by countably many ε-porous sets. Now Lemma 2.18 follows from Theorem 2.20. □

Lemma 2.22. *Let $\varepsilon \in (0, 1/8)$, $\alpha_n, \delta_n \in (0, 1)$ for every $n \in \mathbb{N}$, B be a closed ball and let $(\mathcal{U}_n)_{n=0}^{\infty}$ be a sequence of B-systems such that*

(i) $\mathcal{U}_0 = \{B\}$,

(ii) $\mathcal{U}_{n+1} = \bigcup\{\mathcal{U}_{n+1}(C);\ C \in \mathcal{U}_n\}$, *where* $\mathcal{U}_{n+1}(C)$ *has the* $\mathcal{P}(n+1, \delta_{n+1}, \alpha_{n+1}, \varepsilon)$*-property in* C, $n \in \mathbb{N}_0$,

(iii) *for every* $n \in \mathbb{N}$ *we have* $\alpha_n < (\delta_{n+1})^{n+1}$.

Then the set $\bigcap_{n=0}^{\infty} \bigcup \mathcal{U}_n$ *is a closed non-σ-porous set.*

Proof: We will define a 4ε-Foran system $\mathcal{F}$. A set F is in $\mathcal{F}$ if and only if there exist $p \in \mathbb{N}_0$, a closed ball $B_1 \in \mathcal{U}_p$ and a sequence $(\mathcal{V}_k)_{k=0}^{\infty}$ of B-systems such that

- $F = \bigcap_{k=0}^{\infty} \bigcup \mathcal{V}_k$,
- $\mathcal{V}_0 = \{B_1\}$,
- $\mathcal{V}_{k+1} = \bigcup\{\mathcal{V}_{k+1}(C);\ C \in \mathcal{V}_k\}$, where $\mathcal{V}_{k+1}(C)$ has the $\mathcal{P}(k+1, \delta_{p+k+1}, \alpha_{p+k+1}, \varepsilon)$-property in C, $k \in \mathbb{N}_0$,
- $\mathcal{V}_k \subset \mathcal{U}_{p+k}$ for every $k \in \mathbb{N}_0$.

Observe that such an F is a closed set by Lemma 2.4(iii) and is nonempty because of the completeness of P and property $(\mathrm{P3})_{k+1}$ of each B-system $\mathcal{V}_{k+1}(C)$ ($C \in \mathcal{V}_k$, $k \in \mathbb{N}_0$).

Now take $F \in \mathcal{F}$ and an open set B_2 intersecting F. There exist $p \in \mathbb{N}_0$, a closed ball B_1 and a sequence $(\mathcal{V}_k)_{k=0}^{\infty}$ of B-systems witnessing $F \in \mathcal{F}$. Observe that $\alpha_{p+k} < (\delta_{p+k+1})^{p+k+1} < (\delta_{p+k+1})^{k+1}$ for every $k \in \mathbb{N}$. Using Lemma 2.16 we obtain $m \in \mathbb{N}$, $B_3 \in \mathcal{V}_m$ with $B_3 \subset B_2$ and a sequence $(\mathcal{R}_l)_{l=0}^{\infty}$ of B-systems such that

(a) $\mathcal{R}_0 = \{B_3\}$,

(b) $\mathcal{R}_{l+1} = \bigcup\{\mathcal{R}_{l+1}(C);\ C \in \mathcal{R}_l\}$, where $\mathcal{R}_{l+1}(C)$ has the $\mathcal{P}(l+1, \delta_{m+p+l+1}, \alpha_{m+p+l+1}, \varepsilon)$-property in C, $l \in \mathbb{N}_0$,

(c) $\mathcal{R}_l \subset \mathcal{V}_{m+l}$, $l \in \mathbb{N}_0$,

(d) each point of $\bigcap_{l=0}^{\infty} \bigcup \mathcal{R}_l$ is a point of non-4ε-porosity of the set $\bigcap_{k=0}^{\infty} \bigcup \mathcal{V}_k$.

We put

$$F^\star = \bigcap_{l=0}^{\infty} \bigcup \mathcal{R}_l.$$

Conditions (a)–(c) give that $F^\star \in \mathcal{F}$ and $F^\star \subset B_2 \cap F$. Condition (d) implies that each point $x \in F^\star$ is a point of non-4ε-porosity of the set F. This shows that $\mathcal{F}$ is a 4ε-Foran system. Thus each element of $\mathcal{F}$ is not σ-porous by Lemma 2.18. Thus the set $\bigcap_{n=0}^{\infty} \bigcup \mathcal{U}_n$ is not σ-porous and we are done. □

3. Inscribing closed non-σ-porous sets into Suslin non-σ-porous sets

The main aim of this section is to prove the following theorem.

Theorem 3.1. *Let P be a topologically complete metric space and $S \subset P$ be a non-σ-porous Suslin set. Then there exists a closed non-σ-porous subset F of S.*

Our theorem answers positively Zajíček's question ([Za$_2$, Question 4.20]).

Let $\operatorname{Seq}\mathbb{N}$ be the set of all finite sequences from $\mathbb{N}$ and let $\mathcal{N}$ be the set of all infinite sequences from $\mathbb{N}$, i.e. $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$. We denote the concatenation of $s \in \operatorname{Seq}\mathbb{N}$ and $t \in \operatorname{Seq}\mathbb{N} \cup \mathcal{N}$ by $s^\wedge t$. If $\nu \in \mathcal{N}$, $n \in \mathbb{N}$, then ν_n stands for the n-th element of ν and the symbol $\nu|n$ means the finite sequence $(\nu_1, \nu_2, \ldots, \nu_n)$. The symbol $\nu|0$ denotes the empty sequence. If $s \in \operatorname{Seq}\mathbb{N}$ and $n \in \mathbb{N}$, then we write $s^\wedge n$ instead of $s^\wedge(n)$. If $t \in \operatorname{Seq}\mathbb{N}$, then the symbol $|t|$ denotes the length of t.

Now we introduce auxiliary set-valued mappings.

Definition 3.2. Let Y be a metric space, $\omega > 1$, $r > 0$ and $A \subset Y$. Then we define

$$\begin{aligned} \ker(A) &= A \setminus \bigcup\{G;\ G \text{ is an open subset of } Y \text{ such that } G \cap A \text{ is } \sigma\text{-porous}\};\\ D_{\omega,r}(A) &= A \setminus \bigcup\{B(x, \omega s);\ B(x,s) \cap A = \emptyset \text{ and } s \leq r\};\\ N_{\omega,r}(A) &= \ker(D_{\omega,r}(A)). \end{aligned}$$

We will use the notation $(N_{\omega,r})^n = \underbrace{N_{\omega,r} \circ \cdots \circ N_{\omega,r}}_{n\text{-times}}$.

The proof of the following lemma can be found in [Za$_4$].

Lemma 3.3 (Zajíček [Za$_4$, Lemma 3])**.** *Let M be a subset of a metric space Y and let for each $y \in M$ there exists $r > 0$ such that $B(y,r) \cap M$ is σ-porous. Then M is σ-porous.*

Lemma 3.4. *Let Y be a metric space, $\omega > 1$, $r > 0$ and $A \subset Y$.*

(i) *The sets $\ker(A)$, $D_{\omega,r}(A)$ and $N_{\omega,r}(A)$ are closed in A.*
(ii) *If A is a closed (Suslin, respectively) subset of Y, then $\ker(A)$, $D_{\omega,r}(A)$ and $N_{\omega,r}(A)$ are closed (Suslin, respectively) subsets of Y.*
(iii) *If $n \in \mathbb{N}$, then $(N_{\omega,r})^n(A)$ is closed in A.*
(iv) *The set $A \setminus \ker(A)$ is σ-porous and $G \cap \ker(A)$ is non-σ-porous for every open set G intersecting $\ker(A)$, i.e. $\ker(\ker(A)) = \ker(A)$.*
(v) *If $n \in \mathbb{N}$ and A is non-σ-porous, then there exists $r^\star > 0$ such that $(N_{\omega,r^\star})^n(A)$ is non-σ-porous.*

Proof: (i) The assertion follows immediately from the definition of ker, $D_{\omega,r}$ and $N_{\omega,r}$.

(ii), (iii) These assertions follow from (i).

(iv) The set $A \setminus \ker(A)$ is σ-porous by Lemma 3.3. Let $G \subset Y$ be an open set with $G \cap \ker(A) \neq \emptyset$. If $G \cap A$ is σ-porous, then $G \cap \ker(A) = \emptyset$, a contradiction. So $G \cap A$ is non-σ-porous and so $G \cap \ker(A)$ is also non-σ-porous.

(v) First we show that there exists $r' > 0$ such that $D_{\omega,r'}(A)$ is non-σ-porous. The set $A \setminus \bigcup_{n=1}^{\infty} D_{\omega,1/n}(A)$ is σ-porous since it contains only points of porosity of A. Thus there exists $n_0 \in \mathbb{N}$ such that $D_{\omega,1/n_0}(A)$ is non-σ-porous. Then $N_{\omega,1/n_0}(A)$ is also non-σ-porous by (iv). Thus $r' = 1/n_0$ works.

Now it is easy to find a sequence $r_1, r_2, \dots, r_n$ of positive real numbers such that the set

$$C = N_{\omega,r_n} \circ N_{\omega,r_{n-1}} \circ \dots \circ N_{\omega,r_1}(A)$$

is non-σ-porous. Put $r^\star = \min\{r_i;\ i = 1, \dots, n\}$. We have $C \subset (N_{\omega,r^\star})^n(A)$ (since $N_{\omega,a}(A) \subset N_{\omega,b}(A)$ whenever $0 < b < a$) and we are done. □

Lemma 3.5. *Let P be a nonempty complete metric space without isolated points. Let $A \subset P$, $m \in \mathbb{N}_0$, $\omega > 1$, $r > 0$, $\delta, \kappa, \alpha \in (0,1)$, $40\delta < \kappa$, $1/\omega < \alpha\kappa/10$. Let $D \subset P$ be a set such that $D \cap (N_{\omega,r})^j(A)$ is dense in $(N_{\omega,r})^j(A)$ for every $j \in \{1, \dots, m\}$. Let $x \in D \cap (N_{\omega,r})^m(A)$ and $\overline{B}(x,s) \subset \overline{B}(x,r)$. Then there exists $W \subset D$ with the $\mathcal{C}(m, \delta, \kappa, \alpha)$-property in $\overline{B}(x,s)$.*

Proof: Put

$$P_j = D \cap (N_{\omega,r})^{m-j}(A), \qquad j = 0, \dots, m.$$

It is sufficient to check that the P_j's and $\overline{B}(x,s)$ satisfy the assumption of Lemma 2.12. Clearly $x \in P_0$ and $P_0 \subset P_1 \subset \dots \subset P_m$. If $y \in P_j \cap \overline{B}(x,s)$, $j \in \{0, \dots, m-1\}$, and $B(z,d) \subset \overline{B}(x,s) \setminus P_{j+1}$, then $B(z,d) \subset \overline{B}(x,r) \setminus P_{j+1}$ and $B(z,d) \cap (N_{\omega,r})^{m-j-1}(A) = \emptyset$. Since $x \notin B(z,d)$ we have $d \le r$. We obtain $y \notin B(z, \omega d)$. We estimate

$$\frac{d}{\rho(y,z)} \le \frac{d}{\omega d} = \frac{1}{\omega} < \alpha\kappa/10.$$

This gives $\Gamma^\star(y, \overline{B}(x,s), P_{j+1}) < \alpha\kappa/10$.

Using Lemma 2.12 we see that $W := S_m$ works. □

Definition 3.6 (cf. [Ke, Definition 25.4]). Let $\mathcal{S} = \{P(s);\ s \in \operatorname{Seq}\mathbb{N}\}$ be a *Suslin scheme* on a set X, i.e. a family of subsets of X indexed by $\operatorname{Seq}\mathbb{N}$. The *Suslin operation* $\mathcal{A}$ applied to such a scheme produces the set

$$\mathcal{A}_s P(s) = \bigcup_{\nu \in \mathcal{N}} \bigcap_{n=0}^{\infty} P(\nu|n).$$

We say that a Suslin scheme $\mathcal{S}$ is *regular* if for every $s \in \operatorname{Seq}\mathbb{N}$ and $n \in \mathbb{N}$ we have $P(s^\wedge n) \subset P(s)$.

Setting 3.7. For the rest of this section we fix real numbers $\varepsilon, \kappa, \alpha_n, \delta \in (0,1)$, $\omega_n > 1$ ($n \in \mathbb{N}$) such that $10\kappa < \varepsilon < 1/8$, $40\delta < \kappa$ and for every $n \in \mathbb{N}$ we have $\alpha_n < \delta^{n+1}$, $1/\omega_n < \alpha_n\kappa/10$.

Definition 3.8. Let $\mathcal{S} = \{F(s);\ s \in \mathcal{N}\}$ be a Suslin scheme of subsets of a metric space Y. For every $t \in \operatorname{Seq}\mathbb{N}$ we put $S(t) = \mathcal{A}_s F(t^\wedge s)$ and define a set $T(\mathcal{S}, t)$ by

$$x \in T(\mathcal{S}, t) \overset{\text{def}}{\Longleftrightarrow} \text{ there exist a sequence of positive real numbers } (r_k)_{k=1}^{\infty} \text{ and } \nu \in \mathcal{N} \text{ such that } x \in (N_{\omega_{|t|+k}, r_k})^{|t|+k}(S(t^\wedge \nu|k)) \text{ for every } k \in \mathbb{N}.$$

Setting 3.9. Let P and S be as in Theorem 3.1. For the rest of this section we fix a regular Suslin scheme $\mathcal{S} = \{F(s);\ s \in \operatorname{Seq}\mathbb{N}\}$ of closed subsets of P such that $\mathcal{A}_s F(s) = S$. We denote $S(t) = \mathcal{A}_s F(t^\wedge s)$ and we will write $T(t)$ instead of $T(\mathcal{S}, t)$.

We need the following lemmas.

Lemma 3.10. *If $t \in \operatorname{Seq}\mathbb{N}$ and $x \in T(t)$, then there exists $\mu \in \mathcal{N}$ and $r > 0$ such that $\mu||t| = t$, $x \in (N_{\omega_{|t|+1}, r})^{|t|+1}(S(\mu|(|t|+1)))$ and $x \in T(\mu|(|t|+1))$.*

Proof: Let $(r_k)_{k=1}^{\infty}$ and $\nu \in \mathcal{N}$ witness that $x \in T(t)$. Then $r := r_1$ and $\mu := t^\wedge \nu$ work. □

Lemma 3.11. *Let $t \in \operatorname{Seq}\mathbb{N}$ and $A \subset S(t)$ be a set closed in $S(t)$ with $A = \ker(A)$. Then $T(t) \cap A$ is dense in A.*

Proof: If $A = \emptyset$, then there is nothing to prove. Assume that $A \neq \emptyset$. First we prove that $T(t) \cap A \neq \emptyset$. Let τ be an equivalent complete metric on P. Put $P_0 = A$ and assume that we have defined sets $P_0, P_1, \dots, P_k$, positive real numbers $r_1, \dots, r_k$ and natural numbers $j_1, \dots, j_k$ such that

(i)$_k$ $P_l \subset (N_{\omega_{|t|+l}, r_l})^{|t|+l}(P_{l-1} \cap S(t^\wedge(j_1, \dots, j_l)))$, $l = 1, \dots, k$,
(ii)$_k$ $\operatorname{diam}_\tau P_l < 1/l$, $l = 1, \dots, k$,
(iii)$_k$ $\ker(P_l) = P_l \neq \emptyset$, $l = 0, \dots, k$.

Since

$$S(t^\wedge(j_1, \dots, j_k)) = \bigcup_{j=1}^{\infty} S(t^\wedge(j_1, \dots, j_k, j)),$$

we have that

$$P_k \subset \bigcup_{j=1}^{\infty} S(t^\wedge(j_1, \dots, j_k, j)).$$

Thus there exists $j_{k+1} \in \mathbb{N}$ such that the set $P_k \cap S(t^\wedge(j_1, \dots, j_k, j_{k+1}))$ is non-σ-porous. According to Lemma 3.4(v) there exists $r_{k+1} > 0$ such that the set

$$Q := (N_{\omega_{|t|+k+1}, r_{k+1}})^{|t|+k+1}(P_k \cap S(t^\wedge(j_1, \dots, j_k, j_{k+1})))$$

is non-σ-porous. Choose an open ball B such that the set $Q \cap B$ is non-σ-porous and $\operatorname{diam}_\tau B < 1/(k+1)$. Put $P_{k+1} := Q \cap B$. The set P_{k+1} clearly satisfies

$(\mathrm{i})_{k+1}$ and $(\mathrm{ii})_{k+1}$. Property $(\mathrm{iii})_{k+1}$ is implied by Lemma 3.4(iv). Thus we have defined sequences $(P_k)_{k=0}^{\infty}$, $(r_k)_{k=1}^{\infty}$ and $(j_k)_{k=1}^{\infty}$ such that $(\mathrm{i})_k$–$(\mathrm{iii})_k$ are satisfied for every $k \in \mathbb{N}_0$. Put $\nu = (j_1, j_2, \dots) \in \mathcal{N}$.

We have

$$P_{k+1} \subset P_k \subset S(t^\wedge\nu|k) \subset \overline{S(t^\wedge\nu|k)} \subset F(t^\wedge\nu|k) \text{ and also } \lim_{k\to\infty} \operatorname{diam}_\tau P_k = 0.$$

Thus there exists $x \in \bigcap_{k=0}^{\infty} \overline{P_k}$. We have $x \in \bigcap_{k=0}^{\infty} F(t^\wedge\nu|k) \subset S(t^\wedge\nu|l)$ for every $l \in \mathbb{N}_0$. Using $(\mathrm{i})_k$ and Lemma 3.4(iii) we obtain

$$\overline{P_k} \cap S(t^\wedge\nu|k) \subset (N_{\omega_{|t|+k}, r_k})^{|t|+k}(S(t^\wedge\nu|k))$$

and so $x \in (N_{\omega_{|t|+k}, r_k})^{|t|+k}(S(t^\wedge\nu|k))$ for every $k \in \mathbb{N}$. This means that $x \in T(t)$. Since $x \in \overline{P_0} \cap S(t) = A$ we conclude $T(t) \cap A \neq \emptyset$.

Now assume that G is an open set intersecting A. We find an open set H such that $H \cap A \neq \emptyset$ and $\overline{H} \subset G$. Put $\tilde{A} = \ker(A \cap \overline{H})$. We have that $\tilde{A}$ is closed in $S(t)$ and $\ker(\tilde{A}) = \tilde{A} \neq \emptyset$. According to the previous considerations we have $T(t) \cap \tilde{A} \neq \emptyset$. Thus $T(t) \cap A \cap G \neq \emptyset$ and we are done. □

Proof of Theorem 3.1: First we suppose that P is moreover a complete metric space without isolated points.

We will work with finite sequences of closed balls in P and we will employ the following notation. Let H be a finite sequence of closed balls with n elements. If $k \in \mathbb{N}$, $k \leq n$, then H_k stands for the k-th element of H and $H|k = (H_1, \dots, H_k)$. If B is a closed ball in P, then the concatenation of H and the sequence (B) is denoted by $H^\wedge B$.

We will define systems $\mathcal{U}_n^\star$, $n \in \mathbb{N}_0$, of closed balls in P, sets $\mathcal{H}_n$, $n \in \mathbb{N}$, of finite sequences of closed balls in P and mappings $\varphi : \bigcup_{n=1}^{\infty} \mathcal{H}_n \to \mathcal{N}$, $r : \bigcup_{n=1}^{\infty} \mathcal{H}_n \to (0, +\infty)$ such that

(i) $\mathcal{U}_0^\star = \{\overline{B}(z,d)\}$, $\mathcal{H}_1 = \{(\overline{B}(z,d))\}$, where $z \in P$ and $d > 0$,
(ii) $\mathcal{U}_n^\star = \bigcup\{\mathcal{U}_n^\star(H);\ H \in \mathcal{H}_n\}$, where $\mathcal{U}_n^\star(H)$ is a B-system with the $\mathcal{P}(n, \delta, \alpha_n, \varepsilon)$-property in H_n, $\mathcal{H}_{n+1} = \{H^\wedge B;\ H \in \mathcal{H}_n, B \in \mathcal{U}_n^\star(H)\}$, $n \in \mathbb{N}$,

and for every $n \in \mathbb{N}$ and $H \in \mathcal{H}_n$ the following conditions are satisfied

(iii) $c(H_n) \in (N_{\omega_n, r(H)})^n(S(\varphi(H)|n))$,
(iv) $c(H_n) \in T(\varphi(H)|n)$,
(v) $H_n \subset B(c(H_n), r(H))$,
(vi) $\forall\, \tilde{H} \in \mathcal{H}_{n+1} : H = \tilde{H}|n \ \Rightarrow\ \varphi(H)|n = \varphi(\tilde{H})|n$.

Construction of $\mathcal{U}_n^\star$'s, $\mathcal{H}_n$'s, φ and r. The set S $(= S(\emptyset))$ is non-σ-porous and therefore we have $\ker(\ker(S)) = \ker(S) \neq \emptyset$ (Lemma 3.4(iv)). Using Lemma 3.4(i) we have that $\ker(S)$ is closed in S. Thus Lemma 3.11 implies that $T(\emptyset)$ is nonempty. Choose $z \in T(\emptyset)$. Using Lemma 3.10 we find $\mu \in \mathcal{N}$ and $d \in (0,+\infty)$ such that $z \in N_{\omega_1,d}(S(\mu|1))$ and $z \in T(\mu|1)$. Put $\mathcal{U}_0^\star = \{\overline{B}(z,d)\}$ and $\mathcal{H}_1 = \{(\overline{B}(z,d))\}$. We define $\varphi(\overline{B}(z,d)) = \mu$ and $r(\overline{B}(z,d)) = d$.

Let $n \in \mathbb{N}$. Suppose that $\mathcal{U}_{n-1}^\star$, $\mathcal{H}_n$ are defined and that φ, r are defined on $\bigcup_{j=1}^n \mathcal{H}_j$. Take $H = (H_1,\dots,H_n) \in \mathcal{H}_n$. We apply Lemma 3.5 for $A := S(\varphi(H)|n)$, $m := n$, $\omega := \omega_n$, $r := r(H)$, $\delta := \delta$, $\kappa := \kappa$, $\alpha := \alpha_n$, $D := T(\varphi(H)|n)$ and $\overline{B}(x,s) = H_n$. For every $j \in \{1,\dots,n\}$ we have that $T(\varphi(H)|n) \cap (N_{\omega_n,r(H)})^j(S(\varphi(H)|n))$ is dense in $(N_{\omega_n,r(H)})^j(S(\varphi(H)|n))$ by Lemmas 3.4(iii), 3.4(iv) and 3.11. This, conditions (iii), (iv) and (v) show that all assumptions of Lemma 3.5 are satisfied. Thus there exists a set $W \subset T(\varphi(H)|n)$ with the $\mathcal{C}(n,\delta,\kappa,\alpha_n)$-property in H_n.

Using Lemma 3.10 we find, for each $y \in W$, $\psi^y \in \mathcal{N}$ and $s^y \in (0,+\infty)$ such that

- $y \in (N_{\omega_{n+1},s^y})^{n+1}(S(\psi^y|(n+1)))$,
- $y \in T(\psi^y|(n+1))$,
- $\psi^y|n = \varphi(H)|n$.

By Lemma 2.14 there exists a B-system $\mathcal{U}_n^\star(H)$ such that

- $\mathcal{U}_n^\star(H)$ has the $\mathcal{P}(n,\delta,\alpha_n,\varepsilon)$-property in H_n,
- $c(\mathcal{U}_n^\star(H)) = W$,
- if $V \in \mathcal{U}_n^\star(H)$ and $c(V) = y \in W$, then $V \subset B(y,s^y)$.

Take $H_{n+1} \in \mathcal{U}_n^\star(H)$. Denote $y = c(H_{n+1})$ and define

$$\varphi(H_1,H_2,\dots,H_{n+1}) = \psi^y,\ r(H_1,H_2,\dots,H_{n+1}) = s^y.$$

Finally we put

$$\mathcal{U}_n^\star = \bigcup\{\mathcal{U}_n^\star(H);\ H \in \mathcal{H}_n\},$$
$$\mathcal{H}_{n+1} = \{H^\wedge B;\ H \in \mathcal{H}_n, B \in \mathcal{U}_n^\star(H)\}.$$

It is not difficult to see that the $\mathcal{U}_n^\star$'s, $\mathcal{H}_n$'s, φ and r satisfy (i)–(vi).

Put $F = \bigcap_{n=0}^\infty \bigcup \mathcal{U}_n^\star$. Take $n \in \mathbb{N}_0$ and $C \in \mathcal{U}_n^\star$. Each B-system is point finite and so there exist only finitely many sequences $(H_1,\dots,H_{n+1}) \in \mathcal{H}_{n+1}$ with $H_{n+1} = C$. Using this, Lemma 2.4(i) and Lemma 2.4(iv) we easily obtain by induction that $\mathcal{U}_n^\star$ is a B-system for every $n \in \mathbb{N}_0$. Thus $\bigcup \mathcal{U}_n^\star$ is closed by Lemma 2.4(iii) and this implies that F is closed.

We define a sequence $(\mathcal{U}_n)_{n=0}^\infty$ of B-systems as follows:

- $\mathcal{U}_0 = \mathcal{U}_0^\star$,
- $\mathcal{U}_{n+1} = \bigcup\{\mathcal{U}_{n+1}(C);\ C \in \mathcal{U}_n\}$, where $\mathcal{U}_{n+1}(C) = \mathcal{U}_{n+1}^\star(H_1,\dots,H_{n+1})$ for some $(H_1,\dots,H_{n+1}) \in \mathcal{H}_{n+1}$ with $H_{n+1} = C$, $n \in \mathbb{N}_0$.

Using (ii) we have that $\mathcal{U}_{n+1}(C)$, where $n \in \mathbb{N}_0$ and $C \in \mathcal{U}_n$, has the $\mathcal{P}(n+1, \delta, \alpha_{n+1}, \varepsilon)$-property in C. Using Lemma 2.22 we obtain that $\bigcap_{n=0}^{\infty} \bigcup \mathcal{U}_n$ is non-σ-porous. Thus F is non-σ-porous since $\bigcap_{n=0}^{\infty} \bigcup \mathcal{U}_n \subset F$.

It remains to show that $F \subset S$. Choose $x \in F$. Define a tree $\mathcal{T}$ by

$$\mathcal{T} = \{\emptyset\} \cup \{(H_1, \ldots, H_k) \in \bigcup_{n=1}^{\infty} \mathcal{H}_n;\ k \in \mathbb{N},\ x \in H_k\}.$$

The infinite tree $\mathcal{T}$ is finite splitting. So by König's Lemma there exists a sequence of closed balls $(H_n)_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$ we have $(H_1, \ldots, H_n) \in \mathcal{H}_n$ and $x \in H_n$. Using condition (vi) we find $\nu \in \mathcal{N}$ such that $\nu|n = \varphi(H_1, \ldots, H_n)|n$ for every $n \in \mathbb{N}$. Since $\mathcal{U}_n^{\star}(H_1, \ldots, H_n)$ has the $\mathcal{P}(n, \delta, \alpha_n, \varepsilon)$-property in H_n for every $n \in \mathbb{N}$, we have $\operatorname{diam} H_{n+1} \leq \frac{1}{2} \operatorname{diam} H_n$. This implies $\lim_{n\to\infty} \operatorname{diam} H_n = 0$. Thus we have $\lim_{n\to\infty} c(H_n) = x$. Condition (iii) and the regularity of the Suslin scheme $\mathcal{S}$ give $c(H_n) \in S(\nu|n) \subset F(\nu|n) \subset F(\nu|k)$ for every $n, k \in \mathbb{N}$, $n \geq k$. Thus we have $x \in F(\nu|k)$ for every $k \in \mathbb{N}$ and therefore $x \in S$. Hence $F \subset S$.

Now assume that P is an arbitrary complete metric space. We may and do assume that $S = \ker(S) \neq \emptyset$. If S contains an isolated point x of the space P, then we put $F = \{x\}$ and we are done. Suppose that S contains no isolated point of the space P. Since $\ker(S) = S$ we have that S has no isolated point. Define

$$P^{\star} = \overline{S} \quad \text{and} \quad S^{\star} = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} D_{k,1/n}(S).$$

Due to Lemma 3.4(ii) and due to the well-known fact that a countable union (intersection, respectively) of Suslin sets is Suslin (see [R, p. 16]), we have that the set $S^{\star}$ is Suslin. It is easy to see that $S^{\star}$ contains exactly those points of S, which are points of non-porosity of S. Thus each point of $S^{\star}$ is a point of non-porosity of $P^{\star}$. The set $S \setminus S^{\star}$ is porous and therefore $S^{\star}$ is non-σ-porous in the space P. This gives that $S^{\star}$ is non-σ-porous in the space $P^{\star}$. The space $P^{\star}$ is complete and has no isolated point. Thus we find a closed set $F \subset S^{\star}$, which is not σ-porous in the space $P^{\star}$. Lemma 2.1 gives that F is non-σ-porous also in the space P.

Finally suppose that P is a topologically complete metric space and $S \subset P$ is a Suslin non-σ-porous set. Let $\tilde{P}$ be a completion of P. The set S is Suslin in $\tilde{P}$ since P is $\boldsymbol{G}_{\delta}$ in $\tilde{P}$. Each point of P is a point of non-porosity of P in $\tilde{P}$ since P is dense in $\tilde{P}$. Thus the set S is a non-σ-porous Suslin subset of the space $\tilde{P}$ by Lemma 2.1. We find a set $F \subset S$, which is a closed non-σ-porous subset of the space $\tilde{P}$. Such a set is clearly a closed non-σ-porous subset of P. This finishes the proof of Theorem 3.1. $\square$

Remark 3.12. If P is a locally compact metric space, then F in Theorem 3.1 can be clearly found compact. This is impossible in general. For example, each compact subset of the space $\mathbb{N}^{\mathbb{N}}$ (with the usual product metric) is porous. The same is true for each infinite-dimensional Banach space.

Remark 3.13. Solecki ([So]) proved the following interesting theorem.

Let A be an analytic subset of a Polish space X and $\mathcal{F}$ be a family of closed subsets of X. If A cannot be covered by countably many elements of $\mathcal{F}$, then there exists a $\boldsymbol{G}_\delta$ subset H of A with the same property.

We cannot use this result directly to show that each non-σ-porous analytic subset of a separable complete metric space contains a $\boldsymbol{G}_\delta$ non-σ-porous subset, since there exists a porous $\boldsymbol{G}_\delta$ subset of (say) $\mathbb{R}$ which cannot be covered by countably many closed σ-porous sets (see Lemma 6.12).

4. Analytic subsets of a hyperspace and σ-porosity

Let (E,ρ) be a compact metric space. The set $\mathcal{K}(E)$ of all compact subsets of E is equipped with the Vietoris topology. The Hausdorff metric on $\mathcal{K}(E)$ (where the distance of a nonempty compact set and the empty set is equal to $\operatorname{diam} E+1$) is compatible with the Vietoris topology.

If $D \subset E$, then we denote

$$\begin{aligned} \mathcal{K}(D) &= \{K \in \mathcal{K}(E);\ K \subset D\},\\ \mathcal{K}_\omega(D) &= \{K \in \mathcal{K}(E);\ K \text{ is a countable subset of } D\}. \end{aligned}$$

We say that $S \subset \mathcal{K}(E)$ is *hereditary* if for every $L \in \mathcal{K}(E)$, $K \in S$, $L \subset K$ we have $L \in S$. If $S \subset \mathcal{K}(E)$, then $\operatorname{her} S$ denotes the *hereditary closure* of S, i.e.

$$\operatorname{her} S = \{K \in \mathcal{K}(E);\ \text{there exists } L \in S \text{ with } K \subset L\}.$$

Let us summarize several well-known facts, which will be useful for us in the following. Proofs are easy and, therefore, they will be omitted.

Lemma 4.1. *Let E be a compact metric space.*

(i) *Let $F \subset \mathcal{K}(E)$ be a closed set. Then* $\operatorname{her} F$ *is also a closed subset of $\mathcal{K}(E)$.*

(ii) *Let $K, K_1, K_2, \dots \in \mathcal{K}(E) \setminus \{\emptyset\}$ be such that* $\lim_{n\to\infty} \sup\{\operatorname{dist}(y,K);\ y \in K_n\} = 0$. *Then $K \cup \bigcup_{n=1}^\infty K_n \in \mathcal{K}(E)$.*

(iii) *Let φ be a continuous mapping of E into E. Then the mapping Φ of $\mathcal{K}(E)$ into $\mathcal{K}(E)$ defined by*

$$\Phi : K \mapsto \varphi[K]$$

is continuous.

(iv) *The space $\mathcal{K}(E)$ is compact.*
(v) *The mapping $(K,L) \mapsto K \cap L$ from $\mathcal{K}(E) \times \mathcal{K}(E)$ to $\mathcal{K}(E)$ is Borel.*
(vi) *Let $G \subset E$ be an open set. Then the mapping $K \mapsto \overline{K \cap G}$ from $\mathcal{K}(E)$ to $\mathcal{K}(E)$ is Borel.*
(vii) *The set $\{(K,L) \in \mathcal{K}(E) \times \mathcal{K}(E);\ K \subset L\}$ is closed in $\mathcal{K}(E) \times \mathcal{K}(E)$.*

Lemma 4.2. *Let E be a compact metric space. Let $A \subset \mathcal{K}(E)$ be a set such that $A = \mathcal{A}_s F(s)$, where $\{F(s);\ s \in \operatorname{Seq}\mathbb{N}\}$ is a regular Suslin scheme of closed subsets of $\mathcal{K}(E)$. Then we have* $\operatorname{her} A = \mathcal{A}_s \operatorname{her} F(s)$.

Proof: Let $K \in \operatorname{her} A$. Then there exist $\tilde{K} \in A$ and $\nu \in \mathcal{N}$ such that $K \subset \tilde{K}$ and $\tilde{K} \in F(\nu|n)$ for every $n \in \mathbb{N}_0$. Thus we have $K \in \operatorname{her} F(\nu|n)$ for every $n \in \mathbb{N}_0$ and so $K \in \mathcal{A}_s \operatorname{her} F(s)$.

Now suppose that $K \in \mathcal{A}_s \operatorname{her} F(s)$. It means that there exist $\nu \in \mathcal{N}$ and a sequence $(K_n)_{n=0}^{\infty}$ of compact sets such that $K \subset K_n$, $K_n \in F(\nu|n)$. The space $\mathcal{K}(E)$ is compact and therefore there exists a converging subsequence $(K_{n_j})_{j=1}^{\infty}$ of the sequence $(K_n)_{n=0}^{\infty}$. We have $L := \lim_{j\to\infty} K_{n_j} \in F(\nu|n)$ for every $n \in \mathbb{N}_0$ since the $F(\nu|n)$'s are closed and $F(\nu|(n+1)) \subset F(\nu|n)$. This gives that $L \in A$. It is easy to see that $K \subset L$. These two facts imply $K \in \operatorname{her} A$. $\square$

Lemma 4.3. *Let E be a compact metric space. Let $D \subset E$ and $K \in \mathcal{K}_\omega(D) \setminus \{\emptyset\}$. Let $P_n \subset \mathcal{K}(E)$ be a hereditary subset for every $n \in \mathbb{N}$ and $\mathcal{K}_\omega(D) \subset \bigcup_{n=1}^{\infty} P_n$. Then there exists an open set G such that $K \subset G$ and $\mathcal{K}_\omega(D \cap G) \subset P_m$ for some $m \in \mathbb{N}$.*

Proof: Put $G_k = \{x \in E;\ \operatorname{dist}(x,K) < 1/k\}$ for every $k \in \mathbb{N}$. Suppose to the contrary that the assertion of the lemma does not hold. Then for every $k \in \mathbb{N}$ there exists a nonempty set $K_k \in \mathcal{K}_\omega(D \cap G_k) \setminus P_k$. Put $L = K \cup \bigcup_{k=1}^{\infty} K_k$. We have $L \in \mathcal{K}_\omega(D)$ according to Lemma 4.1(ii). We also have that $L \notin P_k$ for every $k \in \mathbb{N}$ since P_k is a hereditary subset of $\mathcal{K}(E)$, $K_k \notin P_k$ and $K_k \subset L$ for every $k \in \mathbb{N}$. This contradiction proves our lemma. $\square$

Setting 4.4. For the rest of this section, let the real numbers $\varepsilon, \kappa, \alpha_n, \delta \in (0,1)$, $\omega_n > 1$ $(n \in \mathbb{N})$ be the same as in Setting 3.7.

Definition 4.5. Let E be a metric space. Let $W \subset E$. We define $Q(W)$ by

$$x \in Q(W) \overset{\text{def}}{\Longleftrightarrow} \text{there exists a sequence of positive real numbers } (r_k)_{k=1}^{\infty} \text{ such that } x \in (N_{\omega_k,r_k})^k(W) \text{ for every } k \in \mathbb{N}.$$

Lemma 4.6. *Let E be a nonempty compact metric space without isolated points. Let W and D be nonempty subsets of E such that*

(a) *$D \subset Q(W)$,*
(b) *$D \cap (N_{\omega_k,r})^m(W)$ is dense in $(N_{\omega_k,r})^m(W)$ whenever $k, m \in \mathbb{N}$, $r > 0$.*

Let A be an analytic subset of $\mathcal{K}(E)$ containing $\mathcal{K}_\omega(D)$. Then there exists $K \in A$, which is not σ-porous.

Proof: For every $x \in Q(W)$ we fix a sequence $(r_k(x))_{k=1}^\infty$ of positive real numbers such that

$$x \in (N_{\omega_k, r_k(x)})^k(W) \text{ for every } k \in \mathbb{N}.$$

We may also assume that A is hereditary (Lemma 4.1(i) and Lemma 4.2). Thus there exists a regular Suslin scheme $\{F(s);\ s \in \operatorname{Seq}\mathbb{N}\}$ consisting of closed hereditary subsets of $\mathcal{K}(E)$ such that $A = \mathcal{A}_s F(s)$. For $t \in \operatorname{Seq}\mathbb{N}$ we define $A(t) = \mathcal{A}_s F(t^\wedge s)$. We have that $A(t)$ is hereditary and $A(t) = \bigcup_{j=1}^\infty A(t^\wedge j)$ for every $t \in \operatorname{Seq}\mathbb{N}$.

Choose a closed ball B_0 with $c(B_0) \in D$ such that $\operatorname{diam} B_0 < r_1(c(B_0))$. We define $\nu = (\nu_1, \nu_2, \dots) \in \mathcal{N}$, a sequence of B-systems $(\mathcal{V}_n)_{n=0}^\infty$ and a sequence of open sets $(G_n)_{n=0}^\infty$ such that $\mathcal{V}_0 = \{B_0\}$ and for every $n \in \mathbb{N}_0$ we have

- $\mathcal{V}_{n+1} = \bigcup\{\mathcal{V}_{n+1}(C);\ C \in \mathcal{V}_n\}$, where $\mathcal{V}_{n+1}(C)$ has the $\mathcal{P}(n+1, \delta, \alpha_{n+1}, \varepsilon)$-property in C,
- $\bigcup \mathcal{V}_n \subset G_n$,
- $\mathcal{K}_\omega(D \cap G_n) \subset A(\nu|n)$,
- $\forall\, C \in \mathcal{V}_n :\ \operatorname{diam} C < r_{n+1}(c(C))$,
- $\mathcal{V}_n$ is countable,
- $c(\mathcal{V}_n) \subset D$.

We put $\mathcal{V}_0 = \{B_0\}$ and $G_0 = E$. Now suppose that we have defined $\mathcal{V}_n$, G_n and $\nu_1, \dots, \nu_n$. Take $C \in \mathcal{V}_n$. We have that $\operatorname{diam} C < r_{n+1}(c(C))$ and $c(C) \in D \cap (N_{\omega_{n+1}, r_{n+1}(c(C))})^{n+1}(W)$. Using Observation 2.8, Lemma 3.5 and assumption (b) of Lemma 4.6, there exists a countable set $S(C) \subset D$ with the $\mathcal{C}(n+1, \delta, \kappa, \alpha_{n+1})$-property in C. Put

$$S = \bigcup\{S(C);\ C \in \mathcal{V}_n\}.$$

The set S is a countable union of countable sets and S is closed by Lemma 2.4(ii). Thus $S \in \mathcal{K}_\omega(D)$. We have also $S \subset G_n$ and, therefore,

$$S \in \mathcal{K}_\omega(D \cap G_n) \subset A(\nu_1, \dots, \nu_n) = \bigcup_{j=1}^\infty A(\nu_1, \dots, \nu_n, j).$$

According to Lemma 4.3 there exist $\nu_{n+1} \in \mathbb{N}$ and an open set H such that $S \subset H$ and $\mathcal{K}_\omega(D \cap H) \subset A(\nu_1, \dots, \nu_n, \nu_{n+1})$. For every $x \in S$ there exists an open ball $B_x \subset H$ centered at x such that $\operatorname{diam} B_x < r_{n+2}(x)$. Put

$$G_{n+1} = \bigcup\{B_x;\ x \in S\}.$$

For every $C \in \mathcal{V}_n$ there exists a B-system $\mathcal{V}_{n+1}(C)$ with the $\mathcal{P}(n+1, \delta, \alpha_{n+1}, \varepsilon)$-property in C such that $c(\mathcal{V}_{n+1}(C)) = S(C)$ and for every $V \in \mathcal{V}_{n+1}(C)$ we have $V \subset B_{c(V)}$ (Lemma 2.14). The system $\mathcal{V}_{n+1}(C)$ is countable since $S(C)$ is countable and $\mathcal{V}_{n+1}(C)$ is point finite. Put

$$\mathcal{V}_{n+1} = \bigcup \{\mathcal{V}_{n+1}(C); C \in \mathcal{V}_n\}.$$

The system $\mathcal{V}_{n+1}$ is countable since $\mathcal{V}_n$ and each $\mathcal{V}_{n+1}(C)$, $C \in \mathcal{V}_n$, are countable. Thus we have defined the desired $\mathcal{V}_n$'s, G_n's and $\nu = (\nu_1, \nu_2, \dots) \in \mathcal{N}$. Put

$$K = \bigcap_{n=0}^{\infty} \bigcup \mathcal{V}_n.$$

The set K is a closed (hence compact) non-σ-porous subset of E by Lemma 2.22. We have that $c(\mathcal{V}_k)$ is closed by Lemma 2.4(ii). Property $(\mathrm{P3})_n$ of $\mathcal{V}_n(C)$, where $n \in \mathbb{N}$, $C \in \mathcal{V}_{n-1}$, gives that the compact sets $c(\mathcal{V}_k)$, $k \in \mathbb{N}$, converge to K in the space $\mathcal{K}(E)$. We have also

$$c(\mathcal{V}_k) \in A(\nu|k) \subset A(\nu|n) \subset F(\nu|n) \text{ for every } k, n \in \mathbb{N}_0,\ k \geq n.$$

This implies $K \in \bigcap_{n=0}^{\infty} F(\nu|n) \subset A$ and we are done. $\square$

Lemma 4.7. *Let E be a compact metric space. Let $W \subset E$ be an analytic set with* $\ker(W) = W \neq \emptyset$. *Then there exists a nonempty set $D \subset E$ satisfying* (a)–(b) *in Lemma* 4.6.

Proof: Let $\mathcal{S} = \{F(s);\ s \in \operatorname{Seq}\mathbb{N}\}$ be a regular Suslin scheme consisting of closed sets such that $W = \mathcal{A}_s F(s)$. Put $D = T(\mathcal{S}, \emptyset)$ (see Definition 3.8). By the definition we have $D \subset Q(W)$. Lemma 3.11 shows that $D \neq \emptyset$ and $D \cap (N_{\omega_k,r})^m(W)$ is dense in $(N_{\omega_k,r})^m(W)$ whenever $k, m \in \mathbb{N}$, $r > 0$. $\square$

Theorem 4.8. *Let E be a nonempty compact metric space, $D \subset E$ be a dense subset of E and A be an analytic subset of $\mathcal{K}(E)$ containing $\mathcal{K}_\omega(D)$. Then there exists $K \in A$, which is not σ-porous.*

Proof: Suppose that E has an isolated point x. Then $x \in D$ and $\{x\} \in A$. The set $\{x\}$ is clearly non-σ-porous. So we may assume that E has no isolated point. Each σ-porous subset of a metric space is a set of the first category. Using the Baire Category Theorem we obtain that each nonempty open subset of a complete metric space is non-σ-porous. Thus we have $Q(E) = E$ and $(N_{\omega,r})^m(E) = E$ for every $\omega > 1$, $m \in \mathbb{N}$ and $r > 0$. Putting $W := E$ we see that the set D satisfies the assumptions of Lemma 4.6 and we are done. $\square$

Theorem 4.9. *Let E be a nonempty compact metric space, K_0 be a non-σ-porous compact subset of E and A be an analytic subset of $\mathcal{K}(E)$ containing $\mathcal{K}_\omega(K_0)$. Then there exists $K \in A$, which is not σ-porous.*

PROOF: First assume that E has no isolated point. Put $K_1 = \ker(K_0)$. According to Lemma 4.7 there exists a nonempty $D \subset K_1$ satisfying (a)–(b) of Lemma 4.6, where W is replaced by K_1. Thus there exists $K \in A$, which is not σ-porous.

Suppose that E is an arbitrary compact metric space. If $K_1 (= \ker(K_0))$ has an isolated point x, then we put $K = \{x\}$. Such a set is non-σ-porous. Now suppose that K_1 has no isolated point. We put

$$S^\star = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} D_{k,1/n}(K_1).$$

The set $S^\star$ has the following properties:

- $S^\star$ is Borel (Lemma 3.4(ii)),
- $S^\star$ contains exactly points of non-porosity of K_1,
- $S^\star$ is not σ-porous.

According to Theorem 3.1 there exists a closed non-σ-porous set $K_2 \subset S^\star$. The space K_1 has no isolated point, the set K_2 is not σ-porous in the space K_1, the set $A \cap \mathcal{K}(K_2)$ is analytic and contains $\mathcal{K}_\omega(K_2)$. According to the first part of this proof we have that there exists $K \in A$, $K \subset K_2$, which is not σ-porous in the space K_1. Using Lemma 2.1 for $P := E$, $T := K_1$, $A := S^\star$, $C := K$ we obtain that K is non-σ-porous in the space E. □

5. Applications

5.1 Measure, meagerness, capacity and σ-porosity. Each σ-porous subset of a metric space P is meager. This follows directly from the definition of σ-porosity. It is not hard to see (using the Lebesgue Density Theorem) that each σ-porous subset of $\mathbb{R}^n$ is a Lebesgue null set. The following theorem is stated in Dolzhenko's paper ([Do]), where the notion of σ-porosity was introduced.

Theorem 5.1. *There exists a closed non-σ-porous subset of $\mathbb{R}^n$ with null n-dimensional Lebesgue measure.*

The first published proof can be found in [Za$_1$]. We prove a more general theorem using our result from the preceding section and Zajíček's theorem from [Za$_3$].

Theorem 5.2. *Let P be a nonempty separable topologically complete metric space without isolated points. Then there exists a nowhere dense closed non-σ-porous subset of P with Hausdorff dimension zero.*

PROOF: According to [Za$_3$, Theorem 3] there exists a nowhere dense $\boldsymbol{G}_\delta$ non-σ-porous set $H \subset P$ of Hausdorff dimension zero. Using Theorem 3.1 we find a closed nowhere dense non-σ-porous set $F \subset H$. □

Remark 5.3. Theorem 5.2 can be proved directly using Lemma 2.22 constructing an appropriate sequence of B-systems.

Now we will deal with capacities. Following [Ke, Definition 30.1] we define the notion of capacity as follows.

Definition 5.4. Let X be a Hausdorff topological space. A *capacity* on X is a mapping $\gamma : \mathrm{Pow}(X) \to [0, \infty]$ such that:

(i) $A \subset B \Rightarrow \gamma(A) \leq \gamma(B)$;
(ii) $A_1 \subset A_2 \subset \cdots \Rightarrow \lim_{n\to\infty} \gamma(A_n) = \gamma(\bigcup A_n)$;
(iii) for any compact $K \subset X$ we have $\gamma(K) < \infty$;
(iv) for any compact $K \subset X$ with $\gamma(K) < r$ there exists an open set $U \supset K$ with $\gamma(U) < r$.

If E is a compact metric space and γ is a capacity on E, then the set

$$\{K \in \mathcal{K}(E);\ \gamma(K) = 0\}$$

is a $\boldsymbol{G}_\delta$ subset of $\mathcal{K}(E)$ (see [Ke, Exercise 30.15]). This fact and Theorem 4.8 give the following result.

Theorem 5.5. *Let E be a compact metric space and γ be a capacity on E such that there exists a set D dense in E with $\gamma(D) = 0$. Then there exists a compact non-σ-porous set $K \subset E$ with $\gamma(K) = 0$.*

5.2 Trigonometric series and σ-porosity. This section deals with U-sets and N-sets. We start with the definition of U-set. The symbol $\mathbb{T}$ stands for the interval $[0, 2\pi]$, where the endpoints are identified.

Definition 5.6. We say that a set $P \subset \mathbb{T}$ is a set of *uniqueness* (or *U-set*) if every trigonometric series $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ converging to zero in $\mathbb{T} \setminus P$ has necessarily all coefficients equal to zero.

Definition 5.7. We denote

$$U = \{K \in \mathcal{K}(\mathbb{T});\ K \text{ is a set of uniqueness}\}.$$

It is not difficult to prove that each measurable set of uniqueness has zero Lebesgue measure. The question, whether each Borel set of uniqueness is necessarily meager, had been open for a long time and was positively solved by Debs and Saint-Raymond.

Theorem 5.8 (Debs, Saint-Raymond [DSR]). *If $P \subset \mathbb{T}$ is a set of uniqueness with the Baire property, then P is meager.*

The following question was posed in [Za_2] and in [BKR]: Is each Borel set of uniqueness σ-porous? The following theorem answers this question negatively.

Theorem 5.9. *There exists a closed set of uniqueness, which is not σ-porous.*

We cannot use Theorem 4.8 directly since U is a non-analytic subset of $\mathcal{K}(\mathbb{T})$ (Solovay, Kaufman ([Ka])). The proof of this can be found also in the monograph [KL, p. 123], which is devoted to sets of uniqueness. Following [KL] we introduce several notions, which enable us to prove Theorem 5.9.

The space of all functions defined on $\mathbb{T}$ with absolutely convergent Fourier series is denoted by $A(\mathbb{T})$. The space $A(\mathbb{T})$ with the usual norm can be identified with $\ell^1(\mathbb{Z})$. The space $\ell^\infty(\mathbb{Z})$ with the sup-norm is denoted by PM — the space of *pseudomeasures*. The space $c_0(\mathbb{Z})$ with the sup-norm is denoted by PF — the space of *pseudofunctions*. So the dual of PF is $A(\mathbb{T})$ and the dual of $A(\mathbb{T})$ is PM. Let $f \in A(\mathbb{T})$, $S \in PM$. Then we define

$$\langle f, S\rangle = \sum_{n\in\mathbb{Z}} \hat{f}(n)S(-n),$$

where

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx,\ n \in \mathbb{Z}.$$

Let $S \in PF$ and $K \in \mathcal{K}(\mathbb{T})$. We say that K *supports* S if for any open interval I, $I \cap K = \emptyset$, and $\varphi \in \mathcal{C}^\infty(\mathbb{T})$ with $\operatorname{supp}\varphi \subset I$ we have $\langle \varphi, S\rangle = 0$. (Recall that $\mathcal{C}^\infty(\mathbb{T}) \subset A(\mathbb{T})$.)

We will use the following theorems.

Theorem 5.10 (Piatetski-Shapiro [PS], see [KL, p. 174]). *Let $K \in \mathcal{K}(\mathbb{T})$. Then K is a set of uniqueness if and only if*

$$A(\mathbb{T}) = \overline{\{f \in A(\mathbb{T});\ \operatorname{supp} f \cap K = \emptyset\}}^{w^\star}.$$

We define a proper subclass of U as follows

$$K \in U' \overset{\text{def}}{\iff} A(\mathbb{T}) = \overline{\{f \in A(\mathbb{T});\ \operatorname{supp} f \cap K = \emptyset\}}^{w^\star-\text{sequential}}.$$

Theorem 5.11 (Kechris, Louveau [KL, p. 129]). *The set U' is a $\boldsymbol{G}_{\delta\sigma}$ subset of $\mathcal{K}(\mathbb{T})$.*

Theorem 5.12 (Loomis [Lo], see [KL, p. 185]). *Each closed countable subset of $\mathbb{T}$ belongs to U'.*

Proof of Theorem 5.9: Theorems 5.10–5.12 give that U' is an analytic subset of $\mathcal{K}(\mathbb{T})$ with $\mathcal{K}_\omega(\mathbb{T}) \subset U' \subset U$. The application of Theorem 4.8 to U' finishes the proof. □

Remark 5.13. Note that all explicit examples (known to the authors) of sets of uniqueness ($H^{(n)}$-sets, Meyer sets, countable sets, symmetric homogeneous sets satisfying Salem-Zygmund conditions, see [KL, p. 90]) are σ-porous. Moreover, this fact is obvious for all these classes except $H^{(n)}$-sets, $n > 1$. The proof of σ-porosity of $H^{(n)}$-sets is non-trivial and is due to Šleich ([Šl], see also [Za$_5$]).

At the end of this section we focus on so-called N-sets. Here we have the definition.

Definition 5.14. A set $P \subset \mathbb{T}$ is an *N-set* if, for some sequence $(a_n)_{n=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty} a_n = \infty$, the series $\sum_{n=1}^{\infty} a_n \sin nt$ converges absolutely on P.

Konjagin (see [Za$_2$]) proved the following result concerning N-sets.

Theorem 5.15 (Konjagin)**.** *The set $K = \{x \in \mathbb{T};\ \sum_{n=1}^{\infty} |\sin(n!x)|/n \leq 1\}$ is a closed non-σ-porous N-set.*

This answers in negative the question, whether each N-set is σ-porous.

Using results due to Bjőrk and Kaufman we obtain the following theorem (see [LP] and [BKL]).

Theorem 5.16 ([BKL])**.** *The set $\{K \in \mathcal{K}(\mathbb{T});\ K$ is an N-set$\}$ is a $\boldsymbol{G}_\delta$ subset of* $\mathcal{K}(\mathbb{T})$.

Using this, the next theorem and Theorem 4.8 we obtain an alternative proof of the existence of a closed non-σ-porous N-set.

Theorem 5.17 ([Ba])**.** *Each countable set is an N-set.*

5.3 Linearly independent non-σ-porous set.

In this section we show that there exists a compact non-σ-porous subset of $\mathbb{R}$, which is linearly independent in the vector space $\mathbb{R}$ over the field of rational numbers. We use this to answer Laczkovich's question concerning analytic subgroups of $\mathbb{R}$.

Theorem 5.18. *There exists a compact non-σ-porous subset of $\mathbb{R}$, which is linearly independent in the vector space $\mathbb{R}$ over $\mathbb{Q}$.*

Proof: We define

$$A = \{K \in \mathcal{K}([0,1]);\ K \text{ is linearly independent}\}.$$

It is easy to construct a countable independent set $D \subset [0,1]$, which is dense in $[0,1]$. Then we have $\mathcal{K}_\omega(D) \subset A$. Now we show that A is a Borel subset of $\mathcal{K}([0,1])$.

Let Seq' $\mathbb{Q}$ be the set of all nonempty finite sequences s of rational numbers such that at least one member of s is non-zero. For $s \in$ Seq' $\mathbb{Q}$ we define a mapping $L_s : \mathbb{R}^{|s|} \to \mathbb{R}$ by

$$L_s(x_1, \dots, x_{|s|}) = s_1 x_1 + \cdots + s_{|s|} x_{|s|}.$$

(The symbol $|s|$ stands for the length of s.) For $m, n \in \mathbb{N}$ we denote

$$D(n,m) = \{(x_1, \dots, x_n) \in \mathbb{R}^n;\ \operatorname{dist}(x_i, x_j) \geq 1/m,\ i, j = 1, \dots, n,\ i \neq j\}.$$

Define mappings C^n and P^n_m of $\mathcal{K}([0,1])$ to $\mathcal{K}([0,1]^n)$ by

$$C^n : K \mapsto \underbrace{K \times \cdots \times K}_{n \text{ times}}, \qquad P^n_m : K \mapsto C^n(K) \cap D(n,m).$$

It is clear that the mapping C^n is continuous and therefore the mapping P^n_m is Borel by Lemma 4.1(v). We obtain that the mapping $T^m_s : \mathcal{K}([0,1]) \to \mathcal{K}([-|s_1| - \cdots - |s_{|s|}|, |s_1| + \cdots + |s_{|s|}|])$, $s \in \operatorname{Seq'}\mathbb{Q}$, $m \in \mathbb{N}$, defined by

$$T^m_s : K \mapsto L_s[P^{|s|}_m(K)]$$

is Borel. (We have used Lemma 4.1(iii).) It is easy to check that

$$A = \bigcap_{s \in \operatorname{Seq'}\mathbb{Q}} \bigcap_{m=1}^{\infty} \{K \in \mathcal{K}([0,1]);\ 0 \notin T^m_s(K)\}.$$

The set A is Borel since the set $\{K \in \mathcal{K}([0,1]);\ 0 \notin T^m_s(K)\}$ is Borel for every $m \in \mathbb{N}$ and $s \in \operatorname{Seq'}\mathbb{Q}$. Thus A satisfies the assumptions of Theorem 4.8 and we are done. $\square$

Now we turn our attention to analytic subgroups of the reals. Let G be an additive proper subgroup of $\mathbb{R}$. It is well-known that if G is analytic, then G is a meager null set. Laczkovich ([La]) proved the following improvement.

Theorem 5.19 (Laczkovich [La]). *Every analytic proper subgroup of $\mathbb{R}$ can be covered by an $\boldsymbol{F_\sigma}$ null set.*

Laczkovich posed (in a private communication) the following question: Is every proper analytic subgroup of the reals necessarily σ-porous? We answer this question negatively.

Theorem 5.20. *There exists an analytic proper subgroup of $\mathbb{R}$, which is not σ-porous.*

Proof: According to Theorem 5.18, there exists a compact non-σ-porous set $K \subset [0,1]$, which is linearly independent. We choose a proper subset $F \subset K$ with the same properties. Let G be a linear envelope of F (in $\mathbb{R}$ over $\mathbb{Q}$). The set G is a vector subspace of the vector space $\mathbb{R}$ over $\mathbb{Q}$, hence G is an additive subgroup. The group G is proper since $K \setminus G \neq \emptyset$. The set G is non-σ-porous since $F \subset G$. Finally, G is analytic (in fact $\boldsymbol{F_\sigma}$) since

$$G = \bigcup \{s_1 F + \cdots + s_k F;\ k \in \mathbb{N}, (s_1, \dots, s_k) \in \mathbb{Q}^k\}.$$

$\square$

Remark 5.21. M. Chlebík observed that Laczkovich's question can be answered using Konjagin's result on N-sets (Theorem 5.15). Indeed, the set

$$K = \{x \in \mathbb{R};\ \sum_{n=1}^{\infty} |\sin(n!x)|/n < \infty\}$$

is a non-σ-porous $\boldsymbol{F}_\sigma$ set forming a proper subgroup of $\mathbb{R}$.

6. The complexity of the σ-ideal of closed σ-porous sets

Let us recall the definition of σ-ideal of compact sets.

Definition 6.1. Let E be a compact metric space. A set $I \subset \mathcal{K}(E)$ is called *σ-ideal* if the following conditions are satisfied:

- if K, $L \in \mathcal{K}(E)$, $K \in I$, $L \subset K$, then $L \in I$,
- if $K, K_1, K_2, \dots \in \mathcal{K}(E)$, $K_n \in I$ for all $n \in \mathbb{N}$ and $K = \bigcup_{n=1}^{\infty} K_n$, then $K \in I$.

The theory of σ-ideals of compact sets was developed by Kechris, Louveau and Woodin in [KLW]. Obtained results were applied in the theory of trigonometric series (see [DSR], [KL]). In this section we investigate the descriptive properties of the following σ-ideal of compact sets

$$I_{\sigma\text{-}p}(E) = \{K \in \mathcal{K}(E);\ K \text{ is a } \sigma\text{-porous subset of } E\},$$

where E is a given compact metric space (with a fixed metric ρ).

We start with basic definitions.

Definition 6.2. Let E be a compact metric space and let $I \subset \mathcal{K}(E)$ be a σ-ideal. We say that

- I is *calibrated* if for every $F \in \mathcal{K}(E)$ and every sequence $(F_n)_{n=1}^{\infty}$ of sets in I with $\mathcal{K}(F \setminus \bigcup_{n=1}^{\infty} F_n) \subset I$ we have $F \in I$;
- I has the *covering property* if every analytic set $A \subset E$ with $\mathcal{K}(A) \subset I$ can be covered by countably many elements from I;
- I has a *Borel basis* if there exists a Borel set $B \subset I$ such that for every $K \in I$ there exist compact sets $K_n \in B$, $n \in \mathbb{N}$, with $K \subset \bigcup_{n=1}^{\infty} K_n$;
- I is *thin* if E contains no uncountable family of pairwise disjoint closed sets which are not in I;
- I is *locally non-Borel* if, for every $F \in \mathcal{K}(E) \setminus I$, the set $I \cap \mathcal{K}(F)$ is not Borel.

Definition 6.3. A subset P of a Polish space X is called $\mathbf{\Pi^1_1}$-*complete* if it is $\mathbf{\Pi^1_1}$ and for any Polish space Y and any $\mathbf{\Pi^1_1}$ subset Q of Y, there is a Borel mapping $f : Y \to X$ such that $Q = f^{-1}(P)$.

It is easy to see that no $\mathbf{\Pi^1_1}$-complete set is analytic. We will use the following theorems.

Theorem 6.4 (Kechris, Louveau, Woodin [KLW])**.** *Let E be a compact metric space. Then every $\mathbf{\Pi^1_1}$ σ-ideal $I \subset \mathcal{K}(E)$ is either $\mathbf{\Pi^1_1}$-complete or it is $\boldsymbol{G_\delta}$.*

This theorem says that $\mathbf{\Pi^1_1}$ σ-ideals can be separated into two groups: simple σ-ideals ($\boldsymbol{G_\delta}$) and complicated ones ($\mathbf{\Pi^1_1}$-complete).

Theorem 6.5 (Debs, Saint-Raymond [DSR])**.** *Let E be a compact metric space and $I \subset \mathcal{K}(E)$ be a $\mathbf{\Pi^1_1}$ σ-ideal such that I is locally non-Borel, calibrated and has a Borel basis. Then I has the covering property.*

This theorem can be used to show that a certain σ-ideal has no Borel basis. It is sufficient to show that the σ-ideal considered is $\mathbf{\Pi^1_1}$, locally non-Borel, calibrated and does not satisfy the covering property. We will use this method, too.

Theorem 6.6 ([Ze$_1$])**.** *Let E be a compact metric space and let $I \subset \mathcal{K}(E)$ be a calibrated thin $\mathbf{\Pi^1_1}$ σ-ideal. Then I is $\boldsymbol{G_\delta}$.*

Let E be a compact metric space. For every $n \in \mathbb{N}$ we define an auxiliary mapping $g_n : \mathcal{K}(E) \to \mathcal{K}(E)$ by

$$g_n(K) = K \setminus \bigcup\{B(x, 4r);\ \overline{B(x,r)} \cap K = \emptyset, r < 1/n\}.$$

The following observation is easy to see.

Observation 6.7. *Suppose that E is a compact metric space and $K \in \mathcal{K}(E)$. Then the set $K \setminus \bigcup_{n=1}^{\infty} g_n(K)$ is porous and $g_n(K) \subset g_m(K)$ whenever $n, m \in \mathbb{N}$, $n \leq m$.*

Lemma 6.8. *Let E be a compact metric space. The mapping $g_n : \mathcal{K}(E) \to \mathcal{K}(E)$ is Borel for every $n \in \mathbb{N}$.*

Proof: Fix $n \in \mathbb{N}$. According to Lemma 4.1(iv), $\mathcal{K}(E)$ is separable and so it is sufficient to show that the sets

$$X = \{K \in \mathcal{K}(E);\ g_n(K) \subset G\}, \quad Y = \{K \in \mathcal{K}(E);\ g_n(K) \cap G \neq \emptyset\}$$

are Borel, whenever $G \subset E$ is open. Fix an open set $G \subset E$ and suppose that we have $L \in \mathcal{K}(E)$ with $g_n(L) \subset G$. Since L is compact there exists a finite system $\mathcal{D}$ of open balls from the space E such that

- $\overline{B} \cap L = \emptyset$ whenever $B \in \mathcal{D}$,
- each ball of $\mathcal{D}$ has its radius less than $1/n$,
- $L \subset G \cup \bigcup\{B(x, 4r);\ B(x,r) \in \mathcal{D}\}$.

This gives that the open set

$$\{K \in \mathcal{K}(E);\ K \subset G \cup \bigcup\{B(x,4r);\ B(x,r) \in \mathcal{D}\} \text{ and } \overline{B} \cap K = \emptyset \text{ whenever } B \in \mathcal{D}\}$$

is a subset of X and contains L. The set X is open and therefore Borel.

To prove borelness of Y we write $G = \bigcup_{j=1}^{\infty} F_j$, where F_j's are closed. Then

$$\begin{aligned}\mathcal{K}(E) \setminus Y &= \{K \in \mathcal{K}(E);\ g_n(K) \cap G = \emptyset\} \\ &= \bigcap_{j=1}^{\infty} \{K \in \mathcal{K}(E);\ g_n(K) \cap F_j = \emptyset\} \\ &= \bigcap_{j=1}^{\infty} \{K \in \mathcal{K}(E);\ g_n(K) \subset E \setminus F_j\}.\end{aligned}$$

This implies that $\mathcal{K}(E) \setminus Y$ is $\boldsymbol{G_\delta}$ and therefore Y is Borel. □

Lemma 6.9. *Let E be a compact metric space, $K \in \mathcal{K}(E)$ and $\ker(K) = K \neq \emptyset$. Let $\mathcal{B}$ be an open basis of E. If B_1 is an open set with $K \cap B_1 \neq \emptyset$, then there exist $l_0 \in \mathbb{N}$ and $B_2 \in \mathcal{B}$ such that for every $l \in \mathbb{N}$, $l \geq l_0$, we have $g_l(\overline{K \cap B_1}) \supset \overline{g_l(K) \cap B_2} \neq \emptyset$.*

Proof: Let B_1 be an open set with $K \cap B_1 \neq \emptyset$. Using Observation 6.7 we find $x \in E$ and $l_0 \in \mathbb{N}$ such that $x \in g_{l_0}(\overline{K \cap B_1})$ and $B(x, 6/l_0) \subset B_1$. Find $B_2 \in \mathcal{B}$ with $x \in B_2$ and $\operatorname{diam} B_2 < 1/l_0$. Fix $l \in \mathbb{N}$, $l \geq l_0$, and take $y \in g_l(K) \cap B_2$. Suppose that $y \notin g_l(\overline{K \cap B_1})$. We have $y \in K \cap B_2 \subset \overline{K \cap B_1}$ and therefore there exists an open ball $B(z,s)$ such that $s < 1/l$, $y \in B(z,4s)$ and $\overline{B(z,s)} \cap \overline{K \cap B_1} = \emptyset$. Since $l \geq l_0$ we have $\overline{B(z,s)} \subset B(x, 6/l_0) \subset B_1$. This implies $\overline{B(z,s)} \cap K = \emptyset$ and therefore $y \notin g_l(K)$, a contradiction. Thus we have $g_l(K) \cap B_2 \subset g_l(\overline{K \cap B_1})$. Then we have $\overline{g_l(K) \cap B_2} \subset g_l(\overline{K \cap B_1})$ since $g_l(\overline{K \cap B_1})$ is closed. The set $g_l(K) \cap B_2$ is nonempty since it contains x. □

Lemma 6.10. *Let E be a compact metric space. Then $I_{\sigma\text{-}p}(E)$ is $\mathbf{\Pi^1_1}$ in $\mathcal{K}(E)$.*

Proof: The set $\mathcal{K}(E) \setminus \{\emptyset\}$ is closed in $\mathcal{K}(E)$. Put $\mathfrak{W} = (\mathcal{K}(E) \setminus \{\emptyset\})^{\mathbb{N}}$. The space $\mathfrak{W}$ with the usual product topology is a compact metrizable space. Fix a countable open basis $\mathcal{B}$ of the space E. We define a set $\mathfrak{S} \subset \mathfrak{W}$ by

$$\begin{aligned}\mathcal{F} \in \mathfrak{S} \overset{\text{def}}{\Longleftrightarrow} \mathcal{F} \in \mathfrak{W} \text{ and } \forall B_1 \in \mathcal{B}\ \forall m \in \mathbb{N}:\ (\mathcal{F}(m) \cap B_1 \neq \emptyset \Rightarrow \\ \exists B_2 \in \mathcal{B}\ \exists l \in \mathbb{N}\ \exists k \in \mathbb{N}:\ g_l(\overline{\mathcal{F}(m) \cap B_1}) \supset \overline{\mathcal{F}(k) \cap B_2} \neq \emptyset).\end{aligned}$$

Using (vi) and (vii) of Lemma 4.1 and Lemma 6.8 it is not difficult to verify that $\mathfrak{S}$ is a Borel subset of $\mathfrak{W}$. We need the following claim.

Claim. The set $K \in \mathcal{K}(E)$ is in $\mathcal{K}(E) \setminus I_{\sigma\text{-}p}(E)$ if and only if there exists $\mathcal{F} \in \mathfrak{S}$ such that for some $n_0 \in \mathbb{N}$ we have $\mathcal{F}(n_0) \subset K$.

Proof of Claim: If $K \in \mathcal{K}(E) \setminus I_{\sigma\text{-}p}(E)$, then we define $K_\emptyset = \ker(K)$ and $K_{s^\wedge n} = \ker(g_n(K_s))$, $s \in \operatorname{Seq}\mathbb{N}$, $n \in \mathbb{N}$. If B is an open set with $K_s \cap B \neq \emptyset$, then there exists $n \in \mathbb{N}$ such that $K_{s^\wedge n} \cap B \neq \emptyset$ (Lemma 3.4(iv), Observation 6.7). Thus the set $\mathcal{A} := \{s \in \operatorname{Seq}\mathbb{N};\ K_s \neq \emptyset\}$ is an infinite countable subset of $\operatorname{Seq}\mathbb{N}$ and there exists a bijection φ of $\mathbb{N}$ onto $\mathcal{A}$. We define $\mathcal{F} \in \mathfrak{M}$ by $\mathcal{F}(n) = K_{\varphi(n)}$, $n \in \mathbb{N}$.

We show that $\mathcal{F} \in \mathfrak{S}$. Let $B_1 \in \mathcal{B}$, $m \in \mathbb{N}$ and $\mathcal{F}(m) \cap B_1 \neq \emptyset$. Then according to Lemma 6.9 there exist $B_2 \in \mathcal{B}$ and $l_0 \in \mathbb{N}$ such that $g_l(\overline{\mathcal{F}(m) \cap B_1}) \supset \overline{g_l(\mathcal{F}(m)) \cap B_2} \neq \emptyset$ for every $l \geq l_0$. Using Observation 6.7 we find $l_1 \geq l_0$ such that $g_{l_1}(\mathcal{F}(m)) \cap B_2$ is not σ-porous. Then we find $k \in \mathbb{N}$ with $\mathcal{F}(k) = \ker(g_{l_1}(\mathcal{F}(m)))$ and we have $g_{l_1}(\overline{\mathcal{F}(m) \cap B_1}) \supset \overline{\mathcal{F}(k) \cap B_2} \neq \emptyset$. Thus $\mathcal{F} \in \mathfrak{S}$. For some $n_0 \in \mathbb{N}$ we have $\mathcal{F}(n_0) = K_\emptyset \subset K$.

Let $\mathcal{F} \in \mathfrak{S}$, $K \in \mathcal{K}(E)$ and $\mathcal{F}(n_0) \subset K$ for some $n_0 \in \mathbb{N}$. Then the set

$$\mathfrak{H} = \{\overline{\mathcal{F}(m) \cap G};\ m \in \mathbb{N},\ G \in \mathcal{B},\ \mathcal{F}(m) \cap G \neq \emptyset\}$$

forms a 1/3-Foran system. Indeed, take $m \in \mathbb{N}$, $G \in \mathcal{B}$ with $\mathcal{F}(m) \cap G \neq \emptyset$ and an open set B such that $\overline{\mathcal{F}(m) \cap G} \cap B \neq \emptyset$. Then there exists $B_1 \in \mathcal{B}$ with $\overline{B_1} \subset G \cap B$ and $B_1 \cap \mathcal{F}(m) \neq \emptyset$. According to the definition of $\mathfrak{S}$ there exist $B_2 \in \mathcal{B}$ and $l, k \in \mathbb{N}$ such that $g_l(\overline{\mathcal{F}(m) \cap B_1}) \supset \overline{\mathcal{F}(k) \cap B_2} \neq \emptyset$. This implies that each point $x \in \overline{\mathcal{F}(k) \cap B_2}$ is a point of non-1/3-porosity of $\overline{\mathcal{F}(m) \cap B_1}$ (in fact $p(x, \overline{\mathcal{F}(m) \cap B_1}) \leq 1/4$). Hence x is also a point of non-1/3-porosity of the set $\overline{\mathcal{F}(m) \cap G}$. We have also $\overline{\mathcal{F}(k) \cap B_2} \subset \overline{\mathcal{F}(m) \cap B_1} \subset \overline{F(m) \cap G} \cap B$. Thus $\mathfrak{H}$ is a 1/3-Foran system and each element of $\mathfrak{H}$ is non-σ-porous by Lemma 2.18. Therefore the set $\mathcal{F}(n_0)$ is non-σ-porous. This implies that K is not σ-porous and Claim is proved. □

Denote

$$A = \{(K, \mathcal{F}) \in \mathcal{K}(E) \times \mathfrak{M}; \mathcal{F} \in \mathfrak{S}, \mathcal{F}(n_0) \subset K \text{ for some } n_0 \in \mathbb{N}\}.$$

Since $\mathfrak{S}$ is Borel it is not difficult to verify that A is Borel in $\mathcal{K}(E) \times \mathfrak{M}$. Claim implies that $\mathcal{K}(E) \setminus I_{\sigma\text{-}p}(E) = \pi(A)$, where π is the projection of $\mathcal{K}(E) \times \mathfrak{M}$ onto $\mathcal{K}(E)$. This gives that $\mathcal{K}(E) \setminus I_{\sigma\text{-}p}(E)$ is analytic in $\mathcal{K}(E)$ and therefore $I_{\sigma\text{-}p}(E)$ is $\mathbf{\Pi}^1_1$ in $\mathcal{K}(E)$. □

The following lemma follows from Theorem 4.9.

Lemma 6.11. *Let E be a nonempty compact metric space with no isolated point. Then $I_{\sigma\text{-}p}(E)$ is locally non-Borel.*

The following lemma seems to be well-known. The proof is included since we do not know any explicit reference.

Lemma 6.12. *Let P be a nonempty separable complete metric space with no isolated point. Then there exists a $\boldsymbol{G}_\delta$ porous set H which cannot be covered by countably many closed σ-porous sets.*

PROOF: According to [Za$_3$, Theorem 2] (or Theorem 5.2) there exists a nowhere dense closed set K, such that $\ker(K) = K \neq \emptyset$. For every $n \in \mathbb{N}$ we define

$$H_n = \bigcup\{B(x,2r);\ B(x,r) \cap K = \emptyset \text{ and } r < 1/n\}.$$

Each point of $H := K \cap \bigcap_{n=1}^{\infty} H_n$ is a point of porosity of K. Therefore H is porous. Since K is nowhere dense, the set $H_n \cap K$ is dense in K for every $n \in \mathbb{N}$ and so H is a dense $\boldsymbol{G}_\delta$ subset of K. Assume that H can be covered by countably many closed σ-porous sets. According to the Baire Category Theorem at least one σ-porous set contains a portion of K. This is a contradiction, since each portion of K is not σ-porous. □

The following theorem summarize the main descriptive properties of the σ-ideal $I_{\sigma\text{-}p}(E)$.

Theorem 6.13. *Let E be a nonempty compact metric space with no isolated point. Then $I_{\sigma\text{-}p}(E)$ is a $\boldsymbol{\Pi}^1_1$-complete non-thin σ-ideal with no Borel basis.*

PROOF: Theorem 6.4, Lemma 6.10 and Lemma 6.11 give that $I_{\sigma\text{-}p}(E)$ is $\boldsymbol{\Pi}^1_1$-complete.

Theorem 3.1 easily implies that $I_{\sigma\text{-}p}(E)$ is calibrated. Lemma 6.12 shows that $I_{\sigma\text{-}p}(E)$ does not satisfy the covering property. Now Theorem 6.5 and Lemma 6.11 imply that $I_{\sigma\text{-}p}(E)$ has no Borel basis.

Non-thinness of $I_{\sigma\text{-}p}(E)$ follows from Theorem 6.6. □

Remark 6.14. (i) $\boldsymbol{\Pi}^1_1$-completeness of $I_{\sigma\text{-}p}(E)$ was firstly proved by Debs and Preiss ([De]), but the proof was not published.

(ii) If a σ-ideal $I \subset \mathcal{K}(E)$ is $\boldsymbol{\Pi}^1_1$ and is not thin, then there exists a family of cardinality of the continuum of pairwise disjoint elements from $\mathcal{K}(E) \setminus I$ (see [KLW]). This is also the case of σ-ideal $I_{\sigma\text{-}p}(E)$. This result was firstly proved by Reclaw ([Re]).

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Department od Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail: zeleny@karlin.mff.cuni.cz

Mathematical Institute of Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic

E-mail: pelant@matsrv.math.cas.cz

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