Note on analytic Moufang loops

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Abstract. It is explicitly shown how the Lie algebras can be associated with the analytic Moufang loops. The resulting Lie algebra commutation relations are well known from the theory of alternative algebras.

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1. Moufang loops

It is well known how Lie algebras are connected with Lie groups. In the present paper, it is explicitly shown how Lie algebras can be associated with analytic Moufang loops.

A *Moufang loop* [1], [2] is a quasigroup G with the unit element $e \in G$ and the Moufang identity

$$(ag)(ha) = a(gh)a, \qquad a, g, h \in G.$$

Here the multiplication is denoted by juxtaposition. In general, the multiplication need not be associative: $gh \cdot a \neq g \cdot ha$. The inverse element g^{-1} of g is defined by

$$gg^{-1} = g^{-1}g = e.$$

The left (L) and right (R) *translations* are defined by

$$gh = L_g h = R_h g, \qquad g, h \in G.$$

Both translations are invertible mappings and

$$L_g^{-1} = L_{g^{-1}}, \qquad R_g^{-1} = R_{g^{-1}}.$$

2. Analytic Moufang loops and infinitesimal Moufang translations

Following the concept of the Lie group, the notion of an analytic Moufang loop can be easily formulated.

A Moufang loop G is said [3] to be *analytic* if G is also a real analytic manifold and the main operations — multiplication and inversion map $g \mapsto g^{-1}$ — are analytic mappings.

Let $T_e(G)$ denote the tangent space of G at the unit e. For $x \in T_e(G)$, *infinitesimal Moufang translations* are defined as vector fields on G as follows:

$$\begin{aligned} L_x &:= L_x(g) := (dL_g)_e x \quad &\in T_g(G), \\ R_x &:= R_x(g) := (dR_g)_e x \quad &\in T_g(G). \end{aligned}$$

Let the local coordinates of g from the vicinity of $e \in G$ be denoted by g^i $(i = 1, \ldots, r := \dim G)$. Define the auxiliary functions

$$L^i_j(g) := \frac{\partial (gh)^i}{\partial h^j}\Big|_{h=e}, \quad R^i_j(g) := \frac{\partial (hg)^i}{\partial h^j}\Big|_{h=e}.$$

The matrices (L^i_j) and (R^i_j) are invertible. Let $b_j := \frac{\partial}{\partial g^j}|_e$ $(j = 1, \ldots, r)$ be a base in $T_e(G)$. Then both

$$\begin{aligned} L_j &:= (dL_g)_e b_j = L^i_j(g) \frac{\partial}{\partial g^i} \quad &\in T_g(G), \\ R_j &:= (dR_g)_e b_j = R^i_j(g) \frac{\partial}{\partial g^i} \quad &\in T_g(G) \end{aligned}$$

form bases at $T_g(G)$. Thus one has two preferred base fields on G. When writing $T_e(G) \ni x = x^j b_j$, one can easily see that

$$\begin{aligned} L_x &:= L_x(g) = x^j L^i_j(g) \frac{\partial}{\partial g^i} \quad &\in T_g(G), \\ R_x &:= R_x(g) = x^j R^i_j(g) \frac{\partial}{\partial g^i} \quad &\in T_g(G). \end{aligned}$$

3. Structure constants and tangent Mal'tsev algebra

As in the case of Lie groups, structure constants c^i_{jk} of an analytic Moufang loop are defined by

$$c^i_{jk} := \frac{\partial^2 (ghg^{-1}h^{-1})^i}{\partial g^j \partial h^k}\Big|_{g=h=e} = -c^i_{kj}, \qquad i, j, k = 1, \ldots, r.$$

For any $x, y \in T_e(G)$, their (tangent) product $[x, y] \in T_e(G)$ is defined in component form by

$$[x, y]^i := c^i_{jk} x^j y^k = -[y, x]^i, \qquad i = 1, \ldots, r.$$

The tangent space $T_e(G)$ being equipped with such an anti-commutative multiplication is called the *tangent algebra* of the analytic Moufang loop G.

The tangent algebra of G need not be a Lie algebra. There may exist a triple $x, y, z \in T_e(G)$ that does not satisfy the Jacobi identity:

$$J(x, y, z) := [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \neq 0.$$

Instead, for any $x, y, z \in T_e(G)$ one has a more general *Mal'tsev identity* [3]

$$[J(x, y, z), x] = J(x, y, [x, z]).$$

Anti-commutative algebras with this identity are called *Mal'tsev algebras*.

4. Generalized Maurer-Cartan equations

Denote as above $L_x := L_x(g)$ and $R_x := R_x(g)$ for all $x \in T_e(G)$.

It is well known that the infinitesimal translations of a *Lie group* obey the *Maurer-Cartan equations*

$$[L_x, L_y] - L_{[x,y]} = [L_x, R_y] = [R_x, R_y] + R_{[x,y]} = 0.$$

It turns out that for a non-associative analytic Moufang loop these equations are violated minimally. The algebra of infinitesimal Moufang translations reads as *generalized Maurer-Cartan equations* [4]:

$$[L_x, L_y] - L_{[x,y]} = -2[L_x, R_y] = [R_x, R_y] + R_{[x,y]}.$$

We outline a way of closing of this algebra (generalized Maurer-Cartan equations), which in fact means construction of a *finite* dimensional Lie algebra generated by infinitesimal Moufang translations.

Start by rewriting the generalized Maurer-Cartan equations as follows:

$$[L_x, L_y] = 2Y(x; y) + \frac{1}{3}L_{[x,y]} + \frac{2}{3}R_{[x,y]}, \tag{1}$$

$$[L_x, R_y] = -Y(x; y) + \frac{1}{3}L_{[x,y]} - \frac{1}{3}R_{[x,y]}, \tag{2}$$

$$[R_x, R_y] = 2Y(x; y) - \frac{2}{3}L_{[x,y]} - \frac{1}{3}R_{[x,y]}. \tag{3}$$

Here (1) or (2) or (3) can be assumed as a definition (recapitulation) of the *Yamagutian* Y. It can be shown [4] that

$$Y(x;y)+Y(y;x)=0, \tag{4}$$

$$Y([x,y];z)+Y([y,z];x)+Y([z,x];y)=0. \tag{5}$$

The constraints (4) trivially descend from the anti-commutativity of the commutator bracketing, but the proof of (5) needs certain effort. Further, it turns out that the following *reductivity* conditions hold [4]:

$$6[Y(x;y),L_z]=L_{[x,y,z]}, \qquad 6[Y(x;y),R_z]=R_{[x,y,z]} \tag{6}$$

where the trilinear *Yamaguti brackets* $[\cdot,\cdot,\cdot]$ are defined ([6], [7]) in $T_e(G)$ by

$$[x,y,z]\colon=[x,[y,z]]-[y,[x,z]]+[[x,y],z].$$

Finally, the Yamagutian obeys the Lie algebra

$$6[Y(x;y),Y(z;w)]=Y([x,y,z];w)+Y(z;[x,y,w]). \tag{7}$$

The full proof of the Lie algebra commutation relations (1)–(7) has been presented in [4]. The idea of proof is as follows: one must find the *generalized Lie equations* of an analytic Moufang loop and consider their *integrability conditions*. The dimension of the Lie algebra (1)–(7) does not exceed $2r+r(r-1)/2$. The Jacobi identities are guaranteed by the defining identities of the Lie [5] and general Lie [6], [7] *triple systems* associated with the tangent Mal'tsev algebra $T_e(G)$ of G.

We call the pair (L,R) of the maps $x\mapsto L_x, x\mapsto R_x$ a *birepresentation* of the tangent Mal'tsev algebra $T_e(G)$ of G if it satisfies the Lie algebra commutation relations (1)–(7). The Lie subalgebra (7) is a *generalized representation* [6] of the tangent Mal'tsev algebra $T_e(G)$ of G.

The commutation relations of form (1)–(7) are also well known from the theory of alternative algebras [8].

5. Triality

Define M_x by

$$L_x+R_x+M_x=0, \qquad x\in T_e(G).$$

If (L,R) is a birepresentation of $T_e(G)$, then the following pairs are birepresentations as well [4]:

$$(-R,-L),\ (R,M),\ (-M,-R),\ (M,L),\ (-L,-M).$$

In particular, the Yamagutian can be expressed in triality invariant form

$$6Y(x;y)=[L_x,L_y]+[R_x,R_y]+[M_x,M_y].$$

6. Loos brackets and triple closure

Define the Loos brackets $\{\cdot,\cdot,\cdot\}$ by [5]

$$3\{x,y,x\}:=[x,[y,z]]-[y,[x,z]]+2[[x,y],z].$$

Then one has the Lie algebra commutation relations

$$\begin{aligned}&[L_x,L_y]:=L(x;y)\\&[L(x;y),L_z]=L_{\{x,y,z\}}\\&[L(x;y),L(z;w)]=L(\{x,y,z\};w)+L(z;\{x,y,w\}).\end{aligned}$$

By triality, the analogous commutation relations hold for R-operators and M-operators. The Jacobi identities are guaranteed by the defining identities of the *Lie triple systems* [5] associated with the tangent Mal'tsev algebra $T_e(G)$ of G.

7. Weak representations

Define the *triality conjugated* operators

$$M_x^\dagger:=L_x-R_x,\quad L_x^\dagger:=R_x-M_x,\quad R_x^\dagger:=M_x-L_x.$$

Evidently,

$$R_x^\dagger+L_x^\dagger+M_x^\dagger=0.$$

Then we have the following *weak representation* commutation relations of the tangent Mal'tsev algebra $T_e(G)$ of G [7]:

$$\begin{aligned}&[L_x^\dagger,L_y^\dagger]=-L_{[x,y]}^\dagger+6Y(x;y)\\&6[Y(x;y),L_z^\dagger]=L_{[x,y,z]}^\dagger.\end{aligned}$$

By triality, the analogous relations hold for $R^\dagger$-operators and $M^\dagger$-operators.

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