On introduction of two diffeomorphism invariant Colombeau algebras

Jiří Jelínek

Abstract. Equivalent definitions of two diffeomorphism invariant Colombeau algebras introduced in [7] and [5] (Grosser et al.) are listed and some new equivalent definitions are presented. The paper can be treated as tools for proving in [8] the equality of both algebras.

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In [4] a diffeomorphism invariant Colombeau-type algebra was proposed. Such an algebra was consistently introduced in [7], then the authors of [5] have very carefully examined it and, in addition to this algebra denoted by $\mathcal{G}^{\mathrm{d}}$, they have introduced another diffeomorphism invariant Colombeau algebra $\mathcal{G}^2$, apparently larger than $\mathcal{G}^{\mathrm{d}}$ and more close to the algebra that Colombeau and Meril intended in [4]. However, it was not discovered that these two algebras are identical. Thanks to this equality, we can use the simpler definition of $\mathcal{G}^{\mathrm{d}}$ knowing that we do not loose generality. As the proof of equality of both algebras is rather complicated, we postpone it in a separate paper [8]. In this paper, we recapitulate basic definitions and notations and give new equivalent definitions of these algebras. Although the aim of this paper is to give tools for proving the identity $\mathcal{G}^2 = \mathcal{G}^{\mathrm{d}}$, the transparent list of equivalent definitions can be useful also for readers that do not take interest in this identity. E.g. the condition (0°) in §8 discovered by the authors of [5] is a surprisingly simple tool for verifying that a representative is negligible: in [5] the equivalence is proved for $\mathcal{E}^{\mathrm{d}}_M$, here for $\mathcal{E}^2_M$, too.

Basic definitions and notations

We will use mostly the same notations as in [7], [5]. In [5, p. 14], operators T_x, S_ε on $\mathscr{D}$ and T on $\mathscr{D} \times \mathbb{R}^d$ are introduced: If φ is a test function on an

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Euclidean space $\mathbb{R}^d$, $x \in \mathbb{R}^d$, $\varepsilon > 0$, then the functions $T_x\varphi$ and $S_\varepsilon\varphi$ on $\mathbb{R}^d$ and $T(\varphi, x) \in \mathscr{D} \times \mathbb{R}^d$ are defined as follows:

$$T_x\varphi(y) := \varphi(y - x)\,, \quad S_\varepsilon\varphi(y) := \varepsilon^{-d}\varphi\big(\tfrac{y}{\varepsilon}\big)\,, \quad T(\varphi, x) := (T_x\varphi, x).$$

Thanks to this notation we do not need to use Colombeau's notation φ_ε meaning $S_\varepsilon\varphi$.

We deal with test functions $\varphi \in \mathscr{D}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is an open set. The notation $\mathcal{A}_q(\Omega)$ has its usual sense by Colombeau and we write $\mathcal{A}_q$ instead if Ω is clear from the context or not important. We denote $\mathcal{A} := \mathcal{A}_0 - \mathcal{A}_0 = \big\{\varphi \in \mathscr{D}\,;\ \int \varphi = 0\big\}$. The topologies on $\mathcal{A}_q$ and $\mathcal{A}$ are induced by $\mathscr{D}$.

Note that in [7] a different formalism is used assigning representatives to a generalized function. In [5] this is called J-formalism unlike Colombeau's C-formalism: A function $(\varphi, x) \mapsto R(\varphi, x)$ is considered in [7] to be a representative of a generalized function in the case when $R \circ T : \big\{(\varphi, x) \mapsto R(T_x\varphi, x)\big\}$ is a representative of this generalized function in Colombeau's sense. The new formalism is convenient when dealing with generalized functions on a $\mathscr{C}^\infty$ manifold different from $\mathbb{R}^d$ and is used e.g. in [6]. In this paper we will use the classical Colombeau's formalism, because it is sufficient for our aim and the calculations will be simpler. However, while referring to [7], a change of formalism is needed.

§**1. Definition.** If R is a representative, we denote by $(R)_\varepsilon$ or simply by R_ε the function $(R)_\varepsilon(\varphi, x) = R(S_\varepsilon\varphi, x)$ while in [7] $(R)_\varepsilon(\varphi, x) = R(T_x \circ S_\varepsilon\varphi, x)$ as a consequence of another formalism and thus, for a given generalized function, the notation $(R)_\varepsilon(\varphi, x)$ has the same meaning in both formalisms.

In this paper a representative R of a generalized function is a function of specific properties (see below) on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$, while in [5] (similarly in [7] with another formalism) a representative is defined only on $U(\Omega) := \{(\varphi, x);\ \varphi \in \mathcal{A}_0(\Omega - x),\ x \in \Omega\}$. This is legitimized by the following

Proposition. *Every generalized function in $\mathcal{G}^{\mathrm{d}}(\Omega)$ resp. $\mathcal{G}^2(\Omega)$ with a representative $R_0 \in \mathcal{E}^{\mathrm{d}}_M(\Omega)$ resp. $\in \mathcal{E}^2_M(\Omega)$ defined on $U(\Omega)$ has another representative $R \in \mathcal{E}^{\mathrm{d}}_M(\Omega)$ resp. $\in \mathcal{E}^2_M(\Omega)$ that is defined on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$. The equivalence means that after restriction on $U(\Omega)$ it is $R - R_0 \in \mathcal{N}$.*

The proof is below.

Remarks. For representatives defined on $U(\Omega)$ moderateness is defined in [5, 7.2 resp. 17.1] while for representatives defined on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$ the definitions are below §4, (1°) resp. §7 (1°). However these definitions are the same or equivalent. The only difference is that in the former case on a given bounded set resp. path in $\mathscr{C}^\infty\big(\Omega \to \mathcal{A}_0(\mathbb{R}^d)\big)$ and a given $K \Subset \Omega$ (means compact subset), $(R_0)_\varepsilon(\varphi, x)$ is only defined for sufficiently small ε, while in the latter case this is defined always.

So for moderateness of a representative defined on $\mathcal{A}_0(\mathbb{R}^d)\times\Omega$, only its values on $U(\Omega)$ matter.

Proposition says that we obtain the same algebra if we admit only representatives defined on $\mathcal{A}_0(\mathbb{R}^d)\times\Omega$. For $\mathcal{G}^{\mathrm{d}}$ this follows directly from [7, Theorem 21]. In our formalism this theorem can be formulated as follows. For a family of numbers $\{q_i\in\mathbb{N}_0\}_{i\in I}$ and an open covering $\{V_i\}_{i\in I}$ of Ω with $V_i\subseteq\Omega$ denote

$$\begin{aligned}\mathfrak{V}\big((V_i,q_i)_{i\in I}\big) &:= \big\{\,(\varphi,x)\,;\,\exists\, i\in I \ \text{ such that } \ x\in V_i,\ \varphi\in\mathcal{A}_{q_i}(V_i-x)\big\}\\ &= \bigcup_i U(V_i)\cap\mathcal{A}_{q_i}\,.\end{aligned}$$

If R_0 is a $\mathscr{C}^\infty$ function on $\mathfrak{V}\big((V_i,q_i)_{i\in I}\big)$, moderate in a certain way defined in that theorem, then there is a moderate smooth function R on $\mathcal{A}_0(\mathbb{R}^d)\times\Omega$ coinciding with R_0 on some set $\mathfrak{V}\big((V_i',q_i')_{i\in I'}\big)$ of the above type.

It follows from this assertion that R and R_0 define the same generalized function. There is a lack in [7] that the notion of smoothness on $\mathfrak{V}\big((V_i,q_i)_{i\in I}\big)$ is not explained and with the formalism used in [7] we cannot apply the differentiation theory used there. Here we can follow the method of [5, Chapter 5] for defining differentials of R_0 on $U(V_i)\cap\mathcal{A}_{q_i}$ $(\forall\, i)$. The appropriate topology on $U(V_i)$ is τ_2 but we can simply choose the topology τ_1 induced by $\mathscr{D}(\mathbb{R}^d)\times\Omega$. This follows from the fact that we can choose a finer covering $\{V'_{i'}\}_{i'\in I'}$ such that every $\overline{V'_{i'}}$ is compact in some V_i. On the other hand, in [7] with the formalism used there we use no tools to define differentials on $\mathfrak{V}$, but fortunately it is not needed to do so. It suffices to suppose (approach of [9]) that R_0 is smooth on smooth curves in $\mathfrak{V}$ (see Remark 3 below) because the only property concerning smoothness we need is: the composition of smooth mappings on smooth curves is smooth on smooth curves.

Theorem 21 in [7] is stronger than we need. $q_i=0$ would satisfy our task and the reasoning would be much simpler. The authors of [5] used this method in Chapter 8 for verifying chief properties of $\mathcal{G}^{\mathrm{d}}$ and by way they proved our assertion, too. More precisely: The representative R obtained on $U(\Omega)$ while proving S2 is in fact defined on $\mathcal{A}_0(\mathbb{R}^d)\times\Omega$. R is even continuously infinitely differentiable, but we will not use this result; we only note that the same algebras can be constructed with continuously infinitely differentiable representatives.

In [5] this method is not applied to $\mathcal{G}^2$. So we are going to give in brief a proof that is a copy of the proof in [5, Chapter 8]. The details are left to the reader.

Proof of the proposition for $\mathcal{G}^2$: Choose a locally finite covering $\big(W_j\big)_{j\in\mathbb{N}}$ of Ω with $\overline{W}_j\Subset\Omega$ and a partition of unity $(\chi_j)_{j\in\mathbb{N}}$ subordinate to $\big(W_j\big)_{j\in\mathbb{N}}$. Moreover, for each $j\in\mathbb{N}$ choose functions $\vartheta_j\in\mathscr{D}$, $\vartheta_j=1$ on a neighbourhood of $\overline{W}_j$, and

$\psi_j \in \mathcal{A}_0(W_j)$. The map $\pi_j : \mathcal{A}_0(\mathbb{R}^d) \to \mathcal{A}_0(\Omega)$ defined by

$$\pi_j(\varphi) := \vartheta_j \varphi + \big(1 - \textstyle\int \vartheta_j \varphi\big)\, \psi_j$$

is smooth on $\mathcal{A}_0(\mathbb{R}^d)$ and identical on $\mathcal{A}_0(W_j)$. Then for each j the function R_j on $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$ defined by

$$R_j(\varphi, x) := \begin{cases} \chi_j(x) R_0\big(T_{-x} \circ \pi_j \circ T_x(\varphi), x\big) & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega \end{cases}$$

is smooth. To show that $R := \sum R_j$ is moderate we first note that in a neighbourhood of any $K \Subset \Omega$ only finitely many R_j do not vanish identically, so it is enough to show that one single R_j is moderate. For this, it is enough to show that the function (element of $\mathcal{E}(W_j)$ by the following definition)

$$\mathcal{A}_0(\mathbb{R}^d) \times W_j \ni (\varphi, x) \mapsto R_0\big(T_{-x} \circ \pi_j \circ T_x(\varphi), x\big)$$

is moderate. If $W \subset \Omega$ is open and R_0 is defined on $U(\Omega)$, following Grosser et al. [5] we denote by $R_0|_W$ the restriction of R_0 to $U(W)$. We left to the reader to prove that $R_0|_W$ is moderate provided R_0 is moderate. To see that $R_0\big(T_{-x} \circ \pi_j \circ T_x(\varphi), x\big)$ is moderate, it is enough to realize that for a given compact $K \Subset W_j$ and a given bounded path

$$\big\{(\varphi_x^\varepsilon)_{x \in \Omega};\, \varepsilon \in\,]0,1]\big\} \subset \mathscr{C}^\infty\big(\Omega \to \mathcal{A}_0(\mathbb{R}^d)\big),$$

$\forall x \in K$ and ε small enough, we have $S_\varepsilon \varphi_x^\varepsilon \in \mathcal{A}_0(W_j - x)$, so $T_x S_\varepsilon \varphi_x^\varepsilon \in \mathcal{A}_0(W_j)$, where π_j is identical. Thus $R_0\big(T_{-x} \circ \pi_j \circ T_x(\varphi), x\big) = R_0(\varphi, x)$ for $\varphi = \varphi_x^\varepsilon$, $R(\varphi, x) = R_0(\varphi, x)$ is moderate and $R - R_0$ is negligible. $\square$

§**2. Definition.** We denote by $\mathcal{E}[\Omega]$ or $\mathcal{E}(\Omega)$ the space of functions

$$\begin{aligned} \mathcal{A}_0(\mathbb{R}^d) \times \Omega &\to \mathbb{C} \\ (\varphi, x) &\mapsto R(\varphi, x) \end{aligned}$$

that are $\mathscr{C}^\infty$ simultaneously in both variables. As we do not use Schwartz's notation $\mathcal{E}(\Omega)$ for $\mathscr{C}^\infty(\Omega)$, we can use the notation $\mathcal{E}(\Omega)$ (unlike Colombeau) with this meaning. Like in [7], we denote by $\mathbf{d}R$ the total differential of the function R of two variables and by $\mathrm{d}R$ the partial differential with respect to the first variable running mostly over a part of $\mathcal{A}_0$. The derivatives with respect to the second variable are denoted ∂^α and we distinguish them from $\big(\frac{\partial}{\partial x}\big)^\alpha$ e.g. if the first variable depends on x, too. So we do not use indices for distinguishing partial differentials and we can use them to indicate the direction of the derivative; e.g. $\mathrm{d}^2_{\psi_1, \psi_2} R(\varphi, x)$ is the same as $\mathrm{d}^2 R(\varphi, x)[\psi_1, \psi_2]$. Moreover, if we denote $\boldsymbol{\psi} = (\psi_1, \psi_2)$, then $\mathrm{d}^2_{\boldsymbol{\psi}} R(\varphi, x)$ denotes the same, as well. If the function is given as a composition, e.g. $R(S(\varphi), x)$, then $\mathrm{d}R(S(\varphi), x)$ signifies the differential of this composition and is thus distinguished from $(\mathrm{d}R)(S(\varphi), x)$.

Remarks. There are divers notions of differentiability of mappings of locally convex spaces; some of them are equivalent in many cases investigated in this paper: we mostly deal with $\mathscr{C}^\infty$ functions defined on an open part of a subspace of $\mathscr{D}$ or $\mathscr{D} \times \mathbb{R}^d$. Without explicitly mentioned, "differential" means the Fréchet differential: If F is a vector-valued function defined on an open part of a locally convex space $\mathscr{F}$, the Fréchet differentiability of F at $\varphi \in \mathscr{F}$ means that $\mathrm{d}F(\varphi)$ is a continuous linear mapping and

$$\lim_{t\searrow 0} \frac{F(\varphi + t\psi) - F(\varphi)}{t} = \mathrm{d}F(\varphi)[\psi] \tag{1}$$

uniformly if ψ runs over any bounded subset $\mathscr{B}$ of $\mathscr{F}$.

Note that a differentiable mapping (at every point of its domain) need not be continuous, but it is continuous (see Yamamuro [13, §1.7]) in the case $\mathscr{F}$ is metrizable. Following [1] we denote by $\mathscr{C}^n$ the class of differentiable mappings up to order n, unlike [13] where in addition the continuity of the differentials is required. For a $\mathscr{C}^\infty$ mapping on a metrizable space both notions coincide.

The differential of a higher order at a fixed point is a hypo-continuous multilinear mapping. If $\mathscr{F}$ is a Fréchet space, such a mapping is (jointly) continuous (Robertson A.P.-Robertson W.J. [11, VII, Proposition 11]) and evidently this holds for (LF)-spaces, too.

Some authors prefer other notions of differentiability. In Colombeau [1] Silva differential and Silva differential in enlarged sense are introduced and is proved (1.4.7, 1.4.8) that for $\mathscr{C}^\infty$ both notions coincide if $\mathscr{F}$ is a co-Schwartz locally convex space. $\mathscr{D}$ is even co-nuclear, see Pietsch [10, 6.2.6, 4.1.6]. Silva differential in enlarged sense is by definition the Fréchet one with the only exception that $\mathrm{d}F$ is only bounded on bounded sets (not necessarily continuous). However on a bornological space $\mathscr{F}$ (our case) such a mapping is separately continuous; in our case continuous. The authors of [5] choose a direct definition of $\mathscr{C}^\infty$ by Kriegl-Michor [9]: F is by definition $\mathscr{C}^\infty$ iff for every $\mathscr{C}^\infty$ curve C in the domain of F, the curve $F \circ C$ is $\mathscr{C}^\infty$. It is said in Chapter 4 that this notion of smoothness is weaker than Silva-smoothness but is equivalent if $\mathscr{F}$ is a complete Montel space. Hence in our case all the above mentioned notions of $\mathscr{C}^\infty$ smoothness coincide.

The last definition of smoothness has the advantage that it can also be applied when the domain of F is a part of a linear space with a non-induced topology. The domain even need not be open. We distinguish this case saying that F is smooth on smooth curves, regardless if there is any non-trivial curve in its domain. However only in the case the domain is an open subset of $\mathscr{F}$ with the induced topology, it is proved in Kriegl-Michor [9] that F has smooth differentials; only in that case we have the above mentioned equivalence of smoothness.

The following proposition says in brief that continuous differentials on a Fréchet space are locally equi-continuous; this can be easily generalized for mappings into a locally convex space, but we do not need such a generalization. The formulation is a bit complicated in order to correspond to our purposes.

Proposition. *Let $\mathscr{F}$ be a Fréchet space, $\omega \in \mathscr{F}$, $\mathscr{A} \subset \mathscr{F}$ a closed vector subspace* (*with the induced topology*), *F a complex function on an open neighbourhood of ω in the affine space $\omega + \mathscr{A}$, continuously differentiable up to order L ($L \in \mathbb{N}$). Then there is a neighbourhood $\mathcal{U}$ of zero in $\mathscr{A}$ such that for all $\varphi \in \omega + \mathcal{U}$ and $\psi_\ell \in \mathcal{U}$, ($\ell = 1, \ldots, L$) it is $|\mathrm{d}^L_{\psi_1,\ldots,\psi_L} F(\varphi)| \le 1$.*

More generally, if $\mathscr{K} \Subset \omega + \mathscr{A}$ is a compact contained in the domain of F, $L \in \mathbb{N}$, under the same hypotheses there is a neighbourhood $\mathcal{U}$ of zero in $\mathscr{A}$ such that for all $\varphi \in \mathscr{K} + \mathcal{U}$ and $\psi_\ell \in \mathcal{U}$, ($\ell = 1, \ldots, L$) it is $|\mathrm{d}^L_{\psi_1,\ldots,\psi_L} F(\varphi)| \le 1$.

Proof by induction: We change the last inequality with $|\mathrm{d}^L_{\psi_1,\ldots,\psi_L} F(\varphi)| \le 1 + |F(\omega)|$. This is equivalent and holds evidently for $L = 0$, too. Let $L \in \mathbb{N}$ be given, and let (induction assumption) for any $\mathscr{C}^{L-1}$ function F it is $|\mathrm{d}^{L-1}_{\psi_1,\ldots,\psi_{L-1}} F(\varphi)| \le 1 + |F(\omega)|$ under the hypotheses of the proposition. Now, let F be a $\mathscr{C}^L$ function, $\omega \in \mathscr{F}$. Choose a basis of absolutely convex neighbourhoods of zero $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \ldots$ in $\mathscr{A}$ and denote (for $n \in \mathbb{N}$)

$$\mathscr{B}_n := \Big\{\psi \in \mathscr{A};\ \forall\, \varphi \in \omega + \mathcal{U}_n,\ \psi_1, \ldots, \psi_{L-1} \in \mathcal{U}_n : |\mathrm{d}^L_{\psi_1,\ldots,\psi_{L-1},\psi} F(\varphi)| \le 1 + |F(\omega)|\Big\}.$$

$\mathscr{B}_n$ are absolutely convex and closed. $\mathrm{d}_\psi F$ is a $\mathscr{C}^{L-1}$ function, hence by the induction assumption

$$\forall\, \psi \in \mathscr{A} \quad \exists\, \mathcal{U}_n \quad \forall\, \varphi \in \omega + \mathcal{U}_n,\ \psi_1, \ldots, \psi_{L-1} \in \mathcal{U}_n : |\mathrm{d}^L_{\psi_1,\ldots,\psi_{L-1},\psi} F(\varphi)| \le 1.$$

This means $\bigcup \mathscr{B}_n = \mathscr{A}$. It is known for Fréchet spaces that in that case some $\mathscr{B}_n$ is a neighbourhood of zero in $\mathscr{A}$, what we wanted to prove. (Proof: Some $\mathscr{B}_n$ is not nowhere-dense because a Fréchet space is not of the first category. As $\mathscr{B}_n$ is close, it is a neighbourhood of some point. Being absolutely convex, it is a neighbourhood of zero.)

Now we are going to prove the second part. As $\mathscr{K}$ is compact, it can be covered with a finite number of sets $\omega_m + \frac{1}{2}\mathcal{U}_m$ where $\mathcal{U}_m$ is an absolutely convex open neighbourhood of zero in $\mathscr{A}$ assigned to ω_m by the first part of Proposition. Then $\mathcal{U} := \bigcap \mathcal{U}_m$ is the desired neighbourhood. □

Corollary. *Under the same hypotheses, if $\lim\limits_{n\to\infty} \varphi_n = \varphi$ in $\omega + \mathscr{A}$ and $\lim\limits_{n\to\infty} \psi_{\ell n} = \psi_\ell$ in $\mathscr{A}$ ($\ell = 1, \ldots, L$), then $\lim\limits_{n\to\infty} \mathrm{d}^L_{\psi_{1n},\ldots,\psi_{Ln}} F(\varphi_n) = \mathrm{d}^L_{\psi_1,\ldots,\psi_L} F(\varphi)$.*

This holds more generally if $\mathscr{F}$ is an (LF)-space, because then the convergent sequences are contained in a Fréchet subspace of $\mathscr{F}$.

§**3. Definition.** For a locally convex space $\mathcal{F}$, we denote by $\mathscr{C}^\infty(\Omega \to \mathcal{F})$ the locally convex space of all $\mathscr{C}^\infty$ maps

$$\Phi = (\varphi_x)_{x\in\Omega} : \begin{array}{l} \Omega \to \mathcal{F} \\ x \mapsto \varphi_x \end{array}$$

with the usual topology of uniform convergence of every derivative with respect to x on every compact $K \Subset \Omega$.

Notation. The diffeomorphism invariant algebra $\mathcal{G}$ that I have defined in [7] will be denoted here following Grosser et al. [5] by $\mathcal{G}^{\mathrm{d}}$. In this paper we investigate the other algebra $\mathcal{G}^2$ as well and denote the algebra of representatives of $\mathcal{G}^{\mathrm{d}}$ resp. $\mathcal{G}^2$ by $\mathcal{E}_M^{\mathrm{d}}$ resp. $\mathcal{E}_M^2$. On the other hand, the ideal of negligible representatives for $\mathcal{G}^2$ will be denoted simply by $\mathcal{N}$ because $\mathcal{N} \cap \mathcal{E}_M^{\mathrm{d}}$ is then the ideal of negligible representatives for $\mathcal{G}^{\mathrm{d}}$.

§**4. Equivalent definitions of $\mathcal{E}_M^{\mathrm{d}}(\Omega)$.** $\mathcal{E}_M^{\mathrm{d}}(\Omega)$ is the set of all $R \in \mathcal{E}[\Omega]$ with moderate growth, which means that one of the following equivalent conditions is satisfied.

(1°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d\ \ \exists N \in \mathbb{N}$:

$$\left(\tfrac{\partial}{\partial x}\right)^{\alpha} R_\varepsilon(\varphi_x, x) = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly if $x \in K$ and $(\varphi_x)_{x\in\Omega}$ runs over any bounded subset of $\mathscr{C}^\infty\big(\Omega \to \mathcal{A}_0(\mathbb{R}^d)\big)$ (this space is the topological subspace of $\mathscr{C}^\infty\big(\Omega \to \mathscr{D}(\mathbb{R}^d)\big)$).

(2°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ k \in \mathbb{N}_0\ \ \exists N \in \mathbb{N}$:

$$\partial^\alpha \mathrm{d}^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly if $x \in K$, φ runs over any bounded subset of $\mathcal{A}_0(\mathbb{R}^d)$ and $\psi_1, \dots, \psi_k$ are in a bounded subset of $\mathcal{A}(\mathbb{R}^d)$.

(3°) $\forall\, K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ k \in \mathbb{N}_0\ \ \exists N \in \mathbb{N}\ \ \forall B \Subset \mathbb{R}^d$, $\mathscr{B}$ (bounded) $\subset \mathcal{A}_0(B)$ $\exists \mathcal{U}$ (absolutely convex open neighbourhood of zero) $\subset \mathcal{A}(B)$, $C > 0$, $C = 1$ if $k \geq 1$, $\forall x \in K$, $\varepsilon \in \,]0,1]$, $\varphi \in \mathscr{B} + \mathcal{U}$, $\psi_1, \dots, \psi_k \in \mathcal{U}$:

$$\partial^\alpha \mathrm{d}^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] \leq C\, \varepsilon^{-N}.$$

Proof of equivalences: The equivalence (1°) ⇔ (2°) is proved in [7, Theorem 17] (with another formalism) or in [5, Theorem 7.12]. (3°) ⇒ (2°) being evident, we only have to prove (3°) ⇐ (2°), first for the case $\mathscr{B}$ is a singleton, $\mathscr{B} = \{\omega\}$, $\omega \in \mathcal{A}_0(B)$. This proof is left to the reader. It could be the same or simpler than the similar proofs in §7 below for the algebra $\mathcal{E}_M^2$. □

§**5.** For the following definition of the null ideal in $\mathcal{G}^2$, we use the notion of bounded path introduced in Colombeau-Meril [4] in order to define the moderate growth and the negligibility of representatives. It is explained in [7] that a bounded path should depend on $x \in \Omega$, so sometimes its values should belong to $\mathscr{C}^\infty(\Omega \to \mathscr{D})$ rather than to $\mathscr{D}$.

Definition. A path in this paper is a mapping of the interval $]0,1]$ into a topological linear space (or its part), mostly

$$\begin{aligned} &]0,1] \to \mathscr{C}^\infty(\Omega{\to}\mathcal{A}_0) \\ \text{or}\quad &]0,1] \to \mathscr{C}^\infty(\Omega{\to}\mathcal{A}) \\ & \varepsilon \ \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega}, \end{aligned}$$

however paths with values in $\mathcal{A}_0$ or in $\mathcal{A}$ (independent of $x \in \Omega$) will be used, too. Adjectives like $\mathscr{C}^q$, $\mathscr{C}^\infty$ refer to this mapping of the variable ε. Like in [4], we use upper indices, however this will be the only case of using an upper index for a variable.

Remark. Evidently, for a locally convex space $\mathcal{F}$, a path $\varepsilon \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega} \in \mathscr{C}^\infty(\Omega{\to}\mathcal{F})$ is $\mathscr{C}^\infty$ iff the mapping $\varepsilon, x \mapsto \varphi_\varepsilon^x \in \mathcal{F}$ is $\mathscr{C}^\infty$.

Also it is useful to consider paths without any smoothness requirement. In that case a path even need not be continuous. A path is said to be *bounded* if its range is bounded; a path $\varepsilon \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega} \in \mathscr{C}^\infty(\Omega{\to}\mathcal{F})$ is bounded iff for every $K \Subset \Omega$, $\alpha \in \mathbb{N}_0^d$ the set $\Big\{\big(\frac{\partial}{\partial x}\big)^\alpha \varphi_x^\varepsilon\,;\ x \in K,\ \varepsilon \in\,]0,1]\Big\}$ is bounded in $\mathcal{F}$.

§**6. Definition.** We say (by [5], introduced in [4]) that a path

$$\varepsilon \ \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega} \in \mathscr{C}^\infty(\Omega{\to}\mathscr{D})$$

has asymptotically vanishing moments of order $N \in \mathbb{N}$ iff for every $K \Subset \Omega$ and $\beta \in \mathbb{N}_0^d$ with $1 \le |\beta| \le N$ it is

$$\sup_{x\in K}\left|\int_{\mathbb{R}^d} \varphi_x^\varepsilon(\xi) xi^\beta \,\mathrm{d}\xi\right| = O(\varepsilon^N) \qquad (\varepsilon \searrow 0).$$

For a path $\varepsilon \mapsto \varphi^\varepsilon \in \mathscr{D}$ the same means that for all $\beta \in \mathbb{N}_0^d$ with $1 \le |\beta| \le N$ it is

$$\int \varphi^\varepsilon(\xi)\xi^\beta \,\mathrm{d}\xi = O(\varepsilon^N) \qquad (\varepsilon \searrow 0).$$

In [5, Theorem 16.5] is proved (formulated only for $\mathcal{A}_0$ instead of $\mathscr{D}$): If $\varepsilon \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega} \in \mathscr{C}^\infty(\Omega{\to}\mathscr{D})$ is a bounded $\mathscr{C}^\infty$ path with asymptotically vanishing moments of order $q \ge 2$, then $\forall\,\alpha$ the path

$$\varepsilon \ \mapsto \Big(\big(\tfrac{\partial}{\partial x}\big)^\alpha \varphi_x^\varepsilon\Big)_{x\in\Omega} \in \mathscr{C}^\infty(\Omega{\to}\mathscr{D})$$

has asymptotically vanishing moments of order $q-1$.

§**7.** Now we could define the negligible ideal and then the algebra $\mathcal{G}^{\mathrm{d}}$ as the quotient algebra. However, the definition of the negligible ideal for both algebras $\mathcal{G}^{\mathrm{d}}$ and $\mathcal{G}^2$ is the same, so we defer it and define first the algebra of representatives for $\mathcal{G}^2$. This one is introduced in [5], is larger than $\mathcal{E}_M^{\mathrm{d}}$ and more closed to the algebra that Colombeau and Meril intended to introduce in [4].

Equivalent definitions of $\mathcal{E}_M^2$. If $\Omega \subset \mathbb{R}^d$ is an open set, $\mathcal{E}_M^2(\Omega)$ is defined to be the set of all elements $R \in \mathcal{E}(\Omega)$ fulfilling one of the following equivalent conditions ($\mathcal{A}_q$ means $\mathcal{A}_q(\mathbb{R}^d)$).

(1°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d\ \ \exists N \in \mathbb{N}$: for every bounded $\mathscr{C}^\infty$ path

$$\varepsilon \ \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega} \in \mathscr{C}^\infty(\Omega\to\mathcal{A}_0) \tag{2}$$

that has asymptotically vanishing moments of order N, we have

$$\big(\tfrac{\partial}{\partial x}\big)^\alpha R_\varepsilon\big(\varphi_x^\varepsilon, x\big) = O(\varepsilon^{-N}) \qquad (\varepsilon \searrow 0) \tag{3}$$

uniformly for $x \in K$.

(1′°) = condition (1°) without $\mathscr{C}^\infty$ requirement for the path $\varepsilon \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega}$. In that case the bounded path even need not be continuous with respect to ε.

(1″°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d\ \ \exists N \in \mathbb{N}$: (3) holds uniformly if $x \in K$ and (2) runs over a set of paths that are uniformly bounded and have uniformly vanishing moments.

For the following equivalent conditions (2′°) and (3′°) similar equivalent conditions like (1°)–(1″°) can be easily formulated and proved; we will not do it for the sake of brevity.

(2′°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ k \in \mathbb{N}_0\ \ \exists N \in \mathbb{N}$: for every bounded paths

$$\varepsilon \ \mapsto \varphi^\varepsilon \in \mathcal{A}_0\,, \quad \varepsilon \ \mapsto \psi_i^\varepsilon \in \mathcal{A} \quad (i = 1, 2, \ldots, k) \tag{4}$$

that all have asymptotically vanishing moments of order N, we have

$$\partial^\alpha \mathrm{d}^k R_\varepsilon(\varphi^\varepsilon, x)[\psi_1^\varepsilon, \ldots, \psi_k^\varepsilon] = O(\varepsilon^{-N}) \qquad (\varepsilon \searrow 0) \tag{5}$$

uniformly for $x \in K$.

(3′°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ k \in \mathbb{N}_0\ \ \exists N \in \mathbb{N}$: (5) holds whenever the first of bounded paths (4) has asymptotically vanishing moments of order N.

For the following equivalent definitions, we use a function V_N on $\mathcal{A}_0$ ($\forall N \in \mathbb{N}$) estimating moments up to order N. This function should satisfy:

$$\forall \mathscr{B} \text{ (bounded) } \subset \mathcal{A}_0 \quad \exists C_1, C_2 > 0 \quad \forall \varphi \in \mathscr{B} \quad \text{we have}$$

$$C_2 \sum_{\substack{\beta\in\mathbb{N}_0^d \\ 1\le|\beta|\le N}} \left|\int \xi^\beta \varphi(\xi)\,\mathrm{d}\xi\right| \ \le\ V_N(\varphi)\ \le\ C_1 \sum_{\substack{\beta\in\mathbb{N}_0^d \\ 1\le|\beta|\le N}} \left|\int \xi^\beta \varphi(\xi)\,\mathrm{d}\xi\right|. \tag{6}$$

(4°) $\forall K \Subset \Omega$, $\alpha \in \mathbb{N}_0^d$, $k \in \mathbb{N}_0$ $\exists N \in \mathbb{N}$ $\forall B \Subset \mathbb{R}^d$, $\omega \in \mathcal{A}_0(\mathbb{R}^d)$, V_N (fulfilling (6)) $\exists \mathcal{U}$ (absolutely convex open neighbourhood of zero) $\subset \mathcal{A}(B)$, $C > 0$, $C = 1$ if $k \geq 1$:

$$\left| \partial^\alpha \mathrm{d}^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] \right| \leq C\varepsilon^{-N} \tag{7}$$

whenever

$$x \in K\,,\ 0 < \varepsilon \leq 1\,,\ \varphi \in \omega + \mathcal{U}\,,\ V_N(\varphi) \leq \varepsilon^N \ \text{ and } \ \psi_1, \dots, \psi_k \in \mathcal{U}. \tag{8}$$

(5°) $\forall K \Subset \Omega$, $\alpha \in \mathbb{N}_0^d$, $k \in \mathbb{N}_0$ $\exists N \in \mathbb{N}$ $\forall B \Subset \mathbb{R}^d$, $\mathscr{B}$ (bounded) $\subset \mathcal{A}_0(\mathbb{R}^d)$, V_N (fulfilling (6)) $\exists \mathcal{U}$ (absolutely convex open neighbourhood of zero) $\subset \mathcal{A}(B)$, $C > 0$, $C = 1$ if $k \geq 1$: (7) holds whenever

$$x \in K\,,\ 0 < \varepsilon \leq 1\,,\ \varphi \in \mathscr{B} + \mathcal{U},\ V_N(\varphi) \leq \varepsilon^N \ \text{ and } \ \psi_1, \dots, \psi_k \in \mathcal{U}.$$

Remark. By §1, Definition of R_ε, we can replace the expression $\mathrm{d}^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k]$ with

$$\mathrm{d}^k R(S_\varepsilon\varphi, x)[\psi_1, \dots, \psi_k] = (\mathrm{d}^k R)(S_\varepsilon\varphi, x)[S_\varepsilon\psi_1, \dots, S_\varepsilon\psi_k].$$

This equality is a special case of the chain rule (formula for the derivation of a composition, e.g. [7, §12] or Yamamuro [13, (1.8.3)]) where the inner function S_ε is linear. In that case the sum in the chain rule has one term only containing the first differentials of the inner function $\mathrm{d}_\psi S_\varepsilon(\varphi) = S_\varepsilon(\psi)$.

Proof of equivalences: The equivalence of (1°), (1′°) and (1″°) can be easily seen (for (1°) ⇒ (1′°) see the proof of Theorem 3 in [7] or [5, 10.5] the proof of (C) ⇒ (A)).

(1°) ⇔ (2°) is said in in Grosser et al. [5, Theorem 17.4] and proved at the end of Chapter 17. The proof is based on the same proof for $\mathcal{G}^{\mathrm{d}}$ in [7].

(3′°) ⇒ (2′°) is evident. □

Proof of (2′°) ⇒ (4°): by contradiction. If (4°) does not hold for some K, α, k, take N for these K, α, k by (2′°). In non(4°) put $(k+1)N + 1$ instead of N and so get $B \Subset \mathbb{R}^d$, $\omega \in \mathcal{A}_0(\mathbb{R}^d)$ and a function V_N fulfilling (6). Choose a basis $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots$ of absolutely convex open neighbourhoods of zero in $\mathcal{A}(B)$. By non(4°), for every $j = 1, 2, \dots$ there are

$$\begin{gathered} \varepsilon_j \in \left]0,1\right],\ x_j \in K\,,\ \varphi_j \in \omega + \mathcal{U}_j \quad \text{with} \quad V_{(k+1)N+1}(\varphi_j) \leq \varepsilon_j^{(k+1)N+1} \\ \text{and} \quad \psi_{ij} \in \mathcal{U}_j \quad (i = 1, 2, \dots, k) \end{gathered} \tag{9}$$

such that

$$\tag{10} \begin{gathered} \left|\partial^\alpha \mathrm{d}^k R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1j}, \dots, \psi_{kj}]\right| > C\varepsilon_j^{-(k+1)N-1} \\ \text{where } \ C = j \ \text{ for } \ k = 0, \qquad C = 1 \ \text{ for } \ k \geq 1. \end{gathered}$$

As $\{\mathcal{U}_j\}$ is an increasing basis, we have by (9)

$$\tag{11} \lim_{j\to\infty} \varphi_j = \omega\,, \quad \lim_{j\to\infty} \psi_{ij} = 0 \quad (i = 1, 2, \dots, k).$$

Consequently, the sets $\{\varphi_j\,;\, j = 1, 2, \dots\}$, $\{\psi_{ij}\,;\, j = 1, 2, \dots\}$ $(i = 1, 2, \dots, k)$ are bounded in $\mathcal{A}(B)$. As we can take subsequences instead, we can suppose without loss of generality that either $\{\varepsilon_j\}$ has a limit $\varepsilon_0 \in\,]0, 1]$, or

$$\tag{12} \varepsilon_1 > \varepsilon_2 > \cdots \searrow 0$$

and (in both cases) $\lim x_j = x_0 \in K$. In the former case, we have by Corollary 2, due to (11),

$$\tag{13} \begin{aligned} \lim_{j\to\infty} \left|\partial^\alpha \mathrm{d}^k R_{\varepsilon_j}(\varphi_j, x_j)[\psi_{1j}, \dots, \psi_{kj}]\right| &= \left|\partial^\alpha \mathrm{d}^k R_{\varepsilon_0}(\omega, x_0)[0, \dots, 0]\right| \\ &= 0 \ \text{ if } \ k \geq 1 \qquad \text{resp.} \ \ = |\partial^\alpha R_{\varepsilon_0}(\omega, x_0)| \ \text{ if } \ k = 0. \end{aligned}$$

This contradicts (10).

Now only the case (12) remains and we can suppose without loss of generality that $\varepsilon_1 = 1$ in (12). In this case we define paths $\varepsilon \mapsto \varphi^\varepsilon$, $\varepsilon \mapsto \psi_i^\varepsilon$, $\quad (i = 1, \dots, k)$ as follows:

$$\tag{14} \begin{gathered} \varphi^\varepsilon = \varphi_j\,,\ \psi_i^\varepsilon = \varepsilon^N \cdot \psi_{ij} \quad \text{for } \ \varepsilon \in [\varepsilon_j, \varepsilon_{j-1}[\\ (j = 2, 3, \dots \ \ \text{resp.} \ \ j = 1 \ \text{ and } \ \varepsilon = 1\,). \end{gathered}$$

By (9), for $\varepsilon \in\ [\varepsilon_j, \varepsilon_{j-1}[$ we have

$$V_{(k+1)N+1}(\varphi^\varepsilon) = V_{(k+1)N+1}(\varphi_j) \ \leq \varepsilon_j^{(k+1)N+1} \ \leq \varepsilon^{(k+1)N+1}$$

and so due to (6) the path $\varepsilon \mapsto \varphi^\varepsilon$ has asymptotically vanishing moments of order $(k+1)N+1$; the more of order N. The paths are bounded. On bounded sets $\{\psi_{ij}\,;\, j = 1, 2, \dots\}$, moments are bounded, so the paths $\varepsilon \mapsto \psi_i^\varepsilon$, $(i = 1, \dots, k)$ have asymptotically vanishing moments of order N, too. On the other hand, if $\varepsilon = \varepsilon_j$, we estimate due to (10):

$$\begin{aligned} \left|\partial^\alpha \mathrm{d}^k R_\varepsilon(\varphi^\varepsilon, x_j)[\psi_1^\varepsilon, \dots, \psi_k^\varepsilon]\right| &= \left|\partial^\alpha \mathrm{d}^k R_{\varepsilon_j}(\varphi_j, x_j)[\varepsilon_j^N \psi_{1j}, \dots, \varepsilon_j^N \psi_{kj}]\right| \\ &> \varepsilon_j^{kN} \cdot C\varepsilon_j^{-(k+1)N-1} = C\varepsilon_j^{-N-1}\,. \end{aligned}$$

This contradicts $(2'^{\circ})$. □

Proof of $(4^{\circ}) \Rightarrow (5^{\circ})$: Bounded sets in the space $\mathscr{D}$ are relatively compact (see [12, III.2.2., Theorem 7]). Hence (5°) follows easily from the fact that the set $\mathscr{B}$ can be covered with a finite number of sets $\omega_1 + \frac{1}{2}\mathcal{U}_1, \dots, \omega_m + \frac{1}{2}\mathcal{U}_m$, where the neighbourhoods $\mathcal{U}_1, \dots, \mathcal{U}_m$ and the points $\omega_1, \dots, \omega_m$ have the properties described in (4°). Put $\mathcal{U} = \frac{1}{2}\bigcap\limits_{j=1}^{m} \mathcal{U}_j$. Then the sets $\omega_1 + \mathcal{U}_1, \dots, \omega_m + \mathcal{U}_m$ cover $\mathscr{B} + \mathcal{U}$ and the proof is evident. □

Proof of $(5^{\circ}) \Rightarrow (3'^{\circ})$: Getting N from (5°), we are proving $(3'^{\circ})$ for $N+1$ instead of N. Let the first of bounded paths (4) has asymptotically vanishing moments of order $N+1$, let the compact $B \Subset \mathbb{R}^d$ contain all supports of the values of the bounded paths (4) and denote $\mathscr{B} = \{\varphi^{\varepsilon}\,;\, \varepsilon \in\,]0,1]\}$. Choose e.g., by (6),

$$V_N = \sum_{1 \le |\beta| \le N} \left| \int \xi^{\beta} \varphi(\xi)\, \mathrm{d}\xi \right|$$

and so we get $\mathcal{U}$ by (5°). As the sets

$$\left\{\psi_i^{\varepsilon}; \varepsilon \in\,]0,1]\right\} \quad (i = 1, 2, \dots k)$$

are bounded, there is a $c > 0$ such that $c\psi_i^{\varepsilon} \in \mathcal{U} \quad (\forall\, i, \varepsilon)$.
Then the condition (5°) gives

$$\left|\partial^{\alpha} \mathrm{d}^k R_{\varepsilon}(\varphi^{\varepsilon}, x)[c\psi_1^{\varepsilon}, \dots, c\psi_k^{\varepsilon}]\right| \le C\varepsilon^{-N}$$

whenever

$$x \in K \quad \text{and} \quad V_N(\varphi^{\varepsilon}) \le \varepsilon^N.$$

Thanks to (6), this condition is fulfilled for ε small enough, as the path $\varepsilon \mapsto \varphi^{\varepsilon}$ has asymptotically vanishing moments of order $N+1$. Hence

$$\begin{aligned}\left|\partial^{\alpha} \mathrm{d}^k R_{\varepsilon}(\varphi^{\varepsilon}, x)[\psi_1^{\varepsilon}, \dots, \psi_k^{\varepsilon}]\right| &= c^{-k} \left|\partial^{\alpha} \mathrm{d}^k R_{\varepsilon}(\varphi^{\varepsilon}, x)[c\psi_1^{\varepsilon}, \dots, c\psi_k^{\varepsilon}]\right| \\ &\le c^{-k} \cdot C\varepsilon^{-N} = O(\varepsilon^{-N-1})\end{aligned}$$

what we had to prove. Thus the equivalence of all equivalent definitions is proved. □

§**8. Equivalent definitions of the null ideal** $\mathcal{N}$, i.e. the ideal of the negligible representatives for algebra $\mathcal{G}^2$, is the set of all $R \in \mathcal{E}_M^2(\Omega)$ fulfilling one of the following equivalent conditions ($\mathcal{A}_q$ means $\mathcal{A}_q(\mathbb{R}^d)$, $\mathscr{D}$ means $\mathscr{D}(\mathbb{R}^d)$, …). As $\mathcal{E}_M^{\mathrm{d}} \subset \mathcal{E}_M^2$, the more this equivalences hold for $R \in \mathcal{E}_M^{\mathrm{d}}$ and we can use any of the following conditions to define the ideal $\mathcal{N} \cap \mathcal{E}_M^{\mathrm{d}}$ of negligible representatives for the algebra $\mathcal{G}^{\mathrm{d}}$.

(0°) $\forall K \Subset \Omega,\ n \in \mathbb{N}\ \ \exists q \in \mathbb{N}\ \ \forall \mathscr{B}$ (bounded) $\subset \mathscr{D}$:

$$R_\varepsilon(\varphi, x) = O(\varepsilon^n) \qquad (\varepsilon \searrow 0)$$

uniformly for $x \in K$, $\varphi \in \mathscr{B} \cap \mathcal{A}_q$.

(1°) (classical Colombeau's definition, only the uniformity with respect to φ is added here) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}\ \ \exists q \in \mathbb{N}\ \forall \mathscr{B}$ (bounded) $\subset \mathscr{D}$:

$$\partial^\alpha R_\varepsilon(\varphi, x) = O(\varepsilon^n) \qquad (\varepsilon \searrow 0)$$

uniformly for $x \in K$, $\varphi \in \mathscr{B} \cap \mathcal{A}_q$.

(2°) (the same for the differentials with respect to φ) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ k \in \mathbb{N}_0,\ n \in \mathbb{N}\ \ \exists q \in \mathbb{N}\ \forall \mathscr{B}$ (bounded) $\subset \mathscr{D}$:

$$\partial^\alpha \mathrm{d}^k R_\varepsilon(\varphi, x)[\psi_1, \dots, \psi_k] = O(\varepsilon^n) \qquad (\varepsilon \searrow 0)$$

uniformly for $x \in K$, $\varphi \in \mathscr{B} \cap \mathcal{A}_q$, $\psi_1, \dots, \psi_k \in \mathscr{B} \cap (\mathcal{A}_q - \mathcal{A}_q)$.

(3°) $\forall K \Subset \Omega,\ n \in \mathbb{N}\ \ \exists q \in \mathbb{N}$: for every bounded $\mathscr{C}^\infty$ path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$ that has asymptotically vanishing moments of order q, we have

$$R_\varepsilon\big(\varphi^\varepsilon, x\big) = O(\varepsilon^n) \qquad (\varepsilon \searrow 0)$$

uniformly for $x \in K$.

(4°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}\ \ \exists q \in \mathbb{N}$: for every bounded $\mathscr{C}^\infty$ path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$ that has asymptotically vanishing moments of order q, we have

$$\partial^\alpha R_\varepsilon\big(\varphi^\varepsilon, x\big) = O(\varepsilon^n) \qquad (\varepsilon \searrow 0)$$

uniformly for $x \in K$.

(5°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ k \in \mathbb{N}_0,\ n \in \mathbb{N}\ \ \exists q \in \mathbb{N}$: for every bounded $\mathscr{C}^\infty$ paths $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$, $\varepsilon \mapsto \psi_i^\varepsilon \in \mathcal{A}$ $(i = 1, \dots, k)$ that all have asymptotically vanishing moments of order q, we have

$$\partial^\alpha \mathrm{d}^k R_\varepsilon\big(\varphi^\varepsilon, x\big)[\psi_1^\varepsilon, \dots, \psi_k^\varepsilon] = O(\varepsilon^n) \qquad (\varepsilon \searrow 0)$$

uniformly for $x \in K$.

Evidently, equivalent conditions (3′°), (4′°), (5′°) resp. (3″°), (4″°), (5″°) can be added where the $\mathscr{C}^\infty$ requirement for paths is omitted resp. in addition the uniformity condition is supplied like in §7, Equivalent definitions.

(6°) $\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ n \in \mathbb{N}\ \ \exists q \in \mathbb{N}$: for every bounded $\mathscr{C}^\infty$ path

$$\varepsilon \ \mapsto \big(\varphi_x^\varepsilon\big)_{x \in \Omega} \in \mathscr{C}^\infty(\Omega \to \mathcal{A}_0)$$

that has asymptotically vanishing moments of order q, we have

$$\big(\tfrac{\partial}{\partial x}\big)^\alpha R_\varepsilon\big(\varphi_x^\varepsilon, x\big) = O(\varepsilon^n) \qquad (\varepsilon \searrow 0).$$

uniformly for $x \in K$.

Remarks. 8.1. The equivalence (1°) ⇔ (2°) is proved in [7, Theorem 18], while the condition (0°) is added only in [5, Theorem 13.1] (both equivalences are proved in [5] and [7] only in $\mathcal{E}_M^{\mathrm{d}}$, here we have to prove them). It is surprising that there is such a simple tool for proving the negligibility that can be applied to the original Colombeau algebra as well (see [5, Chapter 12, 13]).

8.2. Although we have to consider paths depending on $x \in \Omega$ to define the moderateness, we see that paths not depending on x are sufficient for defining the negligibility. There is an error in [7, Theorem 18.4°] discovered and corrected in [5]: first the formulation does not correspond to the definition of negligible representatives in [4], where the paths do not depend on x, second the equivalence does not hold. Now we see that the condition 18.4° in [7], dealing with paths depending on x, need not be corrected, it can be omitted.

Proof of equivalences: The ideas of the proofs are the same that were used already in [7]. $\mathscr{D}$ in these proofs means $\mathscr{D}(\mathbb{R}^d)$, $\mathcal{A}_q$ means $\mathcal{A}_q(\mathbb{R}^d)$, … . (3°) ⇔ (4°) ⇔ (5°) follow from [5, Theorem 17.9]. □

Proof of (0°) ⇔ (3°): We know that (3°) is equivalent to the similar condition (3′°) without the $\mathscr{C}^\infty$ requirement for the path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$. non(0°) ⇒ non(3′°) being evident, we are going to prove (0°) ⇒ (3′°). For a given K take first a number N by 7(2″°) for $\alpha = 0$, $k = 1$ such that for every bounded path $\varepsilon \mapsto \psi^\varepsilon \in \mathcal{A}$ that has asymptotically vanishing moments of order N we have

$$\mathrm{d}_{\psi^\varepsilon} R_\varepsilon\big(\widetilde{\varphi}^\varepsilon, x\big) = O(\varepsilon^{-N}) \qquad (\varepsilon \searrow 0) \tag{15}$$

uniformly if $x \in K$ and $\varepsilon \mapsto \widetilde{\varphi}^\varepsilon$ runs over a set of equi-bounded paths having uniformly asymptotically vanishing moments of order N. Then, having chosen n, let q satisfies (0°) and at the same time

$$q \geq n + 2N. \tag{16}$$

Let a path $\varepsilon \mapsto \varphi^\varepsilon \in \mathcal{A}_0$ satisfy the hypotheses of (3°) and let $B \Subset \mathbb{R}^d$ be a bounded set containing the supports of all φ^ε. Recall a known lemma of functional analysis (Robertson A.P.-Robertson W.J. [11, II.3, Lemma 5]). If linear forms $f_0, f_1, \dots, f_k$ on a linear space E are linearly independent then there is a point $x \in E$ such that $f_0(x) = 1$, $f_1(x) = \dots = f_k(x) = 0$. Since the functions $x \mapsto x^\beta$ $(\beta \in \mathbb{N}_0^d,\ 0 \leq |\beta| \leq q)$ considered as distributions $\in \mathscr{D}'(B)$ are linearly independent, there are test functions $\psi_\alpha \in \mathscr{D}(B)$ $(\alpha \in \mathbb{N}_0^d,\ 1 \leq |\alpha| \leq q)$ fulfilling

$$\int \psi_\alpha(\xi)\xi^\alpha \,\mathrm{d}\xi = 1, \tag{17}$$

$$\int \psi_\alpha(\xi)\xi^\beta \,\mathrm{d}\xi = 0 \quad \text{for} \quad \beta \neq \alpha\,, 0 \leq |\beta| \leq q\,. \tag{18}$$

By (18), $\psi_\alpha \in \mathcal{A}(B)$ (note that $\alpha \neq 0$). If we denote

(19) $$c_{\alpha\varepsilon} := \int \varphi^\varepsilon(\xi)\xi^\alpha \,\mathrm{d}\xi\,,$$

we obtain that

(20) $$\kappa^\varepsilon := \varphi^\varepsilon - \sum_{\substack{\alpha\in\mathbb{N}_0^d\\ 1\le|\alpha|\le q}} c_{\alpha\varepsilon}\psi_\alpha \ \in\ \mathcal{A}_q(B).$$

As $\varepsilon \mapsto \varphi^\varepsilon$ has asymptotically vanishing moments of order q,

(21) $$c_{\alpha\varepsilon} = O(\varepsilon^q) \qquad (\varepsilon \searrow 0).$$

Let us order the summation indices α in (20) into a sequence $\alpha_1, \dots, \alpha_m$. Then

$$R_\varepsilon(\varphi^\varepsilon, x) - R_\varepsilon(\kappa^\varepsilon, x) = \sum_{j=1}^m \Big(R_\varepsilon\big(\kappa^\varepsilon + \textstyle\sum_{i=1}^{j} c_{\alpha_i\varepsilon}\psi_{\alpha_i}, x\big) \;-\; R_\varepsilon\big(\kappa^\varepsilon + \textstyle\sum_{i=1}^{j-1} c_{\alpha_i\varepsilon}\psi_{\alpha_i},\ x\big)\Big)$$

and by the mean value theorem (e.g. [7, Theorem 11) the term of this sum belongs to the closed convex hull of the set

$$\Big\{\mathrm{d}R_\varepsilon\big(\kappa^\varepsilon + \textstyle\sum_{i=1}^{j-1} c_{\alpha_i\varepsilon}\psi_{\alpha_i} + t\cdot c_{\alpha_j\varepsilon}\psi_{\alpha_j},\ x\big)[c_{\alpha_j\varepsilon}\psi_{\alpha_j}];\ t\in\,]0,1[\Big\}$$
$$= \Big\{\varepsilon^{q-N}\mathrm{d}R_\varepsilon\big(\kappa^\varepsilon + \textstyle\sum_{i=1}^{j-1} c_{\alpha_i\varepsilon}\psi_{\alpha_i} + t\cdot c_{\alpha_j\varepsilon}\psi_{\alpha_j},\ x\big)[\varepsilon^{N-q}c_{\alpha_j\varepsilon}\psi_{\alpha_j}];\ t\in\,]0,1[\Big\}.$$

By (21) ($N \le q$ due to (16)) the path $\varepsilon \mapsto \varepsilon^{N-q}c_{\alpha_j\varepsilon}\psi_{\alpha_j}$ has asymptotically vanishing moments of order N, so it follows from (15) that

$$R_\varepsilon(\varphi^\varepsilon, x) - R_\varepsilon(\kappa^\varepsilon, x) = \varepsilon^{q-N}\cdot O(\varepsilon^{-N}) = O(\varepsilon^{q-2N}) = O(\varepsilon^n)$$

(the last equality follows from (16)) uniformly if $x \in K$. By (20) and (0°), we have $R_\varepsilon(\kappa^\varepsilon, x) = O(\varepsilon^n)$ uniformly for $x \in K$, hence so is $R_\varepsilon(\varphi^\varepsilon, x)$. Thus the equivalence (0°) ⇔ (3°) is proved. □

Proof of (2°) ⇔ (1°) ⇔ (0°): (2°) ⇒ (1°) ⇒ (0°) being obvious, we are going to prove (0°) ⇒ (2°). For this purpose, we write (2°) in the following equivalent form using the total differential $\mathbf{d}$ of R:

(2′°) $\forall K \Subset \Omega$, $k \in \mathbb{N}_0$, $n \in \mathbb{N}$ $\exists q \in \mathbb{N}$ such that $\forall \mathscr{B}$ (bounded) $\subset \mathscr{D}$ we have

(22) $$\mathbf{d}^k R_\varepsilon(\varphi, x)[(\psi_1, h_1), \dots, (\psi_k, h_k)] = O(\varepsilon^n) \qquad (\varepsilon \searrow 0)$$

uniformly for

$$x \in K,\ \varphi \in \mathscr{B} \cap \mathcal{A}_q,\ \psi_i \in \mathscr{B} \cap (\mathcal{A}_q - \mathcal{A}_q), \tag{23}$$
$$h_i \in \mathbb{R}^d,\ |h_i| \le 1 \quad (\text{Euclidean norm},\ i = 1, \dots, k)\,.$$

Similarly, we will write §7, the definition $(2'^{\circ})$ in the form using the total differential:
$\forall\, K^* \Subset \Omega$, $k \in \mathbb{N}_0$ $\exists\, N \in \mathbb{N}$ such that for every bounded paths

$$\varepsilon \ \mapsto \varphi^\varepsilon \in \mathcal{A}_0, \quad \varepsilon \ \mapsto \psi_i^\varepsilon \in \mathcal{A} \quad (i = 1, 2, \dots, k)$$

that all have asymptotically vanishing moments of order N, we have

$$\mathbf{d}^k R_\varepsilon(\varphi^\varepsilon, x)[(\psi_1^\varepsilon, h_1), \dots, (\psi_k^\varepsilon, h_k)] = O(\varepsilon^{-N}) \qquad (\varepsilon \searrow 0)$$

uniformly for $x \in K^*$, $h_i \in \mathbb{R}^d$, $|h_i| \le 1$ $(i = 1, \dots, k)$. Let us write $k+1$ instead of k and apply this definition to test functions belonging to $\mathcal{A}_N$ resp. $\mathcal{A}_N - \mathcal{A}_N$ only. We easily obtain the following consequence:
$\forall\, K^* \Subset \Omega$, $k \in \mathbb{N}_0$ $\exists\, N \in \mathbb{N}$ such that for every bounded $\mathscr{B} \subset \mathscr{D}$, we have
(24)

$$\mathbf{d}^{k+1} R_\varepsilon(\varphi, x)[(\psi_1, h_1), \dots, (\psi_{k-1}, h_{k-1}), (\psi_k, h_k), (\psi_k, h_k)] = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly for $x \in K^*$, $\varphi \in \mathscr{B} \cap \mathcal{A}_N$, $\psi_i \in \mathscr{B} \cap (\mathcal{A}_N - \mathcal{A}_N)$, $h_i \in \mathbb{R}^d$, $|h_i| \le 1$ $(i = 1, \dots, k)$.

In the following, we will write Φ for (φ, x) and Ψ_i for (ψ_i, h_i). The proof will be done by induction. Denote by $S(k)$ $(k \in \mathbb{N}_0)$ the statement

$S(k)$: $\forall\, K \Subset \Omega$, $n \in \mathbb{N}$ $\exists\, q \in \mathbb{N}$ such that $\forall\, \mathscr{B}$ (bounded) $\subset \mathscr{D}$, (22) holds uniformly under conditions (23).

$S(0)$ is (0°). Choosing $K \Subset \Omega$, $k \in \mathbb{N}$, $n \in \mathbb{N}$, we have to deduce $S(k)$ from $S(k-1)$. First, for the chosen K and k, we get N from the consequence containing (24), where we substitute a larger compact

$$K^* := \Big\{ x \in \mathbb{R}^d ;\, \mathrm{dist}(x, K) \le \Delta \Big\} \subset \Omega$$

with an appropriate $\Delta > 0$. Then, for this K^* by the statement $S(k-1)$, we get an integer $q \ge N$ such that

$$\mathbf{d}^{k-1} R_\varepsilon(\Phi)[\Psi_1, \dots, \Psi_{k-1}] = O(\varepsilon^{2n+N}) \qquad (\varepsilon \searrow 0) \tag{25}$$

uniformly under conditions: $x \in K^*$, φ, ψ_i, h_i by (23) for any bounded $\mathscr{B} \subset \mathscr{D}$. Under these conditions and for $t \in [0, \Delta]$, we have by (24)

$$\mathbf{d}^{k+1} R_\varepsilon(\Phi + t\Psi_k)[\Psi_1, \dots, \Psi_{k-1}, \Psi_k, \Psi_k] = O(\varepsilon^{-N}) \qquad (\varepsilon \searrow 0)$$

uniformly. From the mean value theorem it follows

$$\begin{aligned}&\left|\mathbf{d}^k R_\varepsilon(\Phi + t\Psi_k)[\Psi_1,\dots,\Psi_{k-1},\Psi_k] - \mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1,\dots,\Psi_{k-1},\Psi_k]\right| \\ &\qquad \le \sup_{t'\in[0,t]} \left|\mathbf{d}^{k+1} R_\varepsilon(\Phi + t'\Psi_k)[\Psi_1,\dots,\Psi_{k-1},\Psi_k,t\Psi_k]\right| = tO(\varepsilon^{-N}) \qquad (\varepsilon\searrow 0)\end{aligned}$$

uniformly under the above conditions. Denoting by $\overline{B}(a,r)\subset\mathbb{C}$ the closed ball of center a and radius r, we can write this

$$\mathbf{d}^k R_\varepsilon(\Phi + t\Psi_k)[\Psi_1,\dots,\Psi_{k-1},\Psi_k] \in \overline{B}\big(\mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1,\dots,\Psi_{k-1},\Psi_k],\ t\varepsilon^{-N}\cdot c\big)$$

with a constant c depending on $\mathscr{B}$ but neither on $t\in[0,\Delta]$ nor on $\varphi,\psi_i\in\mathscr{B}$. It follows from the mean value theorem again:

$$\begin{aligned}&\mathbf{d}^{k-1} R_\varepsilon(\Phi + \varepsilon^{n+N}\Psi_k)[\Psi_1,\dots,\Psi_{k-1}] - \mathbf{d}^{k-1} R_\varepsilon(\Phi)[\Psi_1,\dots,\Psi_{k-1}] \\ &\in \overline{\operatorname{conv}}\left\{\mathbf{d}^k R_\varepsilon(\Phi + t\Psi_k)[\Psi_1,\dots,\Psi_{k-1},\varepsilon^{n+N}\Psi_k];\ t\in[0,\varepsilon^{n+N}]\right\} \\ &\subset \bigcup_{t\in[0,\varepsilon^{n+N}]} \overline{B}\big(\varepsilon^{n+N}\mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1,\dots,\Psi_{k-1},\Psi_k],\ \varepsilon^{n+N}\cdot t\varepsilon^{-N}c\big) \\ &= \overline{B}\big(\varepsilon^{n+N}\mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1,\dots,\Psi_{k-1},\Psi_k],\ \varepsilon^{2n+N}\cdot c\big).\end{aligned}$$

The radius is $O(\varepsilon^{2n+N})$ uniformly under (23); the left-hand side is $O(\varepsilon^{2n+N})$ as well, thanks to (25). Hence the center $\varepsilon^{n+N}\mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1,\dots,\Psi_{k-1},\Psi_k]$ must be $O(\varepsilon^{2n+N})$, too. Thus

$$\mathbf{d}^k R_\varepsilon(\Phi)[\Psi_1,\dots,\Psi_{k-1},\Psi_k] = O(\varepsilon^n) \qquad (\varepsilon\searrow 0)$$

what we had to prove.

It remains to prove the equivalence with (6°). (5°) $\Rightarrow$ (6°) follows from the chain rule (differentiation of the composition, e.g. [7, Theorem 12] or [13, (1.8.3)]). (6°) $\Rightarrow$ (4°) is obvious. □

9. Now, we can define the quotient algebras $\mathcal{G}^2 := \mathcal{E}^2_M/\mathcal{N}$ and $\mathcal{G}^{\mathrm{d}} := \mathcal{E}^{\mathrm{d}}_M/\mathcal{N}\cap\mathcal{E}^{\mathrm{d}}_M$. The equality of both algebras is proved in [8]. The set of representatives $\mathcal{E}^2_M$ is strictly larger than $\mathcal{E}^{\mathrm{d}}_M$, as is shown in [5, 17.11].

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Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail: jelinek@karlin.mff.cuni.cz

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