On a question of E.A. Michael

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Abstract. A negative answer to a question of E.A. Michael is given: A convex G_δ-subset Y of a Hilbert space is constructed together with a l.s.c. map $Y \to Y$ having closed convex values and no continuous selection.

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In [1] E.A. Michael has proved the following fundamental theorem: *Let $F : X \to B$ be a lower semicontinuous multivalued mapping of a paracompact space X into a Banach space B. Let the values of F be convex and closed. Then the mapping F has a continuous selection.*

In [2] E.A. Michael has asked: *Let Y be a convex G_δ-subset of a Banach space B. Does then every lower semicontinuous mapping $F : X \to Y$ of a paracompact space X with convex closed values in Y have a continuous selection?* V.G. Gutev proved in [3] that the answer is affirmative when X is a countably dimensional metric space or a strongly countably dimensional paracompact space. In [4] V.G. Gutev and V. Valov proved that the answer is affirmative for a paracompact C-space X. See also [5].

Here we will construct an example giving a negative answer. We will construct a lower semicontinuous mapping $F : Y \to Y$ of a convex G_δ-subset Y of Hilbert space into itself with convex closed values in Y, which has no continuous singlevalued selection.

Example. We start with the space $P([0,1])$ of all probability measures on the segment $[0,1]$. We identify a measure $m \in P([0,1])$ with the corresponding linear functional $C([0,1]) \to \mathbb{R}$ on the space $C([0,1])$ of real continuous functions on the segment $[0,1]$. So we associate with a measure $m \in P([0,1])$ a point of the Tychonoff product $\prod\{[-\|\phi\|, \|\phi\|] : \phi \in C([0,1])\}$. Conditions defining probability measures describe closed subset of the Tychonoff product. Then the defined topological space $P([0,1])$ is compact. The sets

$$O(m_0; \phi_1, \dots, \phi_k; \varepsilon) = \{m : m \in P([0,1]),\ |m(\phi_i) - m_0(\phi_i)| < \varepsilon, \quad i = 1, \dots, k\},$$

where $\phi_1, \dots, \phi_k \in C([0,1])$ and $\varepsilon > 0$, give us a base at a point $m_0 \in P([0,1])$. There exists a countable dense subset W of the space of continuous functions on

$[0,1]$. The mapping which associates with a measure m the sequence $\{m(w) : w \in W\}$ maps $P([0,1])$ into a countable product of segments, which can be considered as the Hilbert cube embedded in a Hilbert space. This mapping keeps the convex structure. So we may consider $P([0,1])$ as a subset of a Banach space. Denote by ρ an arbitrary metric on $P([0,1])$.

There exists a proper convex G_δ-subset Y of the space $P([0,1])$ of all probability measures on the segment $[0,1]$ which contains all Dirac measures, see [6]. The set Y may be constructed as follows. Let λ denote the Lebesgue measure. Let us denote by A_k, $k = 1, 2, \dots$, the set of all points $m \in P([0,1])$, satisfying the condition: *There exists a point $n \in P([0,1])$, such that $\rho(n, \lambda) \geq 2^{-k}$ and the segment $[m, n]$ contains λ.* It is easy to show that the sets A_k are closed, the set $Y = P([0,1]) \setminus \bigcup\{A_k : k = 1, 2, \dots\}$ is convex, contains all Dirac measures and does not contain the measure λ.

The mapping $H_0 : P([0,1]) \to [0,1]$ which associates with a measure its support is lower semicontinuous. So the mapping $H_1 : P([0,1]) \to Y$ which associates with a measure m the set of all Dirac measures whose supports lie in $H_0(m)$ is lower semicontinuous. So the mapping $H_2 : P([0,1]) \to Y$ which associates with a measure m the convex hull of $H_1(m)$ is lower semicontinuous. So the mapping $H_3 : P([0,1]) \to Y$ which associates with a measure m the closure of $H_2(m)$ is lower semicontinuous. The values of the mapping H_3 are convex and closed in Y ($H_3(m) = [H_2(m)]_Y = [H_2(m)]_{P([0,1])} \cap Y$, where $[H_2(m)]_Y$ denotes the closure of $H_2(m)$ in Y and $[H_2(m)]_{P([0,1])}$ denotes the closure of $H_2(m)$ in $P([0,1])$).

Denote by $\Delta(a_0, \dots, a_n)$ the set of all measures whose supports lie in the finite set $\{a_0, \dots, a_n\}$. It is homeomorphic to an n-dimensional simplex. The mappings H_2 and H_3 associate with a point p of the simplex the minimal face which contains p. So for every selection h and for every point p of the boundary β of the simplex $\Delta(a_0, \dots, a_n)$, the segment connecting the points p and $h(p)$ lies in β. So the identity mapping $i : \beta \to \beta$ and $h\big|_\beta : \beta \to \beta$ are homotopic. So the degree of the mapping $h\big|_\beta$ is equal to 1. So the mapping $h\big|_{\Delta(a_0,\dots,a_n)}$ is surjective. See [7].

So the image I of a selection h must contain the set of all measures with finite supports. This set is dense in $P([0,1])$. But I is compact, so $I = P([0,1])$. On the other hand $I \subset Y$, a contradiction.

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