On proximities generated by countable families of entourages

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Abstract. It is shown that any proximity that is generated by a countable family of entourages is sequential. Metrization theorems for proximities are derived.

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1. Introduction

The notion of a proximity has been successfully used in the framework of metric spaces. Just recall that all metric spaces ([2], [8], [9], [10]) or even their arbitrary Cartesian uniform products ([5]) are proximally fine (i.e. proximally continuous maps are uniformly continuous). Surprisingly, it is still an open problem whether each proximity generated by a countable family of entourages of the diagonal must be metrizable ([1], repeated in [7]). Recall that it has been proven particularly in [6] that if a topology τ is induced by a proximity, generated by a countable family of entourages of the diagonal, then τ is metrizable. It underlines that the problem mentioned above is not topological but it is related to the proximity structure. The first result, significant for examination of this problem, was announced in [4]:

Proposition 1.1. *If a proximity δ on a set X is generated by a decreasing sequence of entourages in $X \times X$, then δ is metrizable.*

So the problem is to remove the monotonicity requirement in the above proposition. A very useful result on metrizability of proximities comes from [1]:

Theorem 1.2. *If an admissible proximity δ is generated by a countable family Ω of entourages of the diagonal such that the family $\{U \circ U^{-1} \circ U \circ U^{-1} : U \in \Omega\}$ also generates δ, then δ is metrizable.*

All needed notions will be defined below. We recall here just the fact that all sequential proximities are admissible. The main purpose of the present note is to establish the following

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Theorem 4.1. *If a proximity δ is generated by a countable family Ω of entourages of the diagonal in $X \times X$, then δ is sequential.*

So our main result transforms Theorem 1.2 to the following

Corollary 4.2. *If a proximity δ on X is generated by a countable family Ω of entourages of the diagonal in $X \times X$, such that the family $\{U \circ U^{-1} \circ U \circ U^{-1} : U \in \Omega\}$ also generates δ, then δ is metrizable.*

However Theorem 4.1 has also other corollaries, including Proposition 1.1. Let us mention here at least one from [7]: *If δ is a proximity generated by a countable family of entourages of the diagonal then there is a maximal uniformity among the uniformities generating δ*; note that this maximal uniformity is proximally fine and, as mentioned above, each metric uniformity is proximally fine.

2. Preliminaries

For a set T, the set of all infinite countable subsets of T will be denoted by $[T]^\omega$, the set of all couples of T will be denoted by $[T]^2$ and the symbol 2^T denotes the set of all subsets of T. As some authors use slightly different definitions and notation, we prefer to make a list of conventions we are going to use.

A *proximity* δ on a set $X \neq \emptyset$ means the Efremovich proximity [2], i.e. a mapping $\delta : 2^X \times 2^X \to \{0,1\}$ satisfying the following conditions (*axioms of proximity*):

(Prx.1): $\delta[A,B] = \delta[B,A]$ for every $A, B \in 2^X$;
(Prx.2): $\delta[A, B \cup C] = \delta[A,B].\delta[A,C]$ for every $A, B, C \in 2^X$;
(Prx.3): $\delta[\{x\},\{y\}] = 0$ if and only if $x = y$ for every $x, y \in X$;
(Prx.4): $\delta[\emptyset, X] = 1$;
(Prx.5): if $\delta[A,B] = 1$, where $A, B \in 2^X$, then there is $C \in 2^X$ such that $A \subset C$, $B \subset X \backslash C$, $\delta[A, X\backslash C] = 1$, $\delta[B, C] = 1$.

Under an entourage of the diagonal in $X \times X$ we mean, as usually, a subset of $X \times X$, containing the diagonal $\triangle = \{(x,x) : x \in X\}$ of $X \times X$.

We say that a proximity δ on X is *generated by a family Ω of entourages of the diagonal* if for each $A, B \in 2^X$, $\delta[A, B] = 0$ iff $(A \times B) \cap U \neq \emptyset$ for every U in Ω. Observe that if Ω generates δ then $\Omega' = \{V^{-1} \circ U : V, U \in \Omega\}$ generates δ as well; the axiom Prx.5 is very crucial for this fact. Another, but related consequence of Prx.5 is: if $\delta[A,B] = 1$ and δ is generated by Ω, then there is $U \in \Omega$ such that $\delta\big[U[A], B\big] = 1$.

A proximity δ on X is said to be *generated by a family $\mathcal{P}$ of pseudometrics* defined on X if for each $A, B \in 2^X$, $\delta(A,B) = 0$ iff $p(A,B) = \inf\{p(x,y) : x \in A,\ y \in B\} = 0$ for every $p \in \mathcal{P}$; δ is *metrizable* provided δ is generated by an one-element family of pseudometrics.

We are going to use this notation: given a sequence $(a_n)_{n\in\mathbb{N}}$, denote a set $\{a_n : n \in F\}$ by a_F for $F \subset \mathbb{N}$.

Finally define $\overline{T} = \{x \in X : \delta(\{x\}, T) = 0\}$ for $T \subset X$.

Let us recall two more definitions from [1]:

1) A proximity δ is said to be *admissible* if $\delta[A, B] = 0$ $(A, B \in 2^X)$ implies there are: a linearly ordered set λ and mappings $f : \lambda \to \overline{A}$, $g : \lambda \to \overline{B}$ such that for every cofinal subset $l' \subset \lambda$, $\delta\big[f(l'), g(l')\big] = 0$.

2) A proximity δ on X is said to be *sequential* if $\delta[A,\, B] = 0$, where $A, B \in 2^X$, implies there are sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ with $x_n \in A$, $y_n \in B$ for every $n \in \mathbb{N}$ such that $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$.

The relation $\sim$ is defined as follows:
for sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X we write that $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if $\delta[x_F,\, y_F] = 0$ for every $F \in [\mathbb{N}]^\omega$, i.e. the sets x_F and y_F are δ-proximal for every $F \in [\mathbb{N}]^\omega$.

We say that a sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n \in X$ for every $n \in \mathbb{N}$, is δ-*fundamental* if $\delta[a_F,\, a_G] = 0$ for every $F, G \in [\mathbb{N}]^\omega$.

We say that a sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n \in X$ for $n \in \mathbb{N}$, is δ-*discrete* if $\delta[a_F,\, a_G] = 1$ for every $F, G \subset \mathbb{N}$ such that $F \cap G = \emptyset$. In this case, we shall sometimes say that the set $a_\mathbb{N} = \{a_n : n \in \mathbb{N}\}$ is δ-discrete. Observe that if a countable family Ω of entourages of the diagonal generates δ and $a_\mathbb{N} = \{a_n : n \in \mathbb{N}\}$ is δ-discrete then there are cofinite set $F \subseteq \mathbb{N}$ (i.e. $\mathbb{N} \setminus F$ is finite) and $U \in \Omega$ such that $(a_i, a_j) \notin U$ for any two distinct elements of F.

Next, let us adopt the following notation: for every $A, B \in 2^X$, we put $A\delta_\omega B$ if either $\overline{A} \cap \overline{B} \neq \emptyset$ or $\delta[A', B] = 0$ for every $A' \in [A]^\omega$.

For U, $V \subset X \times X$, we define the composition of U and V, denoted by $U \circ V$, using the formula:

$$U \circ V = \{(x,\, y) \in X \times X : \text{ there is } z \in X \text{ such that } (x,\, z) \in V \text{ and } (z,\, y) \in U\}.$$

For $W \subset X \times X$ and $A \subset X$, define

$$W[A] = \{y \in X : \text{ there is } x \in A \text{ such that } (x,\, y) \in W\}.$$

These definitions enable us to use the equality:

$$U \circ V[A] = U[V[A]] \quad \text{for} \quad A \subset X.$$

3. Auxiliary lemmas

We shall often use the following assertions:

Lemma 3.1 ([7]). *Let δ be a proximity on X such that δ is generated by a countable family of entourages of the diagonal and let $A, B \in 2^X$. If $\delta[A, B] = 0$ then there is a (finite or) countable subset A' of A such that $A'\delta_\omega B$.*

Notation 3.2. *For $W \subset \mathbb{N} \times \mathbb{N}$, we denote by $W^>$ the set of all $(n, k) \in W$ such that $n > k$ (analogously $W^< = \{(n, k) \in W : n < k\}$).*

The classical Ramsey theorem gives the following

Lemma 3.3. *If $N' \in [\mathbb{N}]^\omega$ and if $T \subset N' \times N'$ then there is an infinite $F \subset N'$ such that $(F \times F)^> \subset T$ or there is an infinite $G \subset N'$ such that $(G \times G)^> \cap T = \emptyset$.*

Note that one can replace $(F \times F)^>$ by $(F \times F)^<$ and $(G \times G)^>$ by $(G \times G)^<$ in above.

The following simple lemma will be a substantial tool for our reasoning. It may be regarded as an analogue of the fact for metric spaces: each sequence of points in a metric space contains either a Cauchy subsequence or a uniformly discrete subsequence. Of course, we have to prove it here without using the technique of metrics.

Lemma 3.4. *Let δ be a proximity on X such that δ is generated by a countable family Ω of entourages of the diagonal in $X \times X$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in X. Then there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ which is either δ-fundamental or δ-discrete.*

Proof: Let us suppose that there is no δ-discrete subsequence in $(a_n)_{n\in\mathbb{N}}$, i.e. there do not exist $U \in \Omega$ and $G \in [\mathbb{N}]^\omega$ such that $(a_i, a_j) \notin U$ for every $i, j \in G$ with $i \neq j$.

Put $\Omega = \{U_n : n \in \mathbb{N}\}$. It follows immediately from Lemma 3.3 that there is a sequence $\mathbb{N} = F_0 \supset F_1 \supset \dots \supset F_n \supset \dots$ of infinite subsets of $\mathbb{N} = \{1, 2, \dots, n, \dots\}$ such that $\{(a_i, a_j) : (i, j) \in (F_n \times F_n)^>\} \subset U_n$ or $\{(a_i, a_j) : (i, j) \in (F_n \times F_n)^<\} \subset U_n$ for every $n \in \mathbb{N}$. Choose $n_k \in F_k$ so that $n_{k+1} > n_k$ for every $k \in \mathbb{N}$. Then $(a_{n_k})_{k\in\mathbb{N}}$ is δ-fundamental and Lemma 3.4 is proved. $\square$

The following facts are quite obvious:

Lemma 3.5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and let U_1, U_2 be entourages of the diagonal in $X \times X$ such that $(x_i, x_j) \notin U_2^{-1} \circ U_1$ for every $i, j \in \mathbb{N}$ with $i \neq j$. Then $U_1[\{x_i\}] \cap U_2[\{x_j\}] = \emptyset$ for every $i, j \in \mathbb{N}$ with $i \neq j$.*

Corollary 3.6. *Assumptions coincide with those in Lemma 3.5.*

1. *Let $G \subset \mathbb{N}$. Recall $\{x_n : n \in G\} = x_G$. We get that $U_1[x_G] \cap U_2[x_G] = \bigcup_{n\in G}(U_1[\{x_n\}] \cap U_2[\{x_n\}])$.*
2. *If $F, G \subset \mathbb{N}$ are such that $F \cap G = \emptyset$ then $(U_1[x_F] \cap U_2[x_F]) \cap (U_1[x_G] \cap U_2[x_G]) = \emptyset$.*

4. The main result and its proof

Theorem 4.1. *If a proximity δ is generated by a countable family Ω of entourages of the diagonal in $X \times X$, then δ is sequential.*

Proof: Let $\Omega = \{U_n : n = 1, 2, 3, \dots\}$ be a countable family of entourages which generates δ, and recall that the countable family of entourages $\Omega_2 = \{U_{n_2}^{-1} \circ U_{n_1} : n_1, n_2 \in \mathbb{N}$ and $U_{n_1}, U_{n_2} \in \Omega\}$ generates δ as well. Let us consider members of Ω_2 numbered and ordered in a sequence. In other words, $\Omega_2 = \{V_k : k \in \mathbb{N}\}$, where for each $V_k \in \Omega_2$, we have that $V_k = U_{k_2}^{-1} \circ U_{k_1}$ for some $U_{k_1}, U_{k_2} \in \Omega$.

We have to prove that δ is sequential. Let $\delta[A, B] = 0$. We may suppose that $\overline{A} \cap \overline{B} = \emptyset$. For, if $x_0 \in \overline{A} \cap \overline{B}$, then taking $x_n \in A \cap (\bigcap_{i=1}^n U_i[\{x_0\}])$, $y_n \in B \cap (\bigcap_{i=1}^n U_i[\{x_0\}])$ for every $n = 1, 2, \dots$, we get sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that $x_n \in A$, $y_n \in B$ for every $n = 1, 2, \dots$ and $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$.

By Lemma 3.1, we can choose a countable set $A' \in [A]^\omega$ such that $A' \delta_\omega B$. Let $A' = \{a_n : n \in \mathbb{N}\}$ where $a_i \neq a_j$ for every $i, j \in \mathbb{N}$ with $i \neq j$.

As Ω_2 generates δ, Lemma 3.4 implies, after choosing a subsequence and changing the index set if necessary, that there are two options: either $a_\mathbb{N}$ is δ-fundamental or there is $V_k \in \Omega_2$ such that $(a_i, a_j) \notin V_k$ for any two distinct $i, j \in \mathbb{N}$. Recall that in both cases, $a_\mathbb{N}\, \delta_\omega B$. In the latter case, put $V_k = U_2^{-1} \circ U_1$ where $U_1, U_2 \in \Omega$.

Suppose firstly that the sequence $(a_n)_{n\in\mathbb{N}}$ is δ-fundamental. By Lemma 3.1, we choose $B' \in [B]^\omega$ such that $B' \delta_\omega a_\mathbb{N}$ and put $B' = \{b_n : n \in N\}$ where $b_i \neq b_j$ for $i, j \in \mathbb{N}$ with $i \neq j$. Then it is easily seen that $(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}}$; recall that $\overline{A} \cap \overline{B} = \emptyset$.

Suppose now that the sequence $(a_n)_{n\in\mathbb{N}}$ is such that $(a_i,\, a_j) \notin U_2^{-1} \circ U_1$ for every $i, j \in \mathbb{N}$ with $i \neq j$. By Lemma 3.1, we choose $B' \in [B]^\omega$ such that $B' \delta_\omega a_\mathbb{N}$. Define $L = \{a_n : U_1[\{a_n\}] \cap U_2[\{a_n\}] \cap B' \neq \emptyset\}$. Put $V_x = U_1[\{x\}] \cap U_2[\{x\}]$ for $x \in X$ and $V_E = U_1[E] \cap U_2[E]$ for $E \subset X$. So $L = \{a_n : V_{a_n} \cap B' \neq \emptyset\}$ and by Lemma 3.5 and Corollary 3.6, $V_{a_\mathbb{N} \setminus L} \cap B' = \emptyset$, therefore $B' \delta_\omega L$. Indeed, $a_\mathbb{N} = (a_\mathbb{N} \setminus L) \cup L$ and $\delta[B'', a_\mathbb{N}] = 0$ for $B'' \in [B']^\omega$ implies $\delta[B'', L] = 0$ because $V_{a_\mathbb{N} \setminus L} \cap B'' = \emptyset$.
As $\overline{A} \cap \overline{B} = \emptyset$ and $B' \delta_\omega L$, both the set $V_L \cap B'$ and the set L are infinite.

Clearly, we can assume that $L = a_\mathbb{N}$, i.e. $V_{a_n} \cap B' \neq \emptyset$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, choose $b_n \in V_{a_n} \cap B'$ and put $M = \{b_n : n \in \mathbb{N}\}$. By Corollary 3.6, $V_{a_i} \cap V_{a_j} = \emptyset$ for every $i, j \in \mathbb{N}$ with $i \neq j$ so $M \in [B']^\omega$ and $M \delta_\omega L$.

We shall show that $(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}}$. Let $F \in [\mathbb{N}]^\omega$. As

$$L = a_\mathbb{N} = a_{\mathbb{N} \setminus F} \cup a_F$$
$$\delta[b_F, L] = 0$$
$$V_{a_{\mathbb{N} \setminus F}} \cap b_F = \emptyset$$

we get that $\delta[a_F, b_F] = 0$. So Theorem 4.1 is proved. □

Corollary 4.2. *If a proximity δ on X is generated by a countable family Ω of entourages of the diagonal in $X \times X$ such that*

$$(*) \qquad \text{the family } \{U \circ U^{-1} \circ U \circ U^{-1} : U \in \Omega\} \text{ also generates } \delta,$$

then δ is metrizable.

PROOF: Theorem 4.1 allows to apply directly Theorem 1.2. □

Corollary 4.3 ([7])**.** *If a proximity δ is generated by a countable family of pseudometrics, then δ is metrizable.*

PROOF: Suppose δ is generated by a countable family $\mathcal{P}$ of pseudometrics. Then, obviously, the family of symmetric entourages of the diagonal in $X \times X$

$$\Omega_{\mathcal{P}} = \big\{U_{p,n} : p \in \mathcal{P}, n = 1, 2, \dots\big\}$$

where

$$U_{p,n} = \left\{(x, y) \in X \times X : p(x, y) < \frac{1}{n}\right\}$$

also generates δ and satisfies $(*)$ from Corollary 4.2. □

PROOF OF PROPOSITION 1.1: Suppose δ is generated by a decreasing sequence $\Omega = \{U_1 \supset U_2 \supset \dots \supset U_n \supset \dots\}$ of entourages of the diagonal in $X \times X$. Then δ is generated by the family $\{U_n \cap U_n^{-1} : \; n = 1, 2, \dots\}$ of symmetric entourages of the diagonal (by virtue of Prx. 5) which satisfies $(*)$ from Corollary 4.2 (again by Prx. 5). □

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