

## Note on the classification theorems of $g$ -natural metrics on the tangent bundle of a Riemannian manifold $(M, g)$

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*Abstract.* In [7], it is proved that all  $g$ -natural metrics on tangent bundles of  $m$ -dimensional Riemannian manifolds depend on arbitrary smooth functions on positive real numbers, whose number depends on  $m$  and on the assumption that the base manifold is oriented, or non-oriented, respectively. The result was originally stated in [8] for the oriented case, but the smoothness was assumed and not explicitly proved. In this note, we shall prove that, both in the oriented and non-oriented cases, the functions generating the  $g$ -natural metrics are, in fact, smooth on the set of all nonnegative real numbers.

*Keywords:* Riemannian manifold, tangent bundle, natural operation,  $g$ -natural metric, curvatures

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If  $(M, g)$  is an  $m$ -dimensional Riemannian manifold, then we use the terminology of “ $g$ -natural metrics” (cf. [2]) on the tangent bundle  $TM$  to describe metrics on  $TM$  which come from  $g$  by a first order natural operator ([8] and [7]). We have studied these metrics in [1], [2] and [3]. The well-known example of such metrics is the Sasaki metric  $g^s$  [11]. All natural metrics are characterized by the following result:

**Theorem 1** ([8]). *There is a bijective correspondence between natural (possibly degenerated) metrics  $G$  on the tangent bundles of (oriented) Riemannian manifolds and the triples of first order natural  $F$ -metrics  $(\zeta_1, \zeta_2, \zeta_3)$ , where  $\zeta_1$  and  $\zeta_3$  are symmetric. The correspondence is given by*

$$G = \zeta_1^s + \zeta_2^h + \zeta_3^v,$$

where  $\zeta^s$ ,  $\zeta^h$  and  $\zeta^v$  denote the Sasaki lift, the horizontal lift and the vertical lift of  $\zeta$ , respectively.

For the definitions of  $F$ -metrics and their lifts, we refer to [8] (see also [7] for more details on the concept of naturality).

It is proved, furthermore, in [7] that all first order natural  $F$ -metrics on (oriented) Riemannian manifolds form a family parameterized by some arbitrary smooth function on positive real numbers, where the number of functions depends on the dimensions of manifolds (the result was originally stated in [8] for the oriented case, but the smoothness was assumed and not explicitly proved). Precisely, with the notations of [7], we have

**Theorem 2** ([7]). 1) All first order natural  $F$ -metrics  $\zeta$  on non-oriented Riemannian manifolds of dimension  $m > 1$  form a family parametrized by two arbitrary smooth functions  $\alpha, \beta : (0, \infty) \rightarrow \mathbb{R}$  in the following way: For every Riemannian manifold  $(M, g)$  and tangent vectors  $u, X, Y \in M_x$

$$(1) \quad \zeta_{(M,g)}(u)(X, Y) = \alpha(g(u, u))g(X, Y) + \beta(g(u, u))g(u, X)g(u, Y).$$

If  $m = 1$ , then the same assertion holds, but we can always choose  $\beta = 0$ . In particular, all first order natural  $F$ -metrics are symmetric.

2) On oriented Riemannian manifolds, we have the same results for dimensions  $m = 1$  and  $m > 3$ , but for  $m = 2$  and  $m = 3$ , there exist other arbitrary smooth functions  $\varphi, \gamma$  and  $\delta : (0, \infty) \rightarrow \mathbb{R}$  such that:

If  $m = 3$ , then

$$(2) \quad \zeta_{(M,g)}(u)(X, Y) = \alpha(g(u, u))g(X, Y) + \beta(g(u, u))g(u, X)g(u, Y) + \varphi(g(u, u))g(u, X \times Y),$$

where  $\times$  means the vector cross-product.

If  $m = 2$ , then

$$(3) \quad \begin{aligned} \zeta_{(M,g)}(u)(X, Y) = & \alpha(g(u, u))g(X, Y) + \beta(g(u, u))g(u, X)g(u, Y) \\ & \gamma(g(u, u))(g(J^g(u), X)g(u, Y) + g(u, X)g(j^g(u), Y)) \\ & \delta(g(u, u))(g(J^g(u), X)g(u, Y) - g(u, X)g(j^g(u), Y)), \end{aligned}$$

where  $J^g$  is the canonical almost complex structure on  $(M, g)$ .

Actually, the arbitrary parameterizing functions are smooth on all the set of nonnegative real numbers:

**Theorem 3.** All basic functions from Theorem 2 can be prolonged, in fact, to smooth functions on the set  $\mathbb{R}^+$  of all nonnegative real numbers.

PROOF: Note that we will use the technique from [7] throughout the whole proof.

1) Using the same arguments as in [7], we have to discuss all  $O(m)$ -equivariant maps  $\zeta : \mathbb{R}^m \rightarrow \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$ . Denote by  $g^0 = \sum_i dx^i \otimes dx^i$  the canonical Euclidean metric, and by  $|\cdot|$  the induced norm. Each vector  $v \in \mathbb{R}^m$  can be transformed in  $|v| \frac{\partial}{\partial x^1}|_0$  by an element of  $O(m)$ . Hence  $\zeta$  is determined by its values on the one-dimensional subspace spanned by  $\frac{\partial}{\partial x^1}|_0$ . Moreover, we can also change the orientation of the first axis by an element of  $O(m)$ , i.e., we have to define  $\zeta$  only on  $\{t \frac{\partial}{\partial x^1}|_0, t \geq 0\}$ .

Let us define a smooth map  $\xi : \mathbb{R} \rightarrow \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$  by  $\xi(t) = \zeta(t \frac{\partial}{\partial x^1}|_0) \in \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$ , for all  $t \in \mathbb{R}$ , and consider the group  $K_m$  of all linear orthogonal

transformations keeping  $\frac{\partial}{\partial x^1}|_0$  fixed. So for  $t \in \mathbb{R}^+$  (or generally  $\mathbb{R}$ ), the tensor  $\xi(t)$  is  $K_m$ -invariant. On the other hand, every such smooth  $\xi$  on  $\mathbb{R}^+$  determines a natural  $F$ -metric.

So let us assume  $s_{ij}dx^i \otimes dx^j$  is  $K_m$ -invariant. Since we can change the orientation of any coordinate axis, except the first one, by elements of  $K_m$ , then  $s_{ij} = 0$  for  $i \neq j$ . Further we can exchange any couple of coordinate axes different from the first one by elements of  $K_m$ , and so  $s_{ii} = s_{jj}$ , for all  $i \neq 1$  and  $j \neq 1$ . Hence all  $K_m$ -invariant tensors are of the form

$$(4) \quad \bar{\nu}dx^1 \otimes dx^1 + \bar{\mu}g^0,$$

the reals  $\bar{\mu}$  and  $\bar{\nu}$  being independent, if  $m > 1$ . In dimension 1, all  $K_1$ -invariant tensors are of the form  $\bar{\mu}g^0 = \bar{\mu}dx^1 \otimes dx^1$ .

Thus, our mapping  $\xi$  is defined by

$$(5) \quad \xi(t) = \bar{\nu}(t)dx^1 \otimes dx^1 + \bar{\mu}(t)g^0,$$

for all  $t \in \mathbb{R}$ , where  $\bar{\mu}$  and  $\bar{\nu}$  are arbitrary smooth functions on  $\mathbb{R}$  (and they reduce to one function if  $m = 1$ ).

For  $t = 0$ , since  $\zeta$  is  $O(m)$ -invariant, then the tensor  $\xi(0)$  is  $O(m)$ -invariant and so it is a multiple of  $g^0$  (cf. [6, I; p. 277]). It follows, by virtue of (5) that  $\bar{\nu}(0) = 0$ . On the other hand, if we consider the linear orthogonal transformation  $A_m$  which changes the orientation of the first coordinate axis, then the equivariance of  $\zeta$  by  $A_m$  implies that for every  $t \in \mathbb{R}$ ,  $\bar{\mu}(-t) = \bar{\mu}(t)$  and  $\bar{\nu}(-t) = \bar{\nu}(t)$ , i.e.,  $\bar{\mu}$  and  $\bar{\nu}$  are even.

Now, given  $v = t\frac{\partial}{\partial x^1}|_0$ ,  $t > 0$ , we can write

$$\begin{aligned} \zeta_{(\mathbb{R}^m, g^0)}(v)(X, Y) &= \xi(|v|)(X, Y) \\ &= \bar{\mu}(|v|)g^0(X, Y) + \bar{\nu}(|v|)|v|^{-2}g^0(v, X)g^0(v, Y). \end{aligned}$$

To complete the proof, we need the following lemma.

**Lemma 4** ([4]). *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function.*

- (a) *If  $f$  is even, then there exists a smooth function  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that  $f(t) = f(0) + t^2.g(t^2)$  for any  $t$ .*
- (b) *If  $f$  is odd, then there exists a smooth function  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that  $f(t) = t.(f'(0) + t^2.g(t^2))$  for any  $t$ .*

Let us define the functions  $\mu(t)$  and  $\nu(t)$  by  $\nu(t) = t^{-1}\bar{\nu}(\sqrt{t})$  and  $\mu(t) = \bar{\mu}(\sqrt{t})$ , for all  $t > 0$ . The functions  $\mu$  and  $\nu$  being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on  $\mathbb{R}^+$ . For this, applying (a) of Lemma 4 to  $\bar{\mu}$  and  $\bar{\nu}$ , there exist two smooth functions  $\alpha, \beta : \mathbb{R}^+ \rightarrow \mathbb{R}$ , such that  $\bar{\mu}(t) = \bar{\mu}(0) + t^2\alpha(t^2)$  and  $\bar{\nu}(t) = \bar{\nu}(0) + t^2\beta(t^2) = t^2\beta(t^2)$

(since  $\bar{\nu}(0) = 0$ ), for all  $t \in \mathbb{R}^+$ . We deduce that  $\mu(t) = \bar{\mu}(\sqrt{t}) = \bar{\mu}(0) + t\alpha(t)$  and  $\nu(t) = t^{-1}\bar{\nu}(\sqrt{t}) = \beta(t)$ , for all  $t > 0$ . In other words,  $\mu$  and  $\nu$  coincide on  $\mathbb{R}_*^+$  with two smooth functions on  $\mathbb{R}^+$ , and the formula (1) of Theorem 2 is extended to  $\mathbb{R}^+$ . Obviously, every such operator is natural and 1) of the Theorem is proved.

2) For the oriented situation, when  $m > 3$  and  $m = 1$ , the same proof remains valid if we replace  $K_m$  by  $K_m^+ := K_m \cap SO(m)$  and  $A_m$  by the element  $B_m$  of  $SO(m)$  which changes the orientations of the first and the second axes.

It remains to extend the formulas (2) and (3) from Theorem 2 to  $\mathbb{R}^+$ . We can use a similar procedure as before.

For  $m = 3$ , let us assume  $s_{ij}dx^i \otimes dx^j$  is  $K_3^+$ -invariant. If we change the orientation of any coordinate axis, different from the first one, by an element of  $K_3^+$ , then we must change the orientation of the other. It follows that  $s_{12} = s_{21} = s_{13} = s_{31} = 0$ . Further the element of  $K_3^+$  which exchanges the couple of second and third coordinate axes must change the orientation of one of them, and so  $s_{22} = s_{33}$  and  $s_{23} = -s_{32}$ . Hence all  $K_3^+$ -invariant tensors are of the form

$$(6) \quad \bar{\nu}dx^1 \otimes dx^1 + \bar{\mu}g^0 + \bar{\kappa}(dx^2 \otimes dx^3 - dx^3 \otimes dx^2),$$

the reals  $\bar{\mu}$ ,  $\bar{\nu}$  and  $\bar{\kappa}$  being independent. Thus, our mapping  $\xi$  is defined by

$$(7) \quad \xi(t) = \bar{\nu}(t)dx^1 \otimes dx^1 + \bar{\mu}(t)g^0 + \bar{\kappa}(t)(dx^2 \otimes dx^3 - dx^3 \otimes dx^2),$$

for all  $t \in \mathbb{R}$ , where  $\bar{\mu}$ ,  $\bar{\nu}$  and  $\bar{\kappa}$  are arbitrary smooth functions on  $\mathbb{R}$ . By similar arguments as in 1) we have  $\bar{\nu}(0) = \bar{\kappa}(0) = 0$  and also, if we consider the equivariance of  $\zeta$  by  $B_3$ , then we deduce that the functions  $\bar{\nu}$  and  $\bar{\nu}$  are even and that the function  $\bar{\kappa}$  is odd.

As in 1), let us define  $\mu(t)$ ,  $\nu(t)$  and  $\kappa(t)$  by  $\mu(t) = \bar{\mu}(\sqrt{t})$ ,  $\nu(t) = t^{-1}\bar{\nu}(\sqrt{t})$  and  $\kappa(t) = t^{-1/2}\bar{\kappa}(\sqrt{t})$  for all  $t > 0$ . The functions  $\mu$ ,  $\nu$  and  $\kappa$  being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on  $\mathbb{R}^+$ . But we can just apply (a) of Lemma 4 to  $\bar{\mu}$  and  $\bar{\nu}$  and (b) of Lemma 4 to  $\bar{\kappa}$ , and the result follows.

For  $m = 2$ , we have  $K_2^+ := K_2 \cap SO(2) = \{I_2, -I_2\}$ , where  $I_2$  denotes the identity matrix in  $GL(2)$ . Since every tensor in  $\mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$  is  $K_2^+$ -invariant, all  $K_2^+$ -invariant tensors are of the form

$$(8) \quad \bar{\nu}dx^1 \otimes dx^1 + \bar{\mu}g^0 + \bar{\lambda}(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) + \bar{\tau}(dx^1 \otimes dx^2 - dx^2 \otimes dx^1),$$

the reals  $\bar{\mu}$ ,  $\bar{\nu}$ ,  $\bar{\lambda}$  and  $\bar{\tau}$  being independent. Thus, our mapping  $\xi$  is defined by

$$(9) \quad \begin{aligned} \xi(t) = & \bar{\nu}(t)dx^1 \otimes dx^1 + \bar{\mu}(t)g^0 \\ & + \bar{\tau}(t)(dx^2 \otimes dx^1 + dx^1 \otimes dx^2) \\ & + \bar{\lambda}(t)(dx^2 \otimes dx^1 - dx^1 \otimes dx^2), \end{aligned}$$

for all  $t \in \mathbb{R}$ , where  $\bar{\mu}$ ,  $\bar{\nu}$ ,  $\bar{\lambda}$  and  $\bar{\tau}$  are arbitrary smooth functions on  $\mathbb{R}$ . By similar arguments as in 1) we have  $\bar{\nu}(0) = \bar{\lambda}(0) = \bar{\tau}(0) = 0$  and also all the functions  $\bar{\mu}$ ,  $\bar{\nu}$ ,  $\bar{\lambda}$  and  $\bar{\tau}$  are even (it suffices to take the equivariance of  $\zeta$  by  $-I_2$ ).

As in 1), let us define  $\mu(t)$ ,  $\nu(t)$ ,  $\lambda(t)$  and  $\tau(t)$  by  $\mu(t) = \bar{\mu}(\sqrt{t})$ ,  $\nu(t) = t^{-1}\bar{\nu}(\sqrt{t})$ ,  $\lambda(t) = t^{-1}\bar{\lambda}(\sqrt{t})$  and  $\tau(t) = t^{-1}\bar{\tau}(\sqrt{t})$  for all  $t > 0$ . The functions  $\mu$ ,  $\nu$ ,  $\lambda$  and  $\tau$  being clearly smooth on the set of positive real numbers, it remains to prove that they prolong to smooth functions on  $\mathbb{R}^+$ . But we can just apply (a) of Lemma 4 to the functions  $\bar{\mu}$ ,  $\bar{\nu}$ ,  $\bar{\lambda}$  and  $\bar{\tau}$  and the result follows.  $\square$

Combining Theorems 1–3, we obtain for the non-oriented case (an analogous result can be stated for the oriented case):

**Corollary 5.** *Let  $(M, g)$  be a non-oriented Riemannian manifold and  $G$  be a  $g$ -natural metric on  $TM$ . Then there are smooth functions  $\alpha_i, \beta_i : \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $i = 1, 2, 3$ , such that for every  $u, X, Y \in M_x$ , we have*

$$(10) \quad \begin{cases} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2)g_x(X, Y) \\ \qquad \qquad \qquad + (\beta_1 + \beta_3)(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = \alpha_2(r^2)g_x(X, Y) + \beta_2(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^h) = \alpha_2(r^2)g_x(X, Y) + \beta_2(r^2)g_x(X, u)g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2)g_x(X, Y) + \beta_1(r^2)g_x(X, u)g_x(Y, u), \end{cases}$$

where  $r^2 = g_x(u, u)$ .

For  $m = 1$ , the same holds with  $\beta_i = 0$ ,  $i = 1, 2, 3$ .

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