\section*{\textbf{Introduction}}

Obviously, every right ideal of a chain ring is a uniserial right module. The question is when there exists a uniserial module generated by more than countably many elements over a chain ring with at most countably generated right ideals. Note that by definition a chain ring $R$ has the linearly ordered lattice of left ideals as well as the lattice of right ideals. So $R$ is a right Ore ring and the right ring of fractions of every completely prime factor of $R$ is uniserial as a right $R$-module.

The class of chain rings is a natural generalization of the well known class of valuation rings. Basic properties of general chain rings are described in [BBT] and in [F, Chapter 6]. In [D1], [D2] and [BBT] there are presented non-trivial examples of a chain ring which contains precisely three two-sided ideals, i.e. of a nearly simple chain ring (non-trivial means that it has the infinitely generated Jacobson radical). Surprisingly, it is possible to construct an $\omega_1$-generated uniserial module over every non-trivial nearly simple chain ring (Theorem 1.8). The “strange” properties of this class of chain rings are partially described in [P1] and [P2].

The occurrence or, on the other hand, non-existence of a big uniserial right module over a chain ring is equivalent to the question whether the ring is right steady or non-steady. Recall that a ring $R$ is called right steady if every right $R$-module which is not a union of a countable infinite strictly increasing chain of
submodules is finitely generated. The basic results concerning steady rings are published in [EGT], [ZT] and [Z].

A ring-theoretic characterization of commutative steady chain rings (valuation rings) is proved in [ZT]. We characterize the steadiness of chain rings in general in the present paper (Theorem 2.4).

An application of the results obtained for chain rings gives us a characterization of steadiness of a natural generalization of the notion of a chain ring, serial rings (for more about serial rings see [F, Chapter 6] and [P4]). Namely, a serial ring is not right steady if and only if there exists an \(\omega_1\)-generated uniserial module (Theorem 3.5).

Throughout the paper a ring \(R\) means an associative ring with unit, a module means a right \(R\)-module, and a bimodule is an \(R-R\)-bimodule. If \(U\) is a bimodule, \(r.\text{ann}(x)\) (resp. \(l.\text{ann}(x)\)), denotes the right (resp. left) annihilator of an element \(x \in U\). The minimal cardinality of a set of generators of an \(R\)-module \(M\) is denoted by \(\text{gen}_R(M)\). The Jacobson radical of the ring \(R\) is denoted by \(J(R)\) and the prime radical is denoted by \(\text{rad}(R)\).

A bimodule \(RB_R\) is said to be right (resp. left) chain provided \(B_R\) (resp. \(RB\)) is a uniserial right (resp. left) module, i.e. its lattice of right (resp. left) submodules is linearly ordered. A bimodule is said to be a chain if it is both right and left chain. A ring is said to be chain provided it is a chain bimodule \(RR_R\). A ring \(R\) is called serial if \(R\) contains a set of orthogonal idempotents \(\{e_i, i \leq n\}\) such that \(1 = \sum_{i \leq n} e_i\) and for every index \(i \leq n\) the ideals \(e_iR\) and \(Re_i\) are uniserial modules. The set \(\{e_i, i \leq n\}\) is called a complete set of orthogonal idempotents. A module is said to be dually slender if it is not a union of a countable strictly increasing chain of submodules and a ring is called right steady if each dually slender right \(R\)-module is finitely generated. As we have remarked, a nearly simple ring is a ring which contains precisely three ideals, i.e. 0, \(J(R)\) and \(R\).

1. Nearly simple chain rings

We start the section with studying properties of a generalization of a two-sided ideal.

**Definition 1.1.** Let \(U\) be a bimodule. Then \(S \subseteq U\) is a subbimodule of \(U\) if \(S\) is both a right and left submodule of \(U\). A non-zero bimodule is said to be simple provided it contains only trivial subbimodules. A bimodule \(U\) is said to be exceptional if for every non-invertible element \(r \in R\) there exist non-zero elements \(u, v \in U\) such that \(ur = rv = 0\).

Note that the bimodule \(R/J(R)\) is an exceptional simple bimodule whenever \(R\) is local i.e. \(R/J(R)\) is a skew-field. The notion of an exceptional chain bimodule generalizes the Dubrovin’s term of an exceptional nearly simple chain ring (cf. [P2]). A natural example of an exceptional chain bimodule is the Jacobson radical
of an exceptional nearly simple chain ring. Later we will show that there exists a non-artinian (as a right module) exceptional simple bimodule over every non-artinian nearly simple chain ring.

For every element \( u \) of a ring denote by \( r.\ker(u) \) (resp. \( l.\ker(u) \)) a right (resp. left) submodule of a bimodule \( U \) containing all elements of \( U \) annihilated from left (resp. right) by \( u \).

**Remark 1.2.** Let \( U \) be an exceptional chain bimodule over a chain ring \( R \) and \( a, b \in U \).

1. If \( a, b \) have the same right (resp. left) annihilator, there exists an invertible element \( u \in R \) for which \( ua = b \) (resp. \( au = b \)).
2. If \( aR \subset bR \), then \( l.\ann(b) \subset l.\ann(a) \).

**Proof:** (1) It is sufficient to prove the right hand version of the remark.

If \( a = 0 \), the claim is trivial. Suppose \( a \neq 0 \). Since the bimodule \( U \) is left chain, there exists an element \( u \in R \) such that \( ua = b \) or \( ub = a \). Without loss of generality suppose that \( ua = b \). Assume that \( u \in J_R \). Then \( r.\ker(u) \neq 0 \) because \( U \) is exceptional. As \( r.\ker(u) \cap aR \neq 0 \), there is \( r \in R \) such that \( ar \neq 0 \) and \( uar = br = 0 \), i.e., \( r \in r.\ann(b) \setminus r.\ann(a) \), a contradiction. Thus \( u \in R \setminus J_R \) is an invertible element.

(2) Obviously, \( l.\ann(b) \subset l.\ann(a) \). The inclusion is strict by (1). \( \square \)

In the sequel, \( R \) is a chain ring and \( U \) is an exceptional simple chain bimodule. First, make an easy observation concerning exceptional chain bimodules.

**Lemma 1.3.** Let \( M \) be a uniserial \( R \)-module with a countable set \( x_0, x_1, x_2, \ldots \) of generators. Suppose that every cyclic submodule \( x_nR \) is isomorphic to a right submodule of \( U \). Then \( M \) is isomorphic to a right submodule of \( U \).

**Proof:** Without loss of generality \( x_0R \subset x_1R \subset x_2R \subset \ldots \). It is sufficient to define by induction a sequence of monomorphisms \( \nu_n : x_nR \to U \) such that \( \nu_{n+1}(x_n) = \nu_n(x_n) \) for each \( n = 0, 1, 2, \ldots \).

First, fix an arbitrary embedding \( \nu_0 : x_0R \to U \).

Suppose that \( \nu_n \) is defined. By hypothesis, there exists an injective homomorphism \( \mu : x_{n+1}R \to U \). Since \( \mu(x_n) \) has the same right annihilator as \( \nu_n(x_n) \), by Remark 1.2 (1) there exists an invertible element \( u \in R \) such that \( u\mu(x_n) = \nu_n(x_n) \). Define a homomorphism \( \nu_{n+1}(x) = u\mu(x) \) for every \( x \in x_{n+1}R \). Obviously, \( \nu_{n+1} \) is injective and \( \nu_{n+1}(x_n) = \nu_n(x_n) \). \( \square \)

**Lemma 1.4.** Suppose that \( U_R \) is not cyclic and that there exists an embedding of \( U_R \) into \( xR \) for a suitable element \( x \in U \). Then there exists an \( \omega_1 \)-generated uniserial right module.

**Proof:** It suffices to construct a directed system of cyclic modules \( \{m_\alpha R; (\phi_{\beta, \alpha}) \mid \beta \leq \alpha \} \mid \alpha < \omega_1 \} \), indexed by all countable ordinals, satisfying the property that the homomorphisms \( \phi_{\beta, \alpha} : m_\beta R \to m_\alpha R \) are injective and not
surjective whenever \( \beta < \alpha < \omega_1 \) and such that all the \( m_\alpha R \) are isomorphic to proper right submodules of \( UR \).

First, fix an arbitrary non-zero element \( m_0 \in UR \).

Let \( 0 \neq \alpha < \omega_1 \) and suppose that we have already defined all the \( m_\beta R \) for \( \beta \leq \alpha \) and all \( \phi_{\gamma, \beta} \) for \( \gamma < \beta \leq \alpha \) with the required properties. Since there is a monomorphism \( \psi : m_\alpha R \to UR \) and \( \text{im}(\psi) \neq UR \), we can fix an element \( m_{\alpha + 1} \in UR \) such that \( \text{im}(\psi) \subseteq m_{\alpha + 1} R \). Now, put \( \phi_{\beta, \alpha + 1} = \psi \phi_{\beta, \alpha} \).

Let \( \alpha \) be a countable limit ordinal. Let \( (M, \{ \phi_\gamma; \gamma < \alpha \}) \) be a direct limit of the directed system \( \{(m_\beta R; \beta < \alpha), (\phi_{\gamma, \beta}; \gamma \leq \beta < \alpha)\} \). Clearly, the homomorphisms \( \phi_\gamma \) are injective for every \( \gamma < \alpha \). Since \( \alpha \) is of countable cofinality, it is possible to express \( M \) as the union of a countable strictly increasing chain of modules \( \phi_{\alpha_n}(m_{\alpha_n} R) \), where \( \sup\{\alpha_n, n < \omega\} = \alpha \). Since the submodules \( \phi_{\alpha_n}(m_{\alpha_n} R) \) are embeddable into \( U \), the hypothesis of Lemma 1.3 is satisfied. Therefore there exists an embedding \( \psi : M \to UR \). Moreover, \( \psi(M) \) is embeddable into a cyclic right submodule \( xR \subseteq UR \) by the hypothesis. Thus the composition of these two monomorphisms gives us the monomorphism \( \rho : M \to xR \). Now put \( m_\alpha = x \) and \( \phi_{\beta, \alpha} = \rho \phi_{\beta} \).

Since the direct limit of the directed system we have just constructed is a union of a strictly increasing chain of submodules which has the length \( \omega_1 \), we get the required example of an \( \omega_1 \)-generated uniserial module. \( \square \)

**Lemma 1.5.** Let \( UR \) be an infinitely countably generated right \( R \)-module. Then every infinitely countably generated right submodule of \( U \) is isomorphic to \( UR \).

**Proof:** Let \( I = \bigcup_{n<\omega} x_n R \) be an infinitely generated right submodule which is not equal to \( UR \). We may suppose that \( 0 \subset x_0 R \subset x_1 R \subset \ldots \). Fix an element \( z_0 \in UR \) such that \( I \subseteq z_0 R \). Since \( UR \) is countably generated we can express \( UR \) as a union of a strictly increasing chain of cyclic right submodules \( z_n R, n < \omega \).

We now show by induction on \( n \) that there exists a sequence \( y_1, y_2, \ldots \) of elements of \( J = J(R) \) such that \( z_n R \subseteq y_n x_n R \) and \( y_{n+1} x_n = y_n x_n \) for every \( n \). As the left socle \( \text{Soc}(R U) \) of a bimodule is a submodule, we must have either \( \text{Soc}(R U) = 0 \) or \( \text{Soc}(R U) = R U \). But \( RU \) is a non-artinian chain bimodule, so that \( \text{Soc}(R U) = 0 \). In particular \( J x_0 \neq 0 \). Hence \( \bigcup_{y \in J} y x_0 R \) is a non-zero submodule, so that \( \bigcup_{y \in J} y x_0 R = UR \). Therefore there is an element \( y_0 \in J \) such that \( z_0 R \subseteq y_0 x_0 R \).

Suppose that the element \( y_n \) has been defined. If \( z_{n+1} R \subseteq y_n x_{n+1} R \), the element \( y_{n+1} = y_n \) trivially satisfies our requirements. Otherwise \( y_n x_{n+1} R \subset z_{n+1} R \). Put \( L = \text{ann}(x_n) \). Since \( x_n R \subset x_{n+1} R \), it follows from Remark 1.2(2) that \( L x_{n+1} \neq 0 \). The fact that \( \bigcup_{a \in L} a x_{n+1} R \) is a submodule implies the equation \( \bigcup_{a \in L} a x_{n+1} R = UR \). Therefore there is an element \( a \in L \) for which \( z_{n+1} R \subset a x_{n+1} R \).
Now, put \( y_{n+1} = y_n + a \). We need to show that \( y_{n+1} \) satisfies our requirements. Since \( a \in \text{ann}(x_n) \), the equations \( y_{n+1}x_n = y_nx_n + ax_n = y_nx_n \) hold true. Finally, we check the strict inclusion \( z_{n+1}R \subseteq y_{n+1}z_{n+1}R \). Assume that \( y_{n+1}x_{n+1} \subseteq z_{n+1}R \). It means that \((y_n + a)x_{n+1} \subseteq z_{n+1}R \). As \( y_nx_{n+1} \) is an element of \( z_{n+1}R \), it holds true that \( ax_{n+1} = y_{n+1}x_{n+1} - y_nx_{n+1} \in z_{n+1}R \). This is a contradiction because \( z_{n+1}R \subseteq ax_{n+1}R \). Hence \( z_{n+1} \subseteq y_{n+1}z_{n+1}R \).

As \( y_n \in J \), the element \( 1 - y_n \) is invertible, so the left multiplication by \( 1 - y_n \) induces a monomorphism \( u_n : x_n R \rightarrow U_R \). Moreover, \( u_{n+1}(x_n) - u_n(x_n) = (y_n - y_{n+1})x_n = 0 \). Thus the monomorphisms \( u_n \) determines in natural way a homomorphism \( u : I \rightarrow U_R \), which is injective. In order to complete the proof it suffices to show that \( u \) is onto \( U_R \).

Assume that \((1 - y_n)x_n \in x_n R \). Then also \( y_n x_n = x_n - (1 - y_n)x_n \in x_n R \subseteq z_0 R \subseteq z_n R \), a contradiction. Hence \( x_n \in (1 - y_n)x_n R \), and so \( y_n x_n = x_n - (1 - y_n)x_n \in (1 - y_n)x_n R \). Since \( y_n x_n R \subseteq (1 - y_n)x_n R \) for every \( n = 1, 2, \ldots \), \( U = \bigcup_{n<\omega} z_n R \subseteq \bigcup_{n<\omega} y_n x_n R \subseteq \bigcup_{n<\omega} (1 - y_n)x_n R \) is contained in the image of the homomorphism \( u \).

The claim of the previous lemma is proved in [P2, Lemma 5.8] for countably generated locally coherent uniserial modules over an exceptional coherent chain ring.

**Corollary 1.6.** If there exists a non-artinian exceptional simple chain bimodule, there exists an \( \omega_1 \)-generated uniserial module.

**Proof:** If an exceptional simple chain bimodule is uncountably generated, the claim is trivial. On the other hand, let \( U_R \) be countably generated. Since an exceptional simple bimodule is strictly embeddable into itself by Lemma 1.5, we obtain an example of an \( \omega_1 \)-generated uniserial module applying Lemma 1.4.

**Lemma 1.7.** Let \( J \) be a completely prime ideal of a prime chain ring \( R \). Then there exists the right localization of \( R \) with respect to \( J \), denote it by \( R_J \), such that the natural homomorphism \( R \rightarrow R_J \) is injective. We may suppose that \( R \subseteq R_J \).

1. Then for every \( x \in R_J \setminus R \) there exists \( s \in R \setminus J \) such that \( x = s^{-1} \).

2. Let \( M \) be a uniserial \( R_J \)-module. Then \( M \) is a uniserial \( R \)-module.

**Proof:** First, we check the existence of the localization \( R_J \). For this purpose we will apply the existence theorem [C, Theorem 0.5.3]. Take arbitrary elements \( r \in R \) and \( t \in R \setminus J \). Note that the multiplicative set \( R \setminus J \) satisfies the Ore condition, i.e. \( r(R \setminus J) \cap tR \neq \emptyset \). Since \( rt, tr \in R \setminus J \) whenever \( r \in R \setminus J \), it suffices to prove that \( rt \neq 0 \) if \( r \in J \setminus \{0\} \). Suppose that \( r \in J \) and \( tr = 0 \) \((rt = 0)\). As \( R \) is a chain ring and \( J \) is a two-sided ideal, we have \( Rt \supseteq J \supseteq rR \) \((tR \supseteq J \supseteq rR)\), and so \( rRt = 0 \). Since \( R \) is a prime ring, \( r = 0 \). Now, applying [C, Theorem 0.5.3] we get that \( R_J \) is a ring and the natural mapping \( R \rightarrow R_J \)
Lemma 1.7, denote it by $Q$ so that $s$ uniserial module, there exists a Jacobson radical is an exceptional simple bimodule. Hence the Jacobson radical is an exceptional simple bimodule. Then the assertion follows from Corollary 1.6.

To prove the theorem it is sufficient to find a non-artinian exceptional simple $\omega_1$-generated uniserial module. 

Proof: If the ring contains an uncountably generated right ideal, the claim is trivial. Otherwise, suppose that all right ideals are countably generated. In order to prove the theorem it is sufficient to find a non-artinian exceptional simple bimodule. Then the assertion follows from Corollary 1.6.

If the ring is not a domain, all elements of the Jacobson radical have nonzero right and left annihilators, because the set of all right (respectively, left) zero-divisors in a chain ring is always a completely prime ideal [BBT, Lemma 2.3(i)]. Hence the Jacobson radical is an exceptional simple bimodule.

If the ring $R$ is a domain, there exists the right ring of fractions of $R$ by Lemma 1.7, denote it by $Q$. Remark that $R$ is a subbimodule of the $R$-bimodule $Q$ so that $Q/R$ is $R$-bimodule. We will show that $Q/R$ is an exceptional simple chain bimodule.

By Lemma 1.7(2) the right and left module $Q$ is uniserial because $R$ is a chain ring. Hence $Q$ and so $Q/R$ is a chain bimodule. Applying Lemma 1.7(1) we see that every element of $Q \setminus R$ is of the form $r^{-1}$ for some $r \in R \setminus \{0\}$. As $j^{-1}j = jj^{-1} = 1 \in R$ for every element $j \in R \setminus \{0\}$, the bimodule $Q/R$ is exceptional.

It remains to prove that $Q/R$ is a simple bimodule. As we have observed, each non-zero cyclic $R$-submodule of $Q$ strictly containing the submodule $R$ is generated by a suitable element $r^{-1}$ for non-invertible $r \in R \setminus \{0\}$. Fix such an element $r^{-1}$. Remark that $rR \subset J(R)$. Moreover, $Q = \bigcup_{x \in R \setminus \{0\}} x^{-1}R$ hence it suffices to show that for each $x \in R \setminus \{0\}$ there exists an element $y \in R$ such that $x^{-1}R \subseteq yr^{-1}R$. Note that $Rx \cap (R \setminus rR) \neq \emptyset$. Indeed, the left ideal $Rx$ is not a subset of the right ideal $rR$ because $rR$ contains no non-zero ideal and so no non-zero left ideal. Fix an element $c \in Rx \cap (R \setminus rR)$. As $c \in Rx$, there exists an element $s \in R$ such that $c = sx$. Hence $c^{-1}s = x^{-1}$ and so $x^{-1}R \subseteq c^{-1}R$. Finally, since $rR \subseteq cR$, there exists $y \in R$ such that $cy = r$. Now $yr^{-1} = c^{-1}$ and $x^{-1}R \subseteq yr^{-1}R = c^{-1}R$ which finishes the proof. \[\square\]
Observe that a nearly simple chain prime ring (in particular, a domain) is not artinian.

Since an artinian nearly simple chain ring \( R \) (clearly, it is neither a domain nor a prime ring) has Loewy length equal to 2, every \( R \)-module is semiartinian with socle length at most 2. Hence every non-zero uniserial module over an artinian nearly simple chain ring is either simple or it is isomorphic to the module \( R_R \).

**Example 1.9.** There exist a countable nearly simple chain domain and a countable exceptional nearly simple chain prime ring which is not a domain.

**Proof:** The both examples are due to Dubrovin, for the first one see [BBT, Section 6.5] or [D1], and for the second see [D2].

The example of a countable non-artinian nearly simple chain domain is the localization of the ring \( K[P] \) with respect to suitable Ore system where \( K \) is a skew-field and \( P \) is a suitable subsemigroup of a semidirect product of the additive group \((Q, +)\) and the multiplicative group \((Q_{>0}, \cdot)\). Hence the ring is countable whenever we chose a countable skew-field \( K \).

The example of a countable non-artinian exceptional nearly simple chain prime ring which is not a domain is constructed as a suitable factor of ring \( R \) which is contained in a rational closure of a group ring \( K[G] \). Since the group \( G \) in the construction in [D2] is countable, the ring \( R \) is countable iff the skew-field \( K \) is countable. \( \square \)

# 2. Chain rings

In this section we show that the question “over which chain rings there exists an \( \omega_1 \)-generated uniserial module” leads to checking two extreme cases. First of them, the case of nearly simple chain rings, is studied in the first section. The second situation is close to the case of commutative rings.

**Proposition 2.1.** Suppose that a chain ring \( R \) properly contains a prime ideal \( P \) and an ideal \( J, P \subset J \), such that \( J/P \) is a simple bimodule. Then there exists an \( \omega_1 \)-generated uniserial right module.

**Proof:** Without loss of generality suppose that \( P = 0 \), so \( R \) is a prime ring. Note that \( J \) is an idempotent because \( J^2 \nsubseteq P \) and \( J/P \) is a simple bimodule. Hence, by [BBT, Theorem 1.15(1)] \( J \) is completely prime. Now, the right localization \( R_J \) of the ring \( R \) with respect to the prime ideal \( J \) (which exists by Lemma 1.7) is a nearly simple chain ring. Since \( R_J \) is prime, the socle of the ring \( R_J \) vanishes. Hence the Jacobson radical \( J_J \) is not artinian. It follows from Theorem 1.8 that there exists an \( \omega_1 \)-generated uniserial \( R_J \)-module \( M \). Applying Lemma 1.7(2), \( M \) has a structure of a uniserial \( R \)-module. Finally, remark that \( \text{gen}_R(M) \geq \text{gen}_{R_J}(M) = \omega_1 \). \( \square \)
Recall an another meaning of the term exceptional as it used in [BBT]. A prime ideal is *exceptional* if it is not completely prime.

**Corollary 2.2.** If a chain ring $R$ contains an exceptional prime ideal, there exists an $\omega_1$-generated uniserial right module.

**Proof:** By [BBT, Theorem 6.2] there exists a completely prime ideal above an exceptional prime ideal which satisfies the hypothesis of Proposition 2.1. \(\square\)

**Lemma 2.3.** Let $R$ be a chain ring. If $R/\text{rad}(R)$ contains an uncountable strictly decreasing chain of left ideals, there exists an $\omega_1$-generated uniserial module.

**Proof:** If there is an exceptional prime ideal, the existence of an $\omega_1$-generated uniserial module follows immediately from Corollary 2.2. Hence let us suppose that all prime ideals are completely prime. In particular, $\text{rad}(R)$ is a completely prime ideal. We may assume that $\text{rad}(R) = 0$.

Since $R$ contains an uncountable strictly decreasing chain of left ideals, it contains an uncountable strictly decreasing chain of left principle ideals $(Rs_\alpha | \alpha < \omega_1)$ as well. Thus the right ring of fractions $Q(R)$ (which exists by Lemma 1.7) contains as right $R$-modules an uncountable strictly increasing chain of cyclic submodules $(s_\alpha^{-1}R)$. The union $\bigcup_{\alpha < \omega_1} s_\alpha^{-1}R$ gives us an $\omega_1$-generated uniserial submodule. \(\square\)

Recall that a module $M$ is said to be *dually slender* provided the covariant functor $\text{hom}_R(M, -)$ commutes with direct sums. Several characterizations of this notion are known. The most important and useful is the following one: The module is dually slender if and only if it is not a union of a countable infinite strictly increasing chain of submodules (for a proof see [T, Lemma 1.2]). Note that the class of all dually slender modules contains all finitely generated modules and it is closed under taking factors and taking finite sums.

A ring is called *right steady* if the class of all dually slender modules coincides with the class of all finitely generated modules. Recall that a factor of a right steady ring is right steady as well.

**Theorem 2.4.** For a chain ring $R$ the following conditions are equivalent:

1. $R$ is right steady;
2. there exists no $\omega_1$-generated uniserial right module;
3. $R/\text{rad}(R)$ contains no uncountable strictly decreasing chain of ideals, $R$ contains no uncountably generated right ideal and for every ideal $I$ and for every prime ideal $P \subseteq I$ there exists an ideal $K$ such that $P \subseteq K \subseteq I$.

**Proof:** The implication (1) → (2) is in obvious.

(2) → (3) Let us suppose that the negation of (3) holds true and we will find an $\omega_1$-generated uniserial module.
If there exists an uncountable strictly decreasing chain of ideals, there exists an \( \omega_1 \)-generated uniserial module by Lemma 2.3. Since \( R \) is a chain ring, an occurrence of an \( \omega_1 \)-generated right ideal gives us an example of an \( \omega_1 \)-generated uniserial module. Finally, suppose that \( R/\text{rad}(R) \) contains no uncountable strictly decreasing chain of ideals and that \( R \) contains no uncountably generated right ideal. Then the negation of the third item of (3) implies the hypothesis of Proposition 2.1 is satisfied. Applying the proposition we get a required \( \omega_1 \)-generated uniserial module.

(3) \( \rightarrow \) (1) Assume that (3) holds and \( R \) is not right steady. Fix an infinitely generated dually slender module \( M \). Since the ring \( R \) is local, by [Z, Corollary 1.5] it is possible to chose \( M \) containing no maximal submodule. Then every non-zero factor of \( M \) is an infinitely generated module. As \( R \) contains only countably generated right ideals, we may apply [ZT, Lemma 11]. Hence there exists a prime ideal \( P \) such that \( M/MP \) is an infinitely generated module and \( M/MI = 0 \), i.e. \( M = MI \) for every ideal \( P \subseteq I \). As no simple bimodule occurs above the prime ideal \( P \), \( P \) is an intersection of a strictly decreasing chain of ideals. Thus by [BBT, Theorem 1.15(1)] \( P \) is a completely prime ideal. By hypothesis every strictly decreasing chain of ideals is countable. In particular, there exists a decreasing sequence of ideals \( I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \cdots \) such that \( P = \bigcap_{n<\omega} I_n \). Hence the right ring of fractions \( Q = Q(R/P) \) (which exists by Lemma 1.7) is countably generated as a right \( R/P \)-module.

Define modules \( M_n = \{ m \in M \mid mI_n \subseteq MP \} \) for every integer \( n \). Note that \( M_n \subseteq M \) for every \( n \). Hence \( N = M/\bigcup_{n<\omega} M_n \) is nonzero so it is an infinitely generated dually slender \( R/P \)-module. We prove that \( N \) is a torsion-free \( R/P \)-module.

Let \( m \in M \) and let \( mx \in \bigcup_{n<\omega} M_n \) for some \( x \in R \setminus P \). Then \( mxI_n \subseteq MP \) for a suitable \( n \). As \( P \) is prime, there exists \( k \geq n \) such that \( I_k \subseteq xI_n \). Thus \( mI_k \subseteq MP \) and \( m \in M_k \). It implies \( N \) is a torsion-free \( R/P \)-module. So that the module \( N \) is embeddable into the module \( N \otimes_{R/P} Q \cong Q^{(k)} \) (recall \( Q \) is the right ring of fractions of \( R/P \)). Since \( N \) is dually slender, \( N \) embeds into a direct sum of finitely many copies of \( Q \). Hence we have an infinitely generated dually slender submodule of a countably generated module. By [ZT, Lemma 5] a suitable cyclic \( R \)-module contains an infinitely generated dually slender submodule which is an \( \omega_1 \)-generated. It is a contradiction because each submodule of any cyclic \( R \)-module is countably generated.

The condition (3) immediately implies the following fact.

**Corollary 2.5.** Let all right and left ideals of a chain ring \( R \) be countably generated. Then \( R \) is right steady if and only if \( R \) is left steady.

Note that each prime ideal over a chain ring with the Krull dimension is completely prime by [F, 7.17 and 7.38]. In particular, the prime radical \( \text{rad}(R) \), i.e. the intersection of all prime ideals of \( R \), is completely prime.
Corollary 2.6. Let $R$ be a chain ring with the Krull dimension. The following conditions are equivalent:

1. $R$ is right steady;
2. there exists no $\omega_1$-generated uniserial right module;
3. the right ring of fractions $Q(R/\text{rad}(R))_R$ contains no $\omega_1$-generated right $R$-submodule;
4. all right ideals of $R$ are countably generated and $R$ is of a countable Krull dimension.

Proof: As there is no idempotent prime ideal in $R$, every prime factor of $R$ contains no non-trivial simple bimodule. Moreover, a chain ring with the Krull dimension possesses an uncountable strictly decreasing chain of ideals if and only if it has an uncountable Krull dimension. Finally, $R/\text{rad}(R)$ is right steady if and only if $R$ is right steady because $\text{rad}(R)$ is a nilpotent ideal [ZT, Lemma 3]. Now the claim is a consequence of Theorem 2.4. □

Example 2.7. There exists a non-steady commutative valuation domain with the Krull dimension such that all ideals are countably generated.

Proof: Take the inverse natural order on the ordinal $\omega_1$ and define on the group $\mathbb{Z}^{(\omega_1)}$ the corresponding lexicographic order $\preceq$ where $\mathbb{Z}$ is equipped with the natural order. By the Krull theorem (see for example [FS, Theorem I.3.4]) there exists a valuation domain $R$ with the value group order-isomorphic to $(\mathbb{Z}^{(\omega_1)}, \preceq)$. It is not hard to prove that $R$ has the Krull dimension which is at least equal $\omega_1$ so $R$ is not steady. Moreover, every filter on $(\mathbb{Z}^{(\omega_1)}, \preceq)$ is countably generated, hence every ideal of the ring $R$ is countably generated. □

3. An application on serial rings

We start this section with two general lemmas which show the correspondence between classes of dually slender modules over rings $R$ and $eRe$ for an idempotent $e \in R$.

Lemma 3.1. Let $R$ be an arbitrary ring and let $e \in R$ be an idempotent. Suppose $M$ is a right $R$-module such that $MeR = M$.

1. $Me$ is an infinitely generated $eRe$-module if $M$ is an infinitely generated $R$-module.
2. $Me$ is a dually slender $eRe$-module if $M$ is a dually slender $R$-module.

Proof: Note that $Me$ has a natural structure of a right $eRe$-module.

1. Suppose $Me$ is a finitely generated $eRe$-module, hence $Me = \sum_{i \leq n} m_i eRe$. Since $M = MeR$, we get $M = \sum_{i \leq n} m_i eReR = \sum_{i \leq n} m_i eR$, i.e. $M$ is a finitely generated $R$-module.
2. Suppose $Me = \bigcup_{i < \omega} N_i e$ where $N_i e$ is an increasing chain of $eRe$-submodules. Then $M = MeR = \bigcup_{i < \omega} N_i eR$. As $M$ is dually slender, there exists $n < \omega$
such that $M = N_i e R$. Thus $Me = N_i e Re = N_i e$ which proves $Me$ is a dually slender $e Re$-module. \hfill \square

**Lemma 3.2.** Let $R$ be an arbitrary ring and let $e \in R$ be an idempotent. Then the functor $- \otimes e Re$ maps 

1. each infinitely generated $e Re$-module onto an infinitely generated $R$-module,
2. each dually slender $e Re$-module onto a dually slender $R$-module.

**Proof:** We will write $- \otimes -$ instead of $- \otimes e Re$ in the whole proof.

1. Suppose $P$ is an infinitely generated $e Re$-module, i.e. $P = \bigcup_{\alpha < \kappa} P_\alpha$ for a suitable infinite cardinal $\kappa$ and a strictly increasing chain of submodules $P_\alpha$, $\alpha < \kappa$. Denote by $\pi_\alpha$ the natural projection $P$ onto $P/P_\alpha$ and put $Q_\alpha = \ker(\pi_\alpha \otimes e R)$ ($= \{ \sum_{j \leq m} p_j \otimes e r_j \mid p_j \in P_\alpha, r_j \in R, j \leq m < \omega \}, \alpha < \kappa$) for every $\alpha < \kappa$. Remark that $(Q_\alpha \mid \alpha < \kappa)$ forms an increasing chain of $R$-submodules of $P \otimes e R$. As the tensor product commutes with direct limits, $P \otimes e R = \bigcup_{\alpha < \kappa} Q_\alpha$.

It remains to prove that $Q_\alpha \neq P$ for each $\alpha < \kappa$. Fix one $\alpha < \kappa$. Note that $(P/P_\alpha) \otimes e R \neq 0$ because $(P/P_\alpha) \otimes e R \cong \mathbb{Z} ((P/P_\alpha) \otimes e Re) \oplus ((P/P_\alpha) \otimes e R(1-e))$ and $(P/P_\alpha) \otimes e Re \cong P/P_\alpha \neq 0$. Since the tensor product is a right exact functor, $(P \otimes e R)/Q_\alpha \cong (P/P_\alpha) \otimes e R \neq 0$. Hence $Q_\alpha \neq P \otimes e R$ and $P \otimes e R$ is an infinitely generated $R$-module.

2. Let $P \otimes e R = \bigcup_{n < \omega} N_n$ for an increasing chain of $R$-submodules $N_i, i < \omega$. Define $e Re$-submodules $P_n = \{ m \in P \mid m \otimes e \in N_n \}$ for each $n < \omega$. Note that the sequence $(P_n \mid n < \omega)$ forms an increasing chain of $e Re$-submodules of $P$ such that $P = \bigcup_{n < \omega} P_n$. Moreover, it is easy to see that $\{ p \otimes e \mid p \in P_n \} R \subseteq N_n$, for each $n < \omega$. As $P$ is a dually slender $e Re$-module, there exists $n < \omega$ such that $P_n = P$, hence $P_n \otimes e R = N_n = P \otimes e R$. Thus $P \otimes e R$ is a dually slender $R$-module. \hfill \square

**Proposition 3.3.** Let $R$ be an arbitrary ring and let $\{ e_i \mid 1 \leq i \leq n \}$ be a set of orthogonal idempotents satisfying $\sum_{i \leq n} e_i = 1$. Then $R$ is right steady if and only if $e_i Re_i$ is right steady for every $i \leq n$.

**Proof:** Suppose that $R$ is not right steady and fix an infinitely generated dually slender $R$-module $M$. Define by induction a sequence of $R$-modules $M_i$: $M_0 = M$, $M_{i+1} = M_i/e_i e_i R$. Clearly, $M_n = 0$. Take a minimal $i < n$ such that $M_{i+1}$ is finitely generated. Then there exists a finitely generated submodule $F \subseteq M_i$ such that $M_i e_i R + F = M_i$. Hence $M_i/F$ is an infinitely generated dually slender module and $(M_i/F)e_i R = M_i/F$. Now we may apply Lemma 3.1 which proves that $(M_i/F)e_i$ is an infinitely generated dually slender $e_i Re_i$-module, hence $e_i Re_i$ is not right steady.

On the other hand, if $e_i Re_i$ is not right steady for at least one $i$ and $N$ is an infinitely generated dually slender $e_i Re_i$-module, then $N \otimes e_i Re_i e_i R$ is a dually slender $R$-module by Lemma 3.2. Thus $R$ is not right steady. \hfill \square
Proposition 3.3 allows us to construct (non-) steady subrings of full matrix rings over a (non-) steady ring:

**Example 3.4.** Let $R$ be a ring and let $I$, $J$ be ideals of $R$. Then the subring \( \begin{pmatrix} R & I \\ J & R \end{pmatrix} \) of the full matrix ring is right steady iff $R$ is right steady.

As a consequence of Proposition 3.3 we reduce the question of steadiness of a serial ring $R$ to the question of steadiness of chain rings $e_i Re_i$ for the complete set of orthogonal idempotents \( \{e_i, \; i \leq n\} \).

**Theorem 3.5.** The following conditions are equivalent for a serial ring $R$ with a complete set of orthogonal idempotents \( \{e_i, \; i \leq n\} \):

1. $R$ is right steady;
2. $e_i Re_i$ is right steady for every $i \leq n$;
3. there exists no $\omega_1$-generated uniserial right $R$-module.

**Proof:** The equivalence of (1) and (2) follows immediately by Proposition 3.3 and the implication (1) $\rightarrow$ (3) holds true by definition.

It remains to show (3) $\rightarrow$ (2). Proving indirectly suppose that $e_i Re_i$ is not right steady for one index $i \leq n$. Applying Theorem 2.4 we get an $\omega_1$-generated uniserial right $e_i Re_i$-module $N = \bigcup_{\alpha < \omega_1} p_{\alpha} e_i Re_i$. Since the functor $- \otimes e_i Re_i e_i R$ (denote it by $- \otimes e_i R$) commutes with direct limits, the right $R$-module $M = N \otimes e_i R$ is the direct limit of a linear directed system of modules $(p_{\alpha} \otimes e_i)R$, $\alpha < \omega_1$. As every $R$-module $(p_{\alpha} \otimes e_i)R$ is a homomorphic image of the uniserial $R$-module $e_i R$, $M$ is a union of an increasing chain of uniserial $R$-modules, hence it is a uniserial $R$-module as well. By the construction $M$ is at most $\omega_1$-generated. Finally, $M$ is infinitely generated dually slender by Lemma 3.2, so it is precisely an $\omega_1$-generated uniserial right $R$-module. \( \square \)

**Example 3.6.** Let $R$ be a chain ring and $I$ be the maximal ideal of $R$. Then the ring $S = \begin{pmatrix} R & I \\ R & R \end{pmatrix}$ is right (left) steady iff $R$ is right (left) steady. Denote $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then every right ideal contained in $e_1 S$ is either of the form $\begin{pmatrix} iR & iR \\ 0 & 0 \end{pmatrix}$ for some $i \in I$ or of the form $\begin{pmatrix} rR & rI \\ 0 & 0 \end{pmatrix}$ for some $r \in R$. Similarly, every right ideal contained in $e_2 S$ is either of the form $\begin{pmatrix} 0 & 0 \\ rR & rR \end{pmatrix}$ or of the form $\begin{pmatrix} 0 & 0 \\ rR & rI \end{pmatrix}$ for some $r \in R$. Since $rI$ is a maximal submodule of $rR$ for every non-zero element $r \in R$, the right ideals $e_1 S$ and $e_2 S$ are uniserial. Applying the symmetric argument we get $Se_1$ and $Se_2$ are uniserial left ideals, hence $S$ is a non-chain serial ring.

**Acknowledgment.** The author thanks Alberto Facchini for valuable comments and references on an earlier version of the paper.
ω1-generated uniserial modules over chain rings

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(Received April 23, 2003, revised January 15, 2004)