UR Birkhoff interpolation with rectangular sets of derivatives

Nicolae Crainic

Abstract. In this paper we characterize the regular UR Birkhoff interpolation schemes (U = uniform, R = rectangular sets of nodes) with rectangular sets of derivatives, and beyond.

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1. Introduction

The Birkhoff interpolation is one of most general form of multivariate polynomial interpolation. For notational simplicity, we will restrict ourselves to the two-dimensional case. Then the problem depends on

(i) a finite set $Z \subset \mathbb{R}^2$ (of "nodes");
(ii) for each $z \in Z$, a set $A(z) \subset \mathbb{N}^2$ (of "derivatives at the node z");
(iii) a lower set $S \subset \mathbb{N}^2$, defining the interpolation space

$$\mathcal{P}_S = \{P \in \mathbb{R}[x, y] : P = \sum_{(i,j)\in S} a_{i,j} x^i y^j\}.$$

Recall that S is called *lower* if it has the property that:

$$(i, j) \in S \Longrightarrow R(i, j) \subset S,$$

where $R(i, j)$ is the rectangle

$$R(i, j) = \{(i', j') \in \mathbb{N}^2 : 0 \le i' \le i, 0 \le j' \le j\}.$$

The interpolation problem consists of finding polynomials $P \in \mathcal{P}_S$ satisfying the equations

$$\frac{\partial^{i+j} P}{\partial x^i \partial y^j}(z) = c_{i,j}(z),\ \forall z \in Z, (i, j) \in A(z), \tag{1.1}$$

where $c_{i,j}(z)$ are given arbitrary constants. When the conditions at each point involve the same set A of derivatives (i.e. the sets $A(z) = A$ do not depend on z), we talk about the *uniform* problem associated to the triple (Z, A, S). One says that (Z, A, S) is *regular* if, for any choice of the constants $c_{i,j}(z)$, the associated equations (1.1) have a unique solution $P \in \mathcal{P}_S$. Of course, all these definitions apply to arbitrary dimensions. However, the one-dimensional case (univariate schemes) does behave differently and it is quite well understood (see e.g. [1], [3]). Looking at particular types of multivariate problems is a necessary step towards a better understanding of what happens in higher dimensions.

On the other hand, although there are several methods for studying Birkhoff interpolation for generic sets of nodes ([4]), little is known in the case where the shape of Z is more degenerate. One of the simplest and important cases is when Z is *rectangular*, i.e. when it is of type:

$$Z = \{(x_i, y_j) : 0 \le i \le p, 0 \le j \le q\},$$

with $p, q \ge 0$ integers, $x_i \in \mathbb{R}$ distinct real numbers, and similarly the y_j's. We also say that Z is a (p, q)*-rectangular set*, and we put

$$Z_x = \{x_i : 0 \le i \le p\},$$

and similarly Z_y. A *UR Birkhoff* scheme is a uniform scheme (Z, A, S) with rectangular set of nodes Z. The study of UR Birkhoff schemes is part of the author's PhD thesis (see [2]).

In this paper we look at UR Birkhoff interpolation where also A has a rectangular shape. However, since we will use only certain properties that rectangular shapes have, the results we derive hold much more generally. Let us state here the main result in the case where A is rectangular. Given $S \subset \mathbb{N}^2$, we will consider the intersection points of S with the coordinate axes, i.e.

$$S_x = \{\alpha : (\alpha, 0) \in S\},$$

and similarly S_y. In particular, any (bivariate) scheme (Z, A, S) will induce two univariate schemes (Z_x, A_x, S_x) and (Z_y, A_y, S_y).

Theorem 1.1. *If Z is a (p, q)-rectangular set of nodes, and A is (s, t)-rectangular, then the UR Birkhoff interpolation scheme (Z, A, S) is regular if and only if*

(i) *$S = R(p', q')$, with $p' = (s + 1)(p + 1) - 1, q' = (t + 1)(q + 1) - 1$;*
(ii) *the univariate schemes (Z_x, A_x, S_x) and (Z_y, A_y, S_y) are regular.*

The next two sections are devoted to this theorem: the next section takes care of (i), while in the last section we present a stronger version of the theorem (Theorem 3.1) together with its proof.

2. Finding the interpolation space

In this section we show that, for regular UR schemes, the rectangularity condition on A determines the interpolation space (i.e. the lower set S) uniquely. Actually, we will use only one property that rectangular sets have: if A is rectangular, then $|A| = |A_x||A_y|$ (we use the notation $|X|$ to denote the cardinality of a point set X). We will prove that:

Proposition 2.1. *If the UR Birkhoff scheme (Z, A, S) is regular and the set of nodes Z is (p, q)-rectangular, then*

$$|A| \leq |A_x||A_y|.$$

Moreover, if the equality holds, then S must be:

$$S = R(p', q'), \quad p' = (p+1)|A_x| - 1, \; q' = (q+1)|A_y| - 1.$$

The proof is based on a sequence of simple remarks. But first, let us introduce some terminology.

Definition 2.1. One says that a scheme (Z, A, S) is solvable if the interpolation problem (1.1) has at least one solution $P \in \mathcal{P}_S$ (for any choice of the constants). One says that (Z, A, S) has the uniqueness property if the equations (1.1) have at most one solution.

The simple remarks we will be using are put together in the following two lemmas.

Lemma 2.1. *For any lower set $S \subset \mathbb{N}^2$ one has*

$$|S| \leq |S_x||S_y|,$$

and equality holds if and only if $S = R(p', q')$, where $p' = |S_x| - 1$, $q' = |S_y| - 1$.

Lemma 2.2. *Let (Z, A, S) be a UR Birkhoff interpolation scheme.*

(i) *If (Z, A, S) is solvable, then $|S| \geq |A||Z|$.*
(ii) *If (Z, A, S) has the uniqueness property, then $|S| \leq |A||Z|$.*
(iii) *If the bivariate scheme (Z, A, S) has the uniqueness property, then so do the induced univariate schemes (Z_x, A_x, S_x), (Z_y, A_y, S_y).*

Proof: For Lemma 2.1, remark that the condition that S is lower implies that $S \subset R(p', q')$, and then one passes to cardinalities. For (i) and (ii) of Lemma 2.2, one remarks that (1.1) is a system of $|A||Z|$ linear equations, on $|S|$ variables (the coefficients of P). For (iii), remark that a non-trivial solution $P = P(x)$ of the uniform problem associated to (Z_x, A_x, S_x) will also be a nontrivial solution of the homogeneous equations associated to (Z, A, S).

We now prove the proposition. From (iii) of Lemma 2.2, and (ii) applied to (Z_x, A_x, S_x), it follows that $|S_x| \leq (p+1)|A_x|$, and, similarly, $|S_y| \leq (q+1)|A_y|$. Multiplying these two inequalities we get $|S_x||S_y| \leq (p+1)(q+1)|A_x||A_y|$. Hence, using also Lemma 2.1, we get $|S| \leq (p+1)(q+1)|A_x||A_y|$. However, since the scheme is regular we must have $|S| = (p+1)(q+1)|A|$ (by (i) and (ii) of Lemma 2.2), and this implies that $|A| \leq |A_x||A_y|$. Equality would force the intermediate equality of Lemma 2.1, hence the rectangularity of S. $\square$

3. A regularity theorem

In this section we clarify the regularity of UR schemes with rectangular sets of nodes. Again, the result is more general (and this is useful in examples [2]).

Given $A \subset \mathbb{N}^2$, we construct a lower set $S_y(A)$ by moving A downwards (parallel to the OY axis), and then to the left. Here is the more detailed description. We cover A with lines $l_0, \ldots, l_k$ parallel to the OY axis, counted so that

$$|l_0 \cap A| \geq \ldots \geq |l_k \cap A|$$

and we mark the points of A on these lines. On each l_i, we move the points of $A \cap l_i$ downwards until they occupy the first positions with non-negative integer coordinates (if l_i corresponds to the equation $x = \alpha_i$, then the new points will be $(\alpha_i, 0), (\alpha_i, 1), \ldots (\alpha_i, k_i)$, where $k_i = |A \cap l_i| - 1$). Next, we move each line l_i over the line $\{x = i\}$. The new positions occupied by the elements of A will define a lower set denoted $S_y(A)$. With the previous notations, $S_y(A)$ consists of the pairs (i, j) with $0 \leq i \leq k$, $0 \leq j \leq k_i$. The set $S_x(A)$ is defined analogously by interchanging the roles of the X- and Y-axes.

Remark 3.1. A is rectangular if and only if both $S_y(A)$ and $S_x(A)$ are rectangular. However, it may happen that $S_y(A)$ is rectangular without A being rectangular. An example is drawn in Figure 1 (the crosses mark the points of A).

To state the general result, we need one more notation: given an integer α, we put

$$A_y[\alpha] = \{\beta : (\alpha, \beta) \in A\}.$$

We then have the following generalization of the theorem in the introduction.

Theorem 3.1. *If Z is a (p, q)-rectangular set of nodes, and A has the property that $S_y(A)$ is (s, t)-rectangular (in particular, if A is (s, t)-rectangular), then the scheme (Z, A, S) is regular if and only if*

(i) *$S = R(p', q')$, with $p' = (s+1)(p+1) - 1, q' = (t+1)(q+1) - 1$;*
(ii) *the univariate scheme (Z_x, A_x, S_x) is regular;*
(iii) *all the univariate schemes $(Z_y, A_y[\alpha], S_y)$, with $\alpha \in A_x$, are regular.*

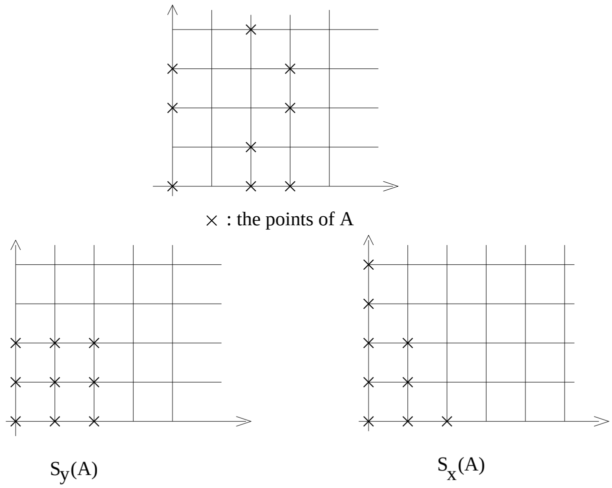

Figure 1

Proof: From Proposition 2.1 we know that the regularity of the scheme implies (i) of our theorem. Hence we have to prove that, if $S = R(p', q')$, and A is as in the statement, then the regularity of (Z, A, S) is equivalent to (ii) and (iii). For this we compute the determinant $D(Z, A, S)$ associated to the system (1.1). But first, note that $S_y(A)$ is (s,t)-rectangular is equivalent to saying that

$$A_x = \{\alpha : (\alpha, 0) \in A\}$$

has $(s+1)$ distinct elements, and that the sets

$$A_y[\alpha] = \{\beta : (\alpha, \beta) \in A\}, \quad \alpha \in A_x$$

all have the $(t+1)$ elements. In other words, $S_y(A)$ is rectangular if and only if A is of form

$$A = \{(\alpha_i, \beta_i^j) : 0 \le i \le s, 0 \le j \le t\},$$

where all the α_i's, as well as all the β_i^j for each i, are distinct. Then $A_x = \{\alpha_0, \ldots, \alpha_s\}$, and $A_y[\alpha_i] = \{\beta_i^0, \ldots, \beta_i^t\}$. We will use this description of A.

The computation of the determinant $D(Z, A, S)$ will be based on several more general remarks. For any matrix $\mathcal{A}$, we denote by $c(\mathcal{A})$, and $l(\mathcal{A})$, the number of its columns, and of its rows, respectively. For any two matrices M, and $\mathcal{A} = (a_{i,j})$, we consider the "tensor product" matrix $M \otimes \mathcal{A}$ which is equal to

$$M \otimes \mathcal{A} = \begin{pmatrix} a_{1,1}M & a_{1,2}M & \ldots & a_{1,c(\mathcal{A})}M \\ a_{2,1}M & a_{2,2}M & \ldots & a_{2,c(\mathcal{A})}M \\ \ldots & \ldots & \ldots & \ldots \\ a_{l(\mathcal{A}),1}M & a_{l(\mathcal{A}),2}M & \ldots & a_{l(\mathcal{A}),c(\mathcal{A})}M \end{pmatrix}.$$

Note that $l(M \otimes \mathcal{A}) = l(M)l(\mathcal{A})$, $c(M \otimes \mathcal{A}) = c(M)c(\mathcal{A})$. With this notation, remark that the matrix $M(Z, A, S)$ of the interpolation problem (and whose determinant is $D(Z, A, S)$) is, up to a re-arrangement of its lines and columns, of type

$$M(Z, A, S) = \begin{pmatrix} M_0 \otimes \mathcal{A}_0 \\ M_1 \otimes \mathcal{A}_1 \\ \dots \\ M_s \otimes \mathcal{A}_s \end{pmatrix}$$

where the matrices M_i and $\mathcal{A}_i$ are defined as follows. To describe $\mathcal{A}_i$, we consider the row

$$l_x(x) = (1 \ x \ \dots \ x^{p'})$$

and the rows of $\mathcal{A}_i$ will be

$$\frac{\partial^\alpha l_x}{\partial x^\alpha}(x_0), \dots, \frac{\partial^\alpha l_x}{\partial x^\alpha}(x_p),$$

with $\alpha = \alpha_i$. Hence

$$c(\mathcal{A}_i) = p' + 1, l(\mathcal{A}_i) = p + 1.$$

To describe M_i, we consider the row $l_y(y) = (1 \ y \ \dots \ y^{q'})$, and the rows of M_i will be

$$\begin{matrix} \frac{\partial^{\beta_i^0} l_y}{\partial y^{\beta_i^0}}(y_0), & \dots, & \frac{\partial^{\beta_i^0} l_y}{\partial y^{\beta_i^0}}(y_q) \\ \dots & \dots & \dots \\ \frac{\partial^{\beta_i^t} l_y}{\partial y^{\beta_i^t}}(y_0), & \dots, & \frac{\partial^{\beta_i^t} l_y}{\partial y^{\beta_i^t}}(y_q) \end{matrix}$$

We now need one more notation. For square matrices $\mathcal{M}_i$, $0 \le i \le k$ (k is any non-negative integer), we consider

$$\operatorname{diag}(\mathcal{M}_0, \dots, \mathcal{M}_k) = \begin{pmatrix} \mathcal{M}_0 & 0 & \dots & 0 \\ 0 & \mathcal{M}_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathcal{M}_k \end{pmatrix},$$

while for any square matrix $\mathcal{M}$ we put

$$\operatorname{diag}_k(\mathcal{M}) = \operatorname{diag}(\underbrace{\mathcal{M}, \dots, \mathcal{M}}_{k}).$$

One clearly has $\det(\operatorname{diag}(\mathcal{M}_0, \dots, \mathcal{M}_k)) = \prod_i \det(\mathcal{M}_i)$. With these, the tensor product of a square matrix M (with m lines and m columns), with an arbitrary matrix $\mathcal{A}$ is

$$M \otimes \mathcal{A} = \operatorname{diag}_{l(\mathcal{A})}(M)(I_m \otimes \mathcal{A}).$$

Also, for any matrices $\mathcal{M}_i$, $\mathcal{A}_i$, $0 \leq i \leq s$, with

$$c(\mathcal{M}_i) = l(\mathcal{A}_i), \ \sum l(\mathcal{M}_i) = c(\mathcal{A}_0) = \ldots = c(\mathcal{A}_s),$$

one has

$$\begin{pmatrix} \mathcal{M}_0 \otimes \mathcal{A}_0 \\ \mathcal{M}_1 \otimes \mathcal{A}_1 \\ \cdots \\ \mathcal{M}_s \otimes \mathcal{A}_s \end{pmatrix} = \mathrm{diag}(\mathcal{M}_0, \ldots, \mathcal{M}_s) \begin{pmatrix} \mathcal{A}_0 \\ \mathcal{A}_1 \\ \cdots \\ \mathcal{A}_s \end{pmatrix}.$$

Coming back to our determinant, we apply the previous formula to $\mathcal{M}_i = \mathrm{diag}_{p+1}(M_i)$, $\mathcal{A}_i = I_{q'+1} \otimes \mathcal{A}_i$, and we get (up to a sign)

$$D(Z, A, S) = \prod_i \det(\mathrm{diag}_{p+1}(M_i)) \det \begin{pmatrix} I_{q'+1} \otimes \mathcal{A}_0 \\ \cdots \\ I_{q'+1} \otimes \mathcal{A}_s \end{pmatrix}.$$

After a rearrangement of the lines and of the columns, the last matrix is precisely $I_{q'+1} \otimes M(Z_x, A_x, S_x)$, hence has determinant $D(Z_x, A_x, S_x)^{q'+1}$. Also, since $M_i = M(Z_y, A_y(\alpha_i), S_y)$, we deduce that, up to a sign,

$$D(Z, A, S) = (\prod_i D(Z_y, A_y(\alpha_i), S_y))^{p+1} D(Z_x, A_x, S_x)^{q'+1}.$$

This clearly implies the assertion in the statement. □

Example 1. The usefulness of Theorem 3 is best seen when combined with other regularity criteria which allow further reductions (e.g. moving the elements of A on the coordinate axes, or elimination of certain points of A). For such examples, we refer to [2]. Here we look at the case where the set A is the one in the picture (Figure 1) and $p = q = 1$. As shown in the picture, $S_y(A)$ is $(2, 2)$-rectangular, hence we can use the version of Theorem 3.1 obtained by replacing the role of the coordinate axes. Condition (i) of the theorem forces $S = R(5, 5)$. On the other hand, all the univariate schemes corresponding to (ii) and (iii) are unidimensional with two nodes, hence their regularity is equivalent to the Polya conditions (see [1], [3], [4]) which, in turn, are clearly satisfied. In conclusion, for a $(1, 1)$-rectangular set of nodes Z, and a lower set S, the scheme (Z, A, S) is regular if and only if $S = R(5, 5)$.

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1 Decembrie 1918 University, Alba Iulia, Romania
E-mail: crainic@math.uu.nl

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