Equality of two diffeomorphism invariant Colombeau algebras

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Abstract. The two diffeomorphism invariant algebras introduced in Grosser M., Farkas E., Kunziger M., Steinbauer R., *On the foundations of nonlinear generalized functions I, II*, Mem. Amer. Math. Soc. **153** (2001), no. 729, 93 pp., are identical.

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The paper is a continuation of [9] and its only aim is to prove that both diffeomorphism invariant Colombeau-type algebras $\mathcal{G}^{\mathrm{d}}$ and $\mathcal{G}^2$ introduced in [8] and [6] (Grosser et al.) coincide. In [6] diverse possibilities to define Colombeau-type algebras are researched; our result shows that there is only one diffeomorphism invariant Colombeau-type algebra among them for a given domain $\Omega \subset \mathbb{R}^d$.

§**1.** In this paper, we use notations introduced in [9] and mostly we refer to [9]. This paper is devoted to prove the following

Theorem. *For every open $\Omega \subset \mathbb{R}^d$, the algebras $\mathcal{G}^2(\Omega)$ and $\mathcal{G}^{\mathrm{d}}(\Omega)$ coincide.*

As the algebras are quotient algebras $\mathcal{G}^2 := \mathcal{E}^2_M/\mathcal{N}$ and $\mathcal{G}^{\mathrm{d}} := \mathcal{E}^{\mathrm{d}}_M/\mathcal{N} \cap \mathcal{E}^{\mathrm{d}}_M$, the theorem says that for any representative $R \in \mathcal{E}^2_M$ another representative $\widetilde{R} \in \mathcal{E}^{\mathrm{d}}_M$ can be found with $R - \widetilde{R} \in \mathcal{N}$. To prove it, in all what follows, we assume that $R \in \mathcal{E}^2_M$ is given and we are going to construct $\widetilde{R}$. This will be done in several steps. In every step functions of variables φ, x are constructed and their properties are presented with the aim to construct at last the required representative $\widetilde{R}$. First we show that it is sufficient to do it for representatives with compact support.

Proposition. *Suppose that for any representative $R \in \mathcal{E}^2_M(\mathbb{R}^d)$ such that there is a compact $K \Subset \mathbb{R}^d$ fulfilling $R(\varphi, x) = 0$ whenever $x \in \mathbb{R}^d \setminus K$, a representative*

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$\widetilde{R} \in \mathcal{E}_M^{\mathrm{d}}(\mathbb{R}^d)$ *can be found with* $R - \widetilde{R} \in \mathcal{N}$*. Then for any open* $\Omega \subset \mathbb{R}^d$ *the algebras* $\mathcal{G}^2(\Omega)$ *and* $\mathcal{G}^{\mathrm{d}}(\Omega)$ *coincide.*

Proof: Let $R \in \mathcal{E}_M^2(\Omega)$. Recall that unlike in [8] and [6] here a representative is defined on $\mathcal{E}(\Omega) = \mathcal{A}_0(\mathbb{R}^d) \times \Omega$ and we do not loose generality with this assumption. Choose a locally finite covering

$$\Omega = \bigcup \Omega_m \quad \text{with} \quad \overline{\Omega_m} \Subset \Omega$$

and a partition of unity $1 = \sum \chi_m$ on Ω subordinated to this covering, $\chi_m \in \mathscr{D}(\Omega_m)$, $K_m := \operatorname{supp} \chi_m$. Then $R = \sum R_m$ if we denote $R_m(\varphi, x) := R(\varphi, x) \cdot \chi_m(x)$ and we have $R_m(\varphi, x) = 0$ whenever $x \in \Omega \setminus K_m$. R_m can be easily extended to belong to $\mathcal{E}_M^2(\mathbb{R}^d)$ putting $R_m(\varphi, x) = 0$ whenever $x \in \mathbb{R}^d \setminus K_m$. By hypothesis, we can find $\widetilde{R}_m \in \mathcal{E}_M^{\mathrm{d}}(\mathbb{R}^d)$ with $\widetilde{R}_m - R_m \in \mathcal{N}$. Then, for every m, we choose a test function $\sigma_m \in \mathscr{D}(\Omega_m)$ that is $= 1$ on a neighbourhood of K_m. The functions χ_m and σ_m are considered to be elements of $\mathcal{E}(\mathbb{R}^d)$ as functions independent of the first variable. Consequently, $\sigma_m \in \mathcal{E}_M^{\mathrm{d}}$. Considering all representatives to be elements of $\mathcal{E}_M^2(\Omega)$ (i.e. restricted to $\mathcal{A}_0(\mathbb{R}^d) \times \Omega$), we have (note that $\mathcal{N}$ is an ideal) $R - \sum \widetilde{R}_m \sigma_m = \sum (R_m - \widetilde{R}_m \sigma_m) = \sum (R_m - \widetilde{R}_m)\sigma_m \in \mathcal{N}$, the sum being locally finite. From the same reason, $\widetilde{R} := \sum \widetilde{R}_m \sigma_m \in \mathcal{E}_M^{\mathrm{d}}$. $\widetilde{R}$ is thus a required representative. □

§**2. Remark.** For $B \Subset \mathbb{R}^d$, it is known that $\mathscr{D}(B)$ is a Fréchet space. Its topology can be generated by a countable system of norms defined by continuous scalar products, e.g.

$$\varphi, \psi \ \mapsto \ \int \frac{\partial^{dm}}{\partial \xi_1^m \ldots \partial \xi_d^m} \varphi(\xi) \cdot \frac{\partial^{dm}}{\partial \xi_1^m \ldots \partial \xi_d^m} \overline{\psi}(\xi) \, \mathrm{d}\xi \, .$$

A continuous scalar product $\varphi, \psi \mapsto (\varphi, \psi)$ is $\mathscr{C}^\infty$, being sesqui-linear, and we have

$$\mathrm{d}_\psi(\varphi, \varphi) = 2\Re(\varphi, \psi),$$
$$\mathrm{d}^2_{\psi_1, \psi_2}(\varphi, \varphi) = 2\Re(\psi_1, \psi_2);$$

the derivatives of higher orders are zero. Hence the norm generated by a continuous scalar product is $\mathscr{C}^\infty$ in all points except of origin. The function $\psi \mapsto (\psi, \psi) = \|\psi\|^2$ is $\mathscr{C}^\infty$ always.

Notation. The function V_N, used in Equivalent Definition [9, §7, (4°) and (5°)] of $\mathcal{E}_M^2$, can be

(1) $$V_N(\varphi) = \Big(\sum_{\substack{\beta \in \mathbb{N}_0^d \\ 1 \le |\beta| \le N}} \big| \textstyle\int \xi^\beta \varphi(\xi) \, \mathrm{d}\xi \big|^2 \Big)^{1/2} \qquad (\varphi \in \mathcal{A}_0).$$

Evidently, this function fulfills [9, (6)]:

$$\forall\, N \in \mathbb{N},\ \mathscr{B}\ (\text{bounded})\ \subset \mathcal{A}_0 \quad \exists\, C_1, C_2 > 0 \quad \forall\, \varphi \in \mathscr{B}:$$
$$C_2 \sum_{\substack{\beta\in\mathbb{N}_0^d \\ 1\le|\beta|\le N}} \left|\int \xi^\beta \varphi(\xi)\,\mathrm{d}\xi\right| \ \le\ V_N(\varphi)\ \le\ C_1 \sum_{\substack{\beta\in\mathbb{N}_0^d \\ 1\le|\beta|\le N}} \left|\int \xi^\beta \varphi(\xi)\,\mathrm{d}\xi\right|.$$

Evidently every multiple $c\cdot V_N(\varphi)$ satisfies these inequalities, so it can be used in Equivalent Definition [9, §7, (4°) and (5°)]. Also the function

$$V_N'(\varphi) = \Big(\sum_{\substack{\beta\in\mathbb{N}_0^d \\ 1\le|\beta|\le N}} \|\varphi\|_{\mathscr{L}^2}^{4|\beta|/d} \big|\textstyle\int \xi^\beta\varphi(\xi)\,\mathrm{d}\xi\big|^2\Big)^{1/2} \qquad (\varphi\in\mathcal{A}_0)$$

fulfills the above inequalities [9, (6)] that allows us to use it in Equivalent Definitions [9, §7, (4°), (5°)]. Indeed, a bounded set is relatively compact, so $\exists\, c_1, c_2 > 0$ (depending on $\mathscr{B}$) $\forall\, \varphi\in\mathscr{B}$ we have $c_2 \le \|\varphi\|_{\mathscr{L}^2} \le c_1$; [9, (6)] follows easily.
It can be checked that

$$\text{(2)}\quad \|S_\varepsilon\varphi\|^{-2/d} = \varepsilon\cdot\|\varphi\|^{-2/d}, \qquad \int S_\varepsilon\varphi(\xi)\xi^\beta\,\mathrm{d}\xi = \varepsilon^{|\beta|}\int\varphi(\xi)\xi^\beta\,\mathrm{d}\xi \quad (\beta\in\mathbb{N}_0^d),$$

so $V_N'(S_\varepsilon\varphi) = V_N'(\varphi)$.

For all what follows, a function $\rho\in\mathcal{A}_0([-1,1])$ is fixed such that

$$\rho(\xi) > 0 \ \text{ iff }\ \xi \in\,]-1,1[\,,$$
$$\rho_\varepsilon := S_\varepsilon\rho\,, \quad \vartheta := \rho_{1/2} * \chi_{[-3/2,\ 3/2]}$$

(convolution with the characteristic function),

$$\vartheta_m(\xi) := \vartheta(2^{-m}\xi) \qquad (m \text{ integer}).$$

So $\vartheta_m(\xi) = 1$ iff $\xi\in[-2^m, 2^m]$, $0 < \vartheta_m(\xi) \le 1$ iff $\xi \in\,]-2^{m+1}, 2^{m+1}[$ and ϑ_m is decreasing on $[2^m, 2^{m+1}]$. Denote furthermore

$$K_m := [-2^m, 2^m]^d$$

and for $\xi = (\xi_1,\dots,\xi_d)\in\mathbb{R}^d$, denote $\vartheta_m^{\otimes d}(\xi) := \vartheta_m(\xi_1)\dots\vartheta_m(\xi_d)$. If there is no danger of confusion, we will write simply $\vartheta_m(\xi)$ instead of $\vartheta_m^{\otimes d}(\xi)$.

§**3.** Thanks to §1, Proposition we can assume that the given representative R belongs to $\mathcal{E}_M^2(\mathbb{R}^d)$ and that there is a compact $K \Subset \Omega$ fulfilling $R(\varphi, x) = 0$ for $x \in \mathbb{R}^d \setminus K$. In this case, in the equivalent definitions of $\mathcal{E}_M^2$ and $\mathcal{N}$ ([9, §§7, 8]) we can omit $\forall K \Subset \Omega$ and replace the uniformity on K with the uniformity on the whole of $\mathbb{R}^d$. Denote by N_L the number N from Equivalent Definition [9, §7, (5°)] holding at the same time for all $|\alpha| \le L$ and for all differentials of order $k \le L$. Certainly, this equivalent definition remains valid if we take any greater number for N_L. We replace our representative with another one determining the same generalized function, if needed, to obtain the following

Properties of R.

(1°) There is an increasing sequence $\{N_L\}_{L\in\mathbb{N}} \subset \mathbb{N}$, $N_L \ge L$, fulfilling: $\forall B \Subset \mathbb{R}^d$, $\mathscr{B}$ (bounded) $\subset \mathcal{A}_0(B)$, $L \in \mathbb{N}$ $\quad \exists \mathcal{U}$ (absolutely convex open neighbourhood of zero) $\subset \mathcal{A}(B)$, $C > 0$ $\quad \forall \ell = 1, 2, \dots, L$, $\psi_1, \dots, \psi_\ell \in \mathcal{U}$, $\varphi \in \mathscr{B} + 2\mathcal{U}$, $\varepsilon \in \,]0, 1]$, $\varepsilon^{N_L} \ge V_{N_L}(\varphi)$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le L$, $x \in \mathbb{R}^d$:

$$\begin{aligned} \big|\partial^\alpha(\mathrm{d}^\ell_{S_\varepsilon\psi_1,\dots,S_\varepsilon\psi_\ell} R)(S_\varepsilon\varphi, x)\big| \,=\, \big|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R(S_\varepsilon\varphi, x)\big| \,&\le\, \varepsilon^{-N_L}, \\ \big|\partial^\alpha R(S_\varepsilon\varphi, x)\big| \,&\le\, C\varepsilon^{-N_L}. \end{aligned} \tag{3}$$

(2°) The first inequality in (3) can be written in the form

$$\big|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R(S_\varepsilon\varphi, x)\big| \le \varepsilon^{-N_L} \|\psi_1\|_{\mathcal{U}} \cdots \|\psi_\ell\|_{\mathcal{U}} \tag{4}$$

if we omit the hypothesis $\psi_1, \dots, \psi_\ell \in \mathcal{U}$, only supposing $\psi_1, \dots, \psi_\ell \in \mathcal{A}(B)$ ($\|\bullet\|_{\mathcal{U}}$ denotes the Minkowski functional assigned to $\mathcal{U}$).

(3°) If $L = 1$, the hypothesis $\varepsilon^{N_L} \ge V_{N_L}(\varphi)$ can be omitted, so that (3) and (4) hold for every $\varepsilon \in \,]0, 1]$.

(4°) Consequently, if $\mathscr{B}$ is convex, $\varphi_1, \varphi_2 \in \mathscr{B} + 2\mathcal{U}$, we have to consider two cases.
If $L = 1$, $|\alpha| \le 1$ then

$$\big|\partial^\alpha\big(R(S_\varepsilon\varphi_2, x) - R(S_\varepsilon\varphi_1, x)\big)\big| \le \varepsilon^{-N_1} \|\varphi_2 - \varphi_1\|_{\mathcal{U}}\,. \tag{5}$$

Otherwise if $|\alpha| \le L$ and $\varepsilon^{N_L} \ge V_{N_L}(\varphi_1)$, $\varepsilon^{N_L} \ge V_{N_L}(\varphi_2)$, $\ell = 1, \dots, L$, $\psi_1, \dots, \psi_{\ell-1} \in \mathcal{A}(B)$, then

$$\begin{aligned} &\big|\partial^\alpha \mathrm{d}^{\ell-1}_{\psi_1,\dots,\psi_{\ell-1}}\big(R(S_\varepsilon\varphi_2, x) - R(S_\varepsilon\varphi_1, x)\big)\big| \\ &\qquad\qquad \le \varepsilon^{-N_L} \|\psi_1\|_{\mathcal{U}} \cdots \|\psi_{\ell-1}\|_{\mathcal{U}}\, \|\varphi_2 - \varphi_1\|_{\mathcal{U}}\,. \end{aligned} \tag{6}$$

Proof of (3°): The items (1°) and (2°) are consequences of [9, §7, (5°)]. The equality in (3) follows from the chain rule where the inner function is linear, so its higher derivatives vanish. (2°) follows from the linearity of differentials. This holds for any representative with compact support. However, for (3°) we have to choose a suitable representative determining a given generalized function and possibly we have to choose the number N_1, too. Let R be a given representative. Applying $\frac{1}{2}\|\chi_B\|^{-2N_1/d}\, V'_{N_1}$ instead of V_{N_1} in Equivalent Definition [9, §7, (5°)] (or in Properties, item (1°)) for $L=1$, we get $\mathcal{U}$ and $C>0$ such that (3) holds if $\ell=1$, $|\alpha|\le 1$, $\psi_1\in\mathcal{U}$, $\varphi\in\mathscr{B}+2\mathcal{U}$ and

$$\varepsilon^{N_1}\ge \tfrac{1}{2}\|\chi_B\|^{-2N_1/d}\, V'_{N_1}(\varphi). \tag{7}$$

Let us define $R'(\varphi,x):=\vartheta(\|\varphi\|^{2N_1/d}\, V'_{N_1}(\varphi))\cdot R(\varphi,x)$ ($\mathscr{L}^2$ norms).

First we prove that $R'\in\mathcal{E}^2_M$ and $R-R'\in\mathcal{N}$. Let $\varepsilon\mapsto\varphi^\varepsilon$ (see [9, §7, (2°), §8, (4°)]) be a bounded path with asymptotically vanishing moments of order N_1+1. This means that the set $\{\varphi^\varepsilon;\ \varepsilon\in\,]0,1]\}$ is bounded and

$$V'_{N_1+1}(\varphi^\varepsilon)=O(\varepsilon^{N_1+1})\qquad(\varepsilon\searrow 0).$$

Consequently the set $\{\|\varphi^\varepsilon\|;\ \varepsilon\in\,]0,1]\}$ is bounded and by (2), as $V'_{N_1}\le V'_{N_1+1}$, we have

$$\|S_\varepsilon\varphi^\varepsilon\|^{2N_1/d}\, V'_{N_1}(S_\varepsilon\varphi^\varepsilon)\le\|\varphi^\varepsilon\|^{2N_1/d}\,\varepsilon^{-N_1}\, V'_{N_1+1}(\varphi^\varepsilon)=O(\varepsilon).$$

It follows that for a sufficiently small ε, $S_\varepsilon\varphi^\varepsilon$ belongs to the open set (independent of x) $\Big\{\varphi;\ \|\varphi\|^{2N_1/d}\, V'_{N_1}(\varphi)<1\Big\}$ where $R'(\bullet,x)=R(\bullet,x)$. Hence the assertions $R'\in\mathcal{E}^2_M$ and $R-R'\in\mathcal{N}$ are proved.

Now we want to prove that R' fulfills (3°). This means that the relations (3) with R' hold for all $\varepsilon\in\,]0,1]$, provided $L=\ell=1$ $(|\alpha|\le 1)$. To this aim, for $\psi\in\mathcal{A}(B)$ we first estimate

$$\begin{aligned}
\mathrm{d}_\psi\partial^\alpha R'(S_\varepsilon\varphi,x)&=\mathrm{d}_\psi\partial^\alpha\Big(\vartheta\big(\|S_\varepsilon\varphi\|^{2N_1/d}\, V'_{N_1}(S_\varepsilon\varphi)\big)\cdot R(S_\varepsilon\varphi,x)\Big)\\
&=\mathrm{d}_\psi\vartheta\Big(\|S_\varepsilon\varphi\|^{2N_1/d}\, V'_{N_1}(S_\varepsilon\varphi)\Big)\cdot\partial^\alpha R(S_\varepsilon\varphi,x)\\
&\quad+\vartheta\Big(\|S_\varepsilon\varphi\|^{2N_1/d}\, V'_{N_1}(S_\varepsilon\varphi)\Big)\cdot\mathrm{d}_\psi\partial^\alpha R(S_\varepsilon\varphi,x)\\
&=\mathrm{d}_\psi\vartheta\Big(\|\varphi\|^{2N_1/d}\,\varepsilon^{-N_1}\, V'_{N_1}(\varphi)\Big)\cdot\partial^\alpha R(S_\varepsilon\varphi,x)\\
&\quad+\vartheta\Big(\|\varphi\|^{2N_1/d}\,\varepsilon^{-N_1}\, V'_{N_1}(\varphi)\Big)\cdot\mathrm{d}_\psi\partial^\alpha R(S_\varepsilon\varphi,x).
\end{aligned}\tag{8}$$

If $\|\varphi\|^{2N_1/d}\,\varepsilon^{-N_1}\, V'_{N_1}(\varphi)>2$ then $R'(S_\varepsilon\varphi,x)=0$, hence we have to estimate (8) only if

$$\tfrac{1}{2}\|\varphi\|^{2N_1/d}\, V'_{N_1}(\varphi)\le\varepsilon^{N_1}.$$

By the Hölder inequality we have (χ denotes the characteristic function):

$$1 = \int \varphi\,\chi_B \le \|\varphi\|\,\|\chi_B\|, \quad \text{i.e.} \quad \|\varphi\| \ge \|\chi_B\|^{-1}. \tag{9}$$

Consequently

$$\tfrac{1}{2}\|\chi_B\|^{-2N_1/d}\, V'_{N_1}(\varphi) \le \varepsilon^{N_1}$$

and this is exactly our hypothesis (7) assuring that (3) holds. Hence two terms of (8) are estimated:

$$\bigl|\partial^\alpha R(S_\varepsilon\varphi, x)\bigr| \le C\varepsilon^{-N_1}, \qquad \bigl|\partial^\alpha \mathrm{d}_\psi R(S_\varepsilon\varphi, x)\bigr| \le C\varepsilon^{-N_1}.$$

It remains to estimate

$$\mathrm{d}_\psi \vartheta\Bigl(\|\varphi\|^{2N_1/d}\,\varepsilon^{-N_1}\, V'_{N_1}(\varphi)\Bigr) \le \max_t \Bigl|\tfrac{\mathrm{d}}{\mathrm{d}t}\vartheta(t)\Bigr|\; \varepsilon^{-N_1}\cdot \mathrm{d}_\psi\bigl(\|\varphi\|^{2N_1/d}\, V'_{N_1}(\varphi)\bigr). \tag{10}$$

By [9, §2, Proposition] about local equicontinuity of differentials there is an absolutely convex open neighbourhood of zero $\mathcal{U} \subset \mathcal{A}(B)$ such that

$$\mathrm{d}_\psi\bigl(\|\varphi\|^{2N_1/d}\, V'_{N_1}(\varphi)\bigr) \le 1 \tag{11}$$

whenever $\varphi \in \mathscr{B} + 2\mathcal{U}$, $\psi \in \mathcal{U}$. Under these hypotheses, we have got

$$\mathrm{d}_\psi \vartheta\Bigl(\|\varphi\|^{2N_1/d}\,\varepsilon^{-N_1}\, V'_{N_1}(\varphi)\Bigr) \le C_1\varepsilon^{-N_1}$$

with a constant C_1 depending only on ϑ. Due to (8) and (3), it follows

$$\mathrm{d}_\psi \partial^\alpha R'(S_\varepsilon\varphi, x) \le C_1 C\,\varepsilon^{-2N_1} + C\varepsilon^{-N_1} \le (C_1+1)C\,\varepsilon^{-2N_1}.$$

Replacing $\mathcal{U}$ with a smaller one, we get $\le \varepsilon^{-2N_1}$. It remains to estimate $\partial^\alpha R'(S_\varepsilon\varphi, x)$. This is similar or simpler, so we let it to the reader. □

Proof of (4°): Using the mean value theorem, we have for some $\tau \in\,]0,1[$

$$\begin{aligned}\Bigl|\partial^\alpha \mathrm{d}^{\ell-1}_{\psi_1,\dots,\psi_{\ell-1}} R(S_{2^{-n}}\,\varphi_2,\, x) - \partial^\alpha \mathrm{d}^{\ell-1}_{\psi_1,\dots,\psi_{\ell-1}} R(S_{2^{-n}}\,\varphi_1,\, x)\Bigr| \\ \le \Bigl|\partial^\alpha \mathrm{d}^\ell R(S_{2^{-n}}\,(\tau\varphi_1 + (1-\tau)\varphi_2),\, x)[\psi_1, \dots, \psi_{\ell-1},\, \varphi_2 - \varphi_1]\Bigr|.\end{aligned}$$

The function $\tau \mapsto V_N(\tau\varphi_1 + (1-\tau)\varphi_2)$ is convex because $V_N(\varphi)$ is the Euclidean norm of the point with coordinates $\int \xi^\beta \varphi(\xi)\,\mathrm{d}\xi$. Thus in the second case of (4°), $V_{N_L}(\tau\varphi_1 + (1-\tau)\varphi_2) \le \varepsilon^{N_L}$ holds for all $\tau \in [0,1]$ and we can apply (4). □

§**4. Notation.** Let a number $r \in \mathbb{N}$ be given, let us consider the Fourier series of a function $\psi \in \mathcal{A}(K_r)$ on the cube $K_{r+1} \subset \mathbb{R}^d$ (§2, Notation, the function $\vartheta_r^{\otimes d}$ will be denoted simply by ϑ_r)

$$(12)\qquad \begin{aligned} \psi(\xi) &= \sum_{\beta \in \mathbb{Z}^d} c'_\beta \, e^{2^{-r-1}\pi i \beta\cdot\xi} \qquad \text{where} \quad \beta\cdot\xi := \beta_1\xi_1 + \cdots + \beta_d\xi_d \,, \\ c'_\beta &= 2^{-d(r+2)} \int \psi(\xi)\, e^{-2^{-r-1}\pi i \beta\cdot\xi} \,\mathrm{d}\xi \,. \end{aligned}$$

As $\int \psi = 0$, we have $c'_0 = 0$. As $\vartheta_r = 1$ on K_r, we have as well

$$\psi(\xi) = \sum_{\beta \neq 0} c'_\beta \, e^{2^{-r-1}\pi i \beta\cdot\xi}\, \vartheta_r(\xi).$$

We will use another expansion $\psi = \sum\limits_{\beta \neq 0} c'_\beta \, \gamma'_\beta$ where the functions γ'_β are defined:

$$\gamma'_\beta(\xi) = e^{2^{-r-1}\pi i \beta\cdot\xi}\, \vartheta_r(\xi) - c''_\beta \, \vartheta_r(\xi)$$

with constants c''_β such that $\gamma'_\beta \in \mathcal{A}$. This means that

$$(13)\qquad c''_\beta \int \vartheta_r(\xi)\,\mathrm{d}\xi = \int e^{2^{-r-1}\pi i \beta\cdot\xi}\, \vartheta_r(\xi)\,\mathrm{d}\xi \,.$$

It is known that the Fourier coefficients (12) of a test function tend rapidly to zero if $|\beta| \to \infty$. By (13), c''_β tend rapidly to zero as well.

We arrange the multi-indices $\beta \neq 0$ into a sequence $\{\beta_j\}_{j=1}^{\infty}$ in such a way that the sequence $\{|\beta_j|\}$ is non-decreasing; then we change the notation writing $\gamma'_j, c'_j, \ldots$ rather than $\gamma'_{\beta_j}, c'_{\beta_j}, \ldots$. Then the above expansion takes the form

$$\psi = \sum_{j=1}^{\infty} c'_j \, \gamma'_j \,.$$

Evidently $|\beta_j| \le j \le (2|\beta_j|+1)^d$ ($\Leftarrow$ j does not exceed the number of the indices β with $|\beta| \le |\beta_j|$). Hence any multi-sequence $\{a_\beta\}_\beta$ is moderated (i.e. $|a_\beta| \le c|\beta|^m$ for some c and m) iff the sequence $\{a_{\beta_j}\}_j$ is moderated. $\{a_\beta\}_\beta$ tends rapidly to zero iff $\{a_{\beta_j}\}_j$ tends rapidly to zero.

If $\mathcal{U} \subset \mathcal{A}(K_{r+1})$ is an absolutely convex open neighbourhood of zero, then $\|\gamma'_j\|_{\mathcal{U}}$ is a moderate sequence (this can be calculated e.g. if $\|\gamma'_\beta\|_{\mathcal{U}}$ is the norm $\|\gamma'_\beta\|$ from §2, Remark). So we get the following

Result. *If $\mathcal{U} \subset \mathcal{A}(K_{r+1})$ is an absolutely convex open neighbourhood of zero, then there are $\gamma_j \in \mathcal{A}(K_{r+1}) \quad (j \in \mathbb{N})$ such that*

$$\sum_{j=1}^{\infty} \|\gamma_j\|_{\mathcal{U}} \leq 1 \tag{14}$$

and any function $\psi \in \mathcal{A}(K_r)$ has an expansion

$$\psi = \sum c_j \, \gamma_j$$

with coefficients c_j tending rapidly to zero.

Indeed, choose a moderate sequence $\lambda_j \nearrow \infty$ such that the functions $\gamma_j := \gamma'_j/\lambda_j \in \mathcal{A}(K_{r+1})$ fulfill (14) and then put $c_j = c'_j \cdot \lambda_j$.

Definition of $R_{krn\omega}$. Let to any $k, r, n \in \mathbb{N}$ and $\omega \in \mathcal{A}_0(K_r)$, a neighbourhood of zero $\mathcal{U}$ in the space $\mathcal{A}(K_{r+1})$ be assigned which is the unit ball for a smooth norm (see §2, Remark), independent of n, following Properties of R (§3) with $\mathscr{B} = \{\omega\} \subset \mathcal{A}(K_{r+1})$ and all $L \leq k$. Assume furthermore that $\mathcal{U}$ is as small as $|\mathrm{d}_\psi V_{N_L}(\varphi)| \leq 1$ whenever $L = 1, \ldots, k$, $\varphi \in \omega + 2\mathcal{U}$, $\psi \in \mathcal{U}$, due to the local equicontinuity of the differentials of $\mathscr{C}^\infty$ functions, [9, §2, Proposition].

Then the function $\varphi, x \mapsto R_{krn\omega}(\varphi, x)$ is defined on the domain

$$S_{2^{-n}}\big(\omega + (\mathcal{U} \cap \mathcal{A}(K_r))\big) \times \mathbb{R}^d = S_{2^{-n}}(\omega + \mathcal{U}) \cap \mathcal{A}_0(K_{r-n}) \times \mathbb{R}^d \tag{15}$$

as follows.

$$R_{krn\omega} := \lim_{\substack{J \in \mathbb{N} \\ J \to \infty}} R_J,$$

$$R_J(S_{2^{-n}}\, \varphi, x) := \int \cdots \int R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j \gamma_j\Big),\ x\Big) \prod_{j=1}^{J} \rho_\delta(t_j)\, \mathrm{d}t_j, \tag{16}$$

$\varphi \in (\omega + \mathcal{U}) \cap \mathcal{A}_0(K_r)$, $\rho_\delta := S_\delta\, \rho$ (§2, following (2)), $\delta = \delta_{kn} := 2^{-n(k+1)N_k}$.

For the sake of simplicity of the notation, we do not indicate the dependence of R_J on k, r, n, ω.

Properties of $R_{krn\omega}$.

(1°) If $k, r, n \in \mathbb{N}$, $\omega \in \mathcal{A}_0(K_r)$, then $R_{krn\omega}$ is well defined on its domain (15). If $x \in \mathbb{R}^d$, $\varphi \in \omega + (\mathcal{U} \cap \mathcal{A}(K_r))$, then

$$R_{krn\omega}(S_{2^{-n}}\varphi, x) = \lim_{J \to \infty} R_J(S_{2^{-n}}\, \varphi, x) \tag{17}$$

uniformly with respect to φ, x and

$$|R_{krn\omega}(S_{2^{-n}}\varphi, x)| \le C \cdot 2^{nN_1}\,, \tag{18}$$

with a constant C not depending on n.

(2°) If $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, $\ell = 0, 1, \ldots, L-1$, $L \le k$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le L$, $\varphi \in \omega + (\mathcal{U} \cap \mathcal{A}(K_r))$, $2^{-nN_L-1} > V_{N_L}(\varphi)$, $\psi_1, \ldots, \psi_\ell \in \mathcal{U} \cap \mathcal{A}(K_r)$, then

$$\partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} R_{krn\omega}(S_{2^{-n}}\varphi, x) = \lim_{J\to\infty} \partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} R_J(S_{2^{-n}}\varphi, x) \tag{19}$$

uniformly with respect to $x, \varphi, \psi_1, \ldots, \psi_\ell$ under the above conditions (k, r, n fixed). Consequently $\varphi, x \mapsto \partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} R_{krn\omega}(S_{2^{-n}}\varphi, x)$ is continuous, hence the order of derivatives (under the above conditions) does not matter. Furthermore we have

$$\left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} R_{krn\omega}(S_{2^{-n}}\varphi, x)\right| \le C\, 2^{nN_L} \tag{20}$$

with a constant C not depending on n, and

$$\left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} \big(R_{krn\omega}(S_{2^{-n}}\varphi, x) - R(S_{2^{-n}}\varphi, x)\big)\right| \le 2^{nN_L} \cdot \delta_{kn}\,. \tag{21}$$

(3°) $R_{krn\omega}$ is $\mathscr{C}^\infty$ with respect to the first variable on its domain (15) and there is an absolutely convex open neighbourhood of zero $\mathcal{V} = \mathcal{V}_{kr\omega} \subset \mathcal{A}(K_r)$ not depending on n such that if $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $L \in \mathbb{N}$, $\psi_\ell \in \mathcal{V}$ $(\ell = 1, \ldots, L)$, $\varphi \in \omega + \mathcal{U}$, then

$$\left|\mathrm{d}^L_{\psi_1,\ldots,\psi_L} R_{krn\omega}(S_{2^{-n}}\varphi, x)\right| \le b_L \cdot 2^{nN_1} \cdot \delta_{kn}^{-L} \tag{22}$$

with a constant b_L depending only on L and ρ.

Proof of (17) and (19): By the definition (16) we have

$$\partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} R_J(S_{2^{-n}}\varphi, x)$$
$$= \int \cdots \int \partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big) \prod_{j=1}^{J} \rho_\delta(t_j)\,\mathrm{d}t_j$$

and we have as well

$$\partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} R_J(S_{2^{-n}}\varphi, x)$$
$$= \int \cdots \int \partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big) \prod_{j=1}^{J+1} \rho_\delta(t_j)\,\mathrm{d}t_j,$$

because $\int \rho_\delta(t_{J+1})\,\mathrm{d}t_{J+1} = 1$. It follows

$$
\begin{aligned}
&\Big|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R_{J+1}(S_{2^{-n}}\varphi, x) - \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R_J(S_{2^{-n}}\varphi, x)\Big| \\
&= \bigg|\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta}\Big[\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J+1} t_j\gamma_j\Big),\ x\Big) \\
&\qquad - \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big)\Big] \cdot \prod_{j=1}^{J+1} \rho_\delta(t_j)\,\mathrm{d}t_j\bigg|.
\end{aligned}
\tag{23}
$$

Now we want to apply §3, Property of R (4°) for $\varepsilon = 2^{-n}$. By hypotheses of §3, (4°), there are two cases. For $L = 1$ this gives estimation

$$
\begin{aligned}
&|R_{J+1}(S_{2^{-n}}\varphi, x) - R_J(S_{2^{-n}}\varphi, x)| \\
&\le \Big|\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta} 2^{nN_1}\|t_{j+1}\gamma_{j+1}\|_{\mathcal{U}} \cdot \textstyle\prod_{j=1}^{J+1}\rho_\delta(t_j)\,\mathrm{d}t_j\Big| \\
&\le 2^{nN_1}\delta_{kn}\|\gamma_{J+1}\|_{\mathcal{U}} \le \|\gamma_{J+1}\|_{\mathcal{U}}
\end{aligned}
\tag{24}
$$

(δ defined in (16)), so by (14) the limit in (17) is uniform. Thus $R_{krn\omega}$ is well defined. Similarly (19) can be deduced from (23): By the local equicontinuity of $\mathrm{d}V_{N_L}(\varphi)$ noted in the definition of $R_{krn\omega}$, we get using the mean value theorem

$$
\Big|V_{N_L}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big) - V_{N_L}(\varphi)\Big| \le \Big\|\sum_{j=1}^{J} t_j\gamma_j\Big\|_{\mathcal{U}} \le \delta = 2^{-n(k+1)N_k} \le 2^{-nN_L-1}.
$$

From the hypothesis in (2°) $V_{N_L}(\varphi) < 2^{-nN_L-1}$, we obtain $\Big|V_{N_L}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big)\Big| \le 2^{-nN_L}$ that is the hypothesis in §3, Property (4°). Thus, by §3, (4°) (for $\ell = 0, 1, \dots, L-1$) we get from (23):

$$
\begin{aligned}
&\Big|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R_{J+1}(S_{2^{-n}}\varphi, x) - \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R_J(S_{2^{-n}}\varphi, x)\Big| \\
&\qquad \le 2^{nN_L}\,\|\psi_1\|_{\mathcal{U}}\cdots\|\psi_\ell\|_{\mathcal{U}} \cdot \delta_{kn}\|\gamma_{J+1}\|_{\mathcal{U}}\,.
\end{aligned}
\tag{25}
$$

As above, thanks to (14), the uniform convergence of the limit in (19) and then the equality (19) follows. □

Proof of (18) and (20): In all cases where the uniform convergence is already proved, we have

$$
\begin{aligned}
\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R_{krn\omega}(S_{2^{-n}}\,\varphi,\, x) &= \lim_{J\to\infty} \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R_J(S_{2^{-n}}\,\varphi, x) \\
&= \lim_{J\to\infty}\int\cdots\int \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big)\prod_{j=1}^{J}\rho_\delta(t_j)\,\mathrm{d}t_j\,.
\end{aligned}
$$

It was shown while proving (19) that the hypothesis in (2°) $V_{N_L}(\varphi) < 2^{-nN_L-1}$ implies $V_{N_L}\Big(\varphi + \sum\limits_{j=1}^{J} t_j\gamma_j\Big) \le 2^{-nN_L}$, and this is the hypothesis in Properties of R (§3) allowing us to use (3) and (4) for estimating the term $\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R\Big(S_{2^{-n}}\Big(\varphi + \sum\limits_{j=1}^{J} t_j\gamma_j\Big),\ x\Big)$ in the last integral. By §3, Property (3°) this hypothesis is not needed for proving (18). Thus (18) and (20) follow from the corresponding properties of R.

□

Proof of (21): (25) holds for $J=0$ as well with $R_J = R$. Adding the inequalities (25), we get

$$\Big|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R_{J+1}(S_{2^{-n}}\varphi, x) - \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R(S_{2^{-n}}\varphi, x)\Big|$$
$$\le 2^{nN_L}\,\|\psi_1\|_{\mathcal{U}}\dots\|\psi_\ell\|_{\mathcal{U}}\cdot\delta_{kn}\sum_{j=1}^{J+1}\|\gamma_j\|_{\mathcal{U}} \le 2^{nN_L}\,\|\psi_1\|_{\mathcal{U}}\dots\|\psi_\ell\|_{\mathcal{U}}\cdot\delta_{kn}$$

due to (14). Hence the inequality (21) is proved. □

Proof of 3°: For $L\in\mathbb{N}$ let $\psi_1,\dots,\psi_L\in\mathcal{A}(K_r)$ be given functions, let

$$\psi_\ell = \sum_{j=1}^{\infty} c_{\ell j}\,\gamma_j\,, \quad \text{i.e.} \quad S_{2^{-n}}\psi_\ell = \sum_{j=1}^{\infty} c_{\ell j}\,S_{2^{-n}}\gamma_j \quad (\ell = 1,\dots,L) \tag{26}$$

be their expansions by §4, Notation with γ_j fulfilling (14). As $\lim\limits_{j\to\infty} c_{\ell j} = 0$ (rapidly), there is an $A>0$ for which

$$|c_{\ell j}| \le A \qquad (\forall\,\ell = 1,\dots,L,\ j\in\mathbb{N}). \tag{27}$$

In the following calculation, $h_1,\dots,h_L$ are real variables with

$$|h_\ell| < \frac{1-\delta}{LA} \tag{28}$$

and we have to put $h_1,\dots,h_L = 0$ to obtain the following equality:

$$\mathrm{d}^L_{\psi_1,\dots,\psi_L} R_{krn\omega}(S_{2^{-n}}\,\varphi,\,x) = \frac{\partial^L}{\partial h_1\dots\partial h_L}\lim_{J\to\infty} R_J\Big(S_{2^{-n}}\Big(\varphi + \sum_{\ell=1}^{L} h_\ell\psi_\ell\Big), x\Big)$$
$$= \frac{\partial^L}{\partial h_1\dots\partial h_L}\lim_{J\to\infty}\int\!\cdots\!\int R\Big(S_{2^{-n}}\Big(\varphi + \sum_{\ell=1}^{L} h_\ell\psi_\ell + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big)\prod_{j=1}^{J}\rho_\delta(t_j)\,\mathrm{d}t_j\,.$$

By (26) this is equal to

$$\frac{\partial^L}{\partial h_1\ldots\partial h_L}\lim_{J\to\infty}\int\cdots\int R\Big(S_{2^{-n}}\Big(\varphi+\sum_{\ell=1}^{L}h_\ell\sum_{j=1}^{\infty}c_{\ell j}\,\gamma_j+\sum_{j=1}^{J}t_j\gamma_j\Big),\ x\Big)\cdot\prod_{j=1}^{J}\rho_\delta(t_j)\,\mathrm{d}t_j$$

$$(29)\qquad =\frac{\partial^L}{\partial h_1\ldots\partial h_L}\lim_{J\to\infty}\int\cdots\int R\Big(S_{2^{-n}}\Big(\varphi+\sum_{\ell=1}^{L}h_\ell\sum_{j=1}^{J}c_{\ell j}\,\gamma_j+\sum_{j=1}^{J}t_j\gamma_j\Big),\ x\Big)\cdot\prod_{j=1}^{J}\rho_\delta(t_j)\,\mathrm{d}t_j$$

because by §3, Property (4°), (27) and (28), the difference of both expressions after $\lim\limits_{J\to\infty}$ is estimated by

$$2^{nN_1}\sum_{\ell=1}^{L}|h_\ell|\sum_{j=J+1}^{\infty}|c_{\ell j}|\cdot\|\gamma_j\|_{\mathcal{U}}\le 2^{nN_1}(1-\delta)\sum_{j=J+1}^{\infty}\|\gamma_j\|_{\mathcal{U}}\,.$$

This tends to zero thanks to (14), only we have to verify the hypothesis in §3, (4°) that $\varphi+\sum\limits_{\ell=1}^{L}h_\ell\sum\limits_{j=1}^{\infty}c_{\ell j}\,\gamma_j+\sum\limits_{j=1}^{J}t_j\gamma_j$ and $\varphi+\sum\limits_{\ell=1}^{L}h_\ell\sum\limits_{j=1}^{J}c_{\ell j}\,\gamma_j+\sum\limits_{j=1}^{J}t_j\gamma_j$ are elements of $\omega+2\mathcal{U}$. Indeed, $\varphi\in\omega+\mathcal{U}$ and for the other terms we have by (28), (27) and (14)

$$\Big\|\sum_{\ell=1}^{L}h_\ell\sum_{j=1}^{\infty}c_{\ell j}\,\gamma_j+\sum_{j=1}^{J}t_j\gamma_j\Big\|_{\mathcal{U}}<\sum_{\ell=1}^{L}\frac{1-\delta}{L\,A}\sum_{j=1}^{\infty}A\|\gamma_j\|_{\mathcal{U}}+\sum_{j=1}^{J}\delta\|\gamma_j\|_{\mathcal{U}}\le 1.$$

Thus (29) is verified. After a substitution in (29) (and putting $h_1,\ldots,h_L=0$) we get

$$(30)\quad \mathrm{d}^L_{\psi_1,\ldots,\psi_L}R_{krn\omega}(S_{2^{-n}}\varphi,\,x)=\frac{\partial^L}{\partial h_1\ldots\partial h_L}\lim_{J\to\infty}\int\cdots\int R\Big(S_{2^{-n}}\Big(\varphi+\sum_{j=1}^{J}t_j\gamma_j\Big),\ x\Big)\cdot\prod_{j=1}^{J}\rho_\delta\Big(t_j-\sum_{\ell=1}^{L}h_\ell c_{\ell j}\Big)\,\mathrm{d}t_j$$

$$=\lim_{J\to\infty}\int\cdots\int R\Big(S_{2^{-n}}\Big(\varphi+\sum_{j=1}^{J}t_j\gamma_j\Big),\ x\Big)\cdot\frac{\partial^L}{\partial h_1\ldots\partial h_L}\prod_{j=1}^{J}\rho_\delta\Big(t_j-\sum_{\ell=1}^{L}h_\ell c_{\ell j}\Big)\,\mathrm{d}t_j$$

provided the last limit is uniform with respect to h_ℓ, $|h_\ell| < \frac{1-\delta}{LA}$ $\quad(\ell = 1, \dots, L)$.

Now we are going to prove it. Let us denote the last integral by I_J. Using the Leibniz rule for the derivation of a product:

$$\frac{\partial}{\partial h_\ell}\prod_{j=1}^{J}\rho_\delta\Big(t_j - \sum_{\ell=1}^{L} h_\ell c_{\ell j}\Big) = \sum_{j_\ell=1}^{J}(-c_{\ell,j_\ell})\frac{\partial}{\partial t_{j_\ell}}\prod_{j=1}^{J}\rho_\delta\Big(t_j - \sum_{\ell=1}^{L} h_\ell c_{\ell j}\Big),$$

we obtain

$$I_J = \int\cdots\int R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big) \cdot \sum_{j_1,\dots,j_L=1}^{J}\Big(\prod_{\ell=1}^{L} -c_{\ell,j_\ell}\frac{\partial}{\partial t_{j_\ell}}\Big)\prod_{j=1}^{J}\rho_\delta\Big(t_j - \sum_{\ell=1}^{L} h_\ell c_{\ell j}\Big)\,\mathrm{d}t_j\,.$$

Using the Kronecker delta ($\boldsymbol{\delta}_j^{j'}$ is the truth value of the statement $j = j'$) we can write

$$I_J = \int\cdots\int R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big) \cdot \sum_{j_1,\dots,j_L=1}^{J}\Big(\prod_{\ell=1}^{L} -c_{\ell,j_\ell}\Big)\prod_{j=1}^{J}\big(\tfrac{\partial}{\partial t_j}\big)^{\sum_{\ell=1}^{L}\boldsymbol{\delta}_j^{j_\ell}}\rho_\delta\Big(t_j - \sum_{\ell=1}^{L} h_\ell c_{\ell j}\Big)\mathrm{d}t_j$$

and we have as well

$$I_J = \int\cdots\int R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big) \cdot \sum_{j_1,\dots,j_L=1}^{J+1}\Big(\prod_{\ell=1}^{L} -c_{\ell,j_\ell}\Big)\prod_{j=1}^{J+1}\big(\tfrac{\partial}{\partial t_j}\big)^{\sum_{\ell=1}^{L}\boldsymbol{\delta}_j^{j_\ell}}\rho_\delta\Big(t_j - \sum_{\ell=1}^{L} h_\ell c_{\ell j}\Big)\,\mathrm{d}t_j\,.$$

Indeed, if some $j_\ell = J+1$, then the term $\rho_\delta\Big(t_{J+1} - \sum_{\ell=1}^{L} h_\ell c_{\ell,J+1}\Big)$ is differentiated and so its integral is equal to 0; else its integral is is equal to 1. It follows

$$I_{J+1} - I_J = \int\cdots\int\Big[R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J+1} t_j\gamma_j\Big),\ x\Big) - R\Big(S_{2^{-n}}\Big(\varphi + \sum_{j=1}^{J} t_j\gamma_j\Big),\ x\Big)\Big] \cdot \sum_{j_1,\dots,j_L=1}^{J+1}\Big(\prod_{\ell=1}^{L} -c_{\ell,j_\ell}\Big)\prod_{j=1}^{J+1}\big(\tfrac{\partial}{\partial t_j}\big)^{\sum_{\ell=1}^{L}\boldsymbol{\delta}_j^{j_\ell}}\rho_\delta\Big(t_j - \sum_{\ell=1}^{L} h_\ell c_{\ell j}\Big)\,\mathrm{d}t_j\,.$$

By (27) and (28) we have $\left|\sum\limits_{\ell=1}^{L} h_\ell c_{\ell j}\right| \le 1-\delta$, so $|t_j| \le 1$ or $\rho_\delta\Bigl(t_j - \sum\limits_{\ell=1}^{L} h_\ell c_{\ell j}\Bigr) = 0$, and we can apply §3, Property of R (4°). We get

$$|I_{J+1} - I_J| \le 2^{nN_1} \|\gamma_{J+1}\|_{\mathcal{U}} \sum_{j_1,\dots,j_L=1}^{J+1} \prod_{\ell=1}^{L} |c_{\ell,j_\ell}| \prod_{j=1}^{J+1} \bigl\|\partial^{\sum_{\ell=1}^{L} \boldsymbol{\delta}_j^{j_\ell}} \rho_\delta\bigr\|_{\mathscr{L}^1}.$$

It is $\bigl\|\rho_\delta\bigr\|_{\mathscr{L}^1} = 1$ (§2, following (2)) and, in the last product, for given $j_1,\dots,j_L$ there are at most L indices j for which ρ_δ is differentiated. Thus this product can be estimated

$$\begin{aligned}\prod_{j=1}^{J+1} \bigl\|\partial^{\sum_{\ell=1}^{L} \boldsymbol{\delta}_j^{j_\ell}} \rho_\delta\bigr\|_{\mathscr{L}^1} &= \prod_{j=1}^{J+1} \delta^{-\sum_{\ell=1}^{L} \boldsymbol{\delta}_j^{j_\ell}} \bigl\|\partial^{\sum_{\ell=1}^{L} \boldsymbol{\delta}_j^{j_\ell}} \rho\bigr\|_{\mathscr{L}^1} \\ &\le \delta^{-\sum_{j=1}^{J+1}\sum_{\ell=1}^{L} \boldsymbol{\delta}_j^{j_\ell}} \Bigl(\max_{0\le \ell\le L} \|\partial^\ell \rho\|_{\mathscr{L}^1}\Bigr)^L = \delta^{-L}\cdot b_L\end{aligned}$$

with a constant b_L depending only on ρ and L. It follows

$$\begin{aligned}|I_{J+1} - I_J| \;&\le\; 2^{nN_1}\delta^{-L} b_L \|\gamma_{J+1}\|_{\mathcal{U}} \sum_{j_1,\dots,j_L=1}^{J+1} \prod_{\ell=1}^{L} |c_{\ell,j_\ell}| \\ &= 2^{nN_1}\delta^{-L} b_L \|\gamma_{J+1}\|_{\mathcal{U}} \prod_{\ell=1}^{L}\sum_{j=1}^{J+1} |c_{\ell j}| \;\le\; 2^{nN_1}\delta^{-L} b_L \|\gamma_{J+1}\|_{\mathcal{U}} \prod_{\ell=1}^{L}\sum_{j=1}^{\infty} |c_{\ell j}| \\ &=\; 2^{nN_1} c^L \delta^{-L} b_L \|\gamma_{J+1}\|_{\mathcal{U}}\end{aligned} \tag{31}$$

where the constant $c = \max\limits_{\ell=1,\dots,L} \sum_{j=1}^{\infty} |c_{\ell j}|$ depends only on $\psi_1,\dots,\psi_L$ and r. Now the uniform convergence of (30) can be deduced from (14). The smoothness is verified; it remains to deduce the estimations.

The inequality (31) holds for $J = 0$ as well with $I_0 = 0$. Adding these inequalities we get

$$|I_J| \;\le\; 2^{nN_1} c^L \delta^{-L} b_L \sum_{j=1}^{J} \|\gamma_j\|_{\mathcal{U}} \;\le\; 2^{nN_1} c^L \delta^{-L} b_L\,.$$

This is an estimation for the last integral in (30). Hence

$$|\mathrm{d}^L_{\psi_1,\dots,\psi_L} R_{krn\omega}(S_{2^{-n}}\,\varphi,\, x)| \le b_L c^L 2^{nN_1}\cdot \delta^{-L}.$$

b_L depends only on ρ and L, c is the constant in (31), $c = 1$ if $\psi_1, \ldots, \psi_L$ belong to

$$\mathcal{V} = \left\{ \psi \in \mathcal{A}(K_r);\ \psi = \sum c_j \gamma_j\,,\ \sum |c_j| \le 1 \right\}.$$

By §4, Result γ_j depends on $\mathcal{U}$ not depending on n. It remains to prove that $\mathcal{V}$ is a neighbourhood of zero in $\mathcal{A}(K_r)$. It is known that the Fourier coefficients of functions ψ running over a bounded set in $\mathcal{A}(K_r)$ tend uniformly rapidly to zero. Evidently the same holds for the coefficients c_j in §4, Notation. Hence any bounded set is absorbed by $\mathcal{V}$. In a metric vector space such sets $\mathcal{V}$ are neighbourhoods of zero. □

§**5. Partition of unity.** The space $\mathscr{D}$ has the property of smooth partition of unity expressed by the following

Theorem. *For $B \Subset \mathbb{R}^d$, let $\{\omega_s + \mathcal{U}_s\}_{s \in S}$ be an open covering of the space $\mathscr{D}(B)$, where S is an arbitrary set of indices, $\omega_s \in \mathscr{D}(B)$ $(\forall\, s \in S)$, $\mathcal{U}_s$ are open neighbourhoods of zero in $\mathscr{D}(B)$. Then there is a locally finite smooth (i.e. $\mathscr{C}^\infty$) partition of unity on $\mathscr{D}(B)$*

$$1 = \sum_{m=1}^{\infty} \Phi_m$$

subordinated to this covering.

This means:

1° The functions $\Phi_m : \mathscr{D}(B) \to [0,1]$, fulfilling this equality, are $\mathscr{C}^\infty$and

$$\forall\, m \quad \exists\, s \in S: \ \operatorname{supp} \Phi_m \subset \omega_s + \mathcal{U}_s\,.$$

2° For every $\omega \in \mathscr{D}(B)$ there is an absolutely convex open neighbourhood of zero $\mathcal{U} \subset \mathscr{D}(B)$ such that $\omega + \mathcal{U}$ meets only a finite number of supports of functions Φ_m.

For the proof, we refer to [13, (5.3.8)], where a more general theorem is proved concerning several cathegories of smoothness, not only $\mathscr{C}^\infty$. Hypotheses: $\mathscr{D}$ is a Lindelöf locally convex space and there are sufficiently many $\mathscr{C}^\infty$ functions on $\mathscr{D}$ so that they generate the original topology on $\dot{\mathscr{D}}$This is fulfilled, see §2, Remark.

Corollary. *For $B \Subset \mathbb{R}^d$, let $\{\omega_s + \mathcal{U}_s\}_{s \in S}$ be an open covering of the space $\mathcal{A}_0(B)$, where $(\forall\, s \in S)$ $\omega_s \in \mathcal{A}_0(B)$ and $\mathcal{U}_s$ is an open neighbourhood of zero in $\mathcal{A}(B)$. Then there is a locally finite smooth partition of unity on $\mathcal{A}_0(B)$*

$$1 = \sum_{m=1}^{\infty} \Phi_m$$

subordinated to this covering.

Proof: We can write $\mathcal{U}_s = \widetilde{\mathcal{U}}_s \cap \mathcal{A}(B)$ where $\widetilde{\mathcal{U}}_s$ are neighbourhoods of zero in the space $\mathscr{D}(B)$. Then we apply Theorem to the covering $\{(\omega_s + \widetilde{\mathcal{U}}_s)\}_s \cup \{\mathscr{D}(B) \smallsetminus \mathcal{A}_0(B)\}$ of $\mathscr{D}(B)$. □

§**6. Notation.** Let us have chosen $k, r \in \mathbb{N}$. Then, for every $\omega \in \mathcal{A}_0(K_r)$, we have a neighbourhood of zero $\mathcal{U}_\omega \subset \mathcal{A}(K_r)$ (independent of n) such that $\forall\, n \in \mathbb{N}$ the function $R_{krn\omega}$ is defined by §4, on $S_{2^{-n}}(\omega + \mathcal{U}_\omega) \times \mathbb{R}^d$. Thus we have a covering of $\mathcal{A}_0(K_r)$ with the sets $\omega + \mathcal{U}_\omega$. We choose a partition of unity $1 = \sum_{m=1}^\infty \Phi_m$ on $\mathcal{A}_0(K_r)$ by the above corollary. For every m, we choose a test function ω_m for which $\operatorname{supp} \Phi_m \subset \omega_m + \mathcal{U}_{\omega_m}$; we will use the notation $\mathcal{U}_m$ rather than $\mathcal{U}_{\omega_m}$.

Definition of R_{krn}. With the above notation, for $\varphi \in \mathcal{A}_0(K_r)$ (so $S_{2^{-n}}\, \varphi \in \mathcal{A}_0(K_{r-n})$) we define

$$R_{krn}(S_{2^{-n}}\varphi, x) := \sum_{m=1}^{\infty} \Phi_m(\varphi) \cdot R_{krn\omega_m}(S_{2^{-n}}\varphi, x). \tag{32}$$

If φ does not belong to $\operatorname{supp} \Phi_m$, the term of this sum is considered to be zero even if $R_{krn\omega_m}(S_{2^{-n}}\varphi, x)$ is not defined.

Properties of R_{krn}. (1°) For every $k, r, n \in \mathbb{N}$, the function R_{krn} is defined on $\mathcal{A}_0(K_{r-n}) \times \mathbb{R}^d$.

Moreover, for every $\omega \in \mathcal{A}_0(K_r)$ and $L \in \mathbb{N}_0$, there exist an absolutely convex open neighbourhood of zero $\mathcal{U} \subset \mathcal{A}(K_r)$ and a constant $C > 0$, both independent of n, such that for every $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ the following hold.

(2°) If $1 \le L \le k$, $\ell = 0, 1, \dots, L-1$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le L$, $\varphi \in \omega + \mathcal{U}$, $2^{-nN_L - 1} > V_{N_L}(\varphi)$, $\psi_1, \dots, \psi_\ell \in \mathcal{U}$, then $\varphi, x \mapsto \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell}\, \partial^\alpha R_{krn}(S_{2^{-n}}\varphi, x)$ is continuous and

$$\left|\mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell}\, \partial^\alpha R_{krn}(S_{2^{-n}}\varphi, x)\right| \le C \cdot 2^{nN_L}, \tag{33}$$

$$\left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} \left(R_{krn}(S_{2^{-n}}\varphi, x) - R(S_{2^{-n}}\varphi, x)\right)\right| \le C\, 2^{nN_L} \cdot \delta_{kn} \tag{34}$$

($\delta = \delta_{kn}$ by (16)).

(3°) R_{krn} is $\mathscr{C}^\infty$ with respect to the first variable on its domain $\mathcal{A}_0(K_{r-n})$ and if $\varphi \in \omega + \mathcal{U}$ and $\psi_\ell \in \mathcal{U}$ $\quad(\ell = 1, \dots, L)$, then

$$\left|\mathrm{d}^L_{\psi_1,\dots,\psi_L} R_{krn}(S_{2^{-n}}\varphi, x)\right| \le C\, 2^{nN_1} \cdot \delta_{kn}^{-L}.$$

Proof of (2°): If $\omega \in \mathcal{A}_0(K_r)$, we choose a neighbourhood of zero $\mathcal{U} \subset \mathcal{A}(K_r)$ such that $\omega + \mathcal{U}$ meets only a finite number of supports of the functions Φ_m (by 5.2°). So $\mathcal{U}$ can be chosen as small as $(\forall\, m)$

$$\text{either} \quad (\omega + \mathcal{U}) \cap \operatorname{supp} \Phi_m = \emptyset \quad \text{or} \quad (\omega + \mathcal{U}) \subset \omega_m + \mathcal{U}_m\,. \tag{35}$$

Furthermore, thanks to [9, §2, Proposition], let $\mathcal{U}$ be as small as

$$\left|\mathrm{d}^{\ell}_{\psi_1,\dots,\psi_\ell}\,\Phi_m(\varphi)\right| \le 1 \tag{36}$$

whenever $1 \le \ell \le k-1$, $m \in \mathbb{N}$, $\varphi \in \omega + \mathcal{U}$, $\psi_1,\dots,\psi_\ell \in \mathcal{U}$; for $\ell = 0$ this is fulfilled, too. Now we use the following Leibniz rule for derivation of a product of two functions: if F_1, F_2 are two smooth functions on (a part of) a locally convex space, then

$$\mathrm{d}^{\ell}\big(F_1(\varphi)F_2(\varphi)\big)[\psi_1,\dots,\psi_\ell] = \sum_{\substack{I_1 \\ I_1\cup I_2=\{1,\dots,\ell\} \\ \text{disjoint}}} \mathrm{d}^{\#I_1}_{\boldsymbol{\psi}_{I_1}} F_1(\varphi)\cdot \mathrm{d}^{\#I_2}_{\boldsymbol{\psi}_{I_2}} F_2(\varphi),$$

where, for $I = \{i_1,\dots,i_{\#I}\} \subset \{1,\dots,\ell\}$, $\boldsymbol{\psi}_I$ denotes the finite sequence $\psi_{i_1},\dots,\psi_{i_{\#I}}$ and the summation is extended over all ordered decompositions of multi-index $(1,\dots,\ell)$ in two disjoint multi-indices I_1, I_2 that are written in the increasing order. Unlike the chain rule (theorem on the derivative of the composition), here the multi-index I_1 or I_2 can be empty. The proof of the Leibniz formula can be deduced easily from the chain rule if the outer function is $F(u,v) = uv$, $u = F_1$, $v = F_2$.

We apply the Leibniz rule for differentiating the product to the defining formula (32):

$$\begin{aligned}\mathrm{d}^{\ell}_{\psi_1,\dots,\psi_\ell}&\partial^\alpha R_{krn}(S_{2^{-n}}\varphi,x)\\ &= \sum_{m=1}^{\infty}\sum_{\substack{I_1 \\ I_1\cup I_2=\{1,\dots,\ell\} \\ \text{disjoint}}} \mathrm{d}^{\#I_1}_{\boldsymbol{\psi}_{I_1}}\Phi_m(\varphi)\cdot \mathrm{d}^{\#I_2}_{\boldsymbol{\psi}_{I_2}}\partial^\alpha R_{krn\omega_m}(S_{2^{-n}}\varphi,x)\end{aligned}$$

and, by (35) as $\varphi \in \omega+\mathcal{U}$, the first sum is extended only over a finite number of m for which $(\omega+\mathcal{U}) \subset \omega_m+\mathcal{U}_m$. Then $(\omega+\mathcal{U})-(\omega+\mathcal{U}) \subset (\omega_m+\mathcal{U}_m)-(\omega_m+\mathcal{U}_m)$; for absolutely convex sets it follows $\mathcal{U} \subset \mathcal{U}_m$. Hence the functions $R_{krn\omega_m}$, fulfilling §4, Properties on $\omega_m+\mathcal{U}_m$ fulfil it the more on $\omega+\mathcal{U}$. By the above Leibniz formula, we estimate $\mathrm{d}^{\ell}_{\psi_1,\dots,\psi_\ell}\partial^\alpha R_{krn}(S_{2^{-n}}\varphi,x)$ using (36) for estimating the term $\mathrm{d}^{\#I_1}_{\boldsymbol{\psi}_{I_1}}\Phi_m(\varphi)$ and using §4, Properties of $R_{krn\omega}$ for estimating the term $\mathrm{d}^{\#I_2}_{\boldsymbol{\psi}_{I_2}}\partial^\alpha R_{krn\omega_m}(S_{2^{-n}}\varphi,x)$. So we deduce (33) from the corresponding inequality (20) in Properties of $R_{krn\omega}$. The constant C in (33) depends only on the constants assigned by §4, Properties to the functions $R_{krn\omega_m}$ and on the used finite set of terms of the sum $\sum_m$, so it is independent of n.

For estimating $\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell}\big(R_{krn}(S_{2^{-n}}\varphi,x)-R(S_{2^{-n}}\varphi,x)\big)$, we apply the Leibniz formula to the product $\sum_{m=1}^\infty \Phi_m(\varphi)\cdot\big(R_{krn\omega_m}(S_{2^{-n}}\varphi,x)-R(S_{2^{-n}}\varphi,x)\big)$ and proceed similarly. □

Proof of (3°): Unlike in §4, Properties of $R_{krn\omega}$, here the neighbourhood $\mathcal{U}$ depends on L. We chose $\mathcal{U}$ as small as (36) hold whenever $1\le \ell\le L$, $m\in\mathbb{N}$, $\varphi\in\omega+\mathcal{U}$, $\psi_1,\dots,\psi_\ell\in\mathcal{U}$. Denote by $\mathcal{V}_m$ the neighbourhood $\mathcal{V}$ defined by §4 (3°) for the function $R_{krn\omega_m}$ and chose furthermore $\mathcal{U}$ such that we have instead of (35):

$$\text{either}\quad (\omega+\mathcal{U})\cap\operatorname{supp}\Phi_m=\emptyset\quad\text{or}\quad(\omega+\mathcal{U})\subset\omega_m+(\mathcal{V}_m\cap\mathcal{U}_m).$$

Then we follow the above proof and deduce (3°) from the corresponding properties of $R_{krn\omega}$, §4: (18) and the item (3°). □

§**7. Notations.** Choose test functions $\psi_\alpha\in\mathscr{D}(K_r\setminus K_{r-1})$ ($\alpha\in\mathbb{N}_0^d$, $0\le|\alpha|\le N_k$), fulfilling (like in [8, (22), (23)])

$$\int\psi_\alpha(\xi)\cdot\xi^\alpha\,\mathrm{d}\xi=1$$
$$\int\psi_\alpha(\xi)\cdot\xi^\beta\,\mathrm{d}\xi=0\quad\text{for}\quad\beta\ne\alpha,\ 0\le|\beta|\le N_k\,.$$

Let $\Lambda_r:\ \mathscr{D}(K_{r+1})\to\mathscr{D}(K_r)$ be a continuous (hence smooth) linear mapping defined (see end of §2, ϑ_r means $\vartheta_r^{\otimes d}$):

$$\Lambda_r\varphi:=\varphi\cdot\vartheta_{r-1}+\sum_{0\le|\alpha|\le N_k}c_\alpha\,\psi_\alpha$$

with such constants c_α depending on φ that

$$\forall\,\beta\in\mathbb{N}_0^d,0\le|\beta|\le N_k:\qquad\int\Lambda_r\,\varphi(\xi)\xi^\beta\,\mathrm{d}\xi=\int\varphi(\xi)\xi^\beta\,\mathrm{d}\xi\,.$$

This means $\int\varphi(\xi)\xi^\beta\,\mathrm{d}\xi=\int\vartheta_{r-1}(\xi)\varphi(\xi)\xi^\beta\,\mathrm{d}\xi+c_\beta$, hence c_β are well determined; Λ_r maps $\mathcal{A}_0(K_{r+1})$ into $\mathcal{A}_0(K_r)$ and $\mathcal{A}(K_{r+1})$ into $\mathcal{A}(K_r)$. Λ_r is identical on $\mathscr{D}(K_{r-1})$; for $\varphi\in\mathcal{A}_0(K_{r+1})$ and $N\le N_k$ we have $V_N(\Lambda_r\varphi)=V_N(\varphi)$.

Definition of R'_{krn}. Let $k,n\in\mathbb{N}$ be given. For $r\in\mathbb{N}$, we define the functions R'_{krn} on $\mathcal{A}_0(K_{r-n})\times\mathbb{R}^d$ by induction as follows. $R'_{k1n}=R_{k1n}$. If R'_{krn} is already defined on $\mathcal{A}_0(K_{r-n})\times\mathbb{R}^d$, we define

$$\begin{aligned}(37)\quad & R'_{k,r+1,n}(S_{2^{-n}}\varphi,x):=\\ & R_{k,r+1,n}(S_{2^{-n}}\varphi,x)+R'_{krn}\big(S_{2^{-n}}(\Lambda_r\varphi),\,x\big)-R_{k,r+1,n}\big(S_{2^{-n}}(\Lambda_r\varphi),\,x\big)\\ &\qquad\qquad\text{for}\quad\varphi\in\mathcal{A}_0(K_{r+1}),\ x\in\mathbb{R}^d\,.\end{aligned}$$

Properties of R'_{krn}. (1°) For every $k, r, n \in \mathbb{N}$ the function R'_{krn} is defined on $\mathcal{A}_0(K_{r-n}) \times \mathbb{R}^d$ and for $\varphi \in \mathcal{A}_0(K_{r-n-1})$ it is $R'_{k,r,n}(\varphi, x) = R'_{k,r+1,n}(\varphi, x)$.

Moreover, for every $\omega' \in \mathcal{A}_0(K_r)$ and $L \in \mathbb{N}_0$ there is an absolutely convex open neighbourhood of zero $\mathcal{U}' \subset \mathcal{A}(K_r)$ and a constant $C'_{kr} > 0$, both independent of n, such that for every $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ the following hold.

(2°) If $1 \le L \le k$, $\ell = 0, 1, \dots, L-1$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le L$, $\varphi' \in \omega' + \mathcal{U}'$, $2^{-nN_L-1} > V_{N_L}(\varphi')$, $\psi'_1, \dots, \psi'_\ell \in \mathcal{U}'$, then $\varphi', x \mapsto \mathrm{d}^\ell_{\psi'_1,\dots,\psi'_\ell} \partial^\alpha R'_{krn}(S_{2^{-n}}\varphi', x)$ is continuous and

$$\left|\mathrm{d}^\ell_{\psi'_1,\dots,\psi'_\ell} \partial^\alpha R'_{krn}(S_{2^{-n}}\varphi', x)\right| \le C'_{kr}\, 2^{nN_L}, \tag{38}$$

$$\left|\partial^\alpha \mathrm{d}^\ell_{\psi'_1,\dots,\psi'_\ell} \left(R'_{krn}(S_{2^{-n}}\varphi', x) - R(S_{2^{-n}}\varphi', x)\right)\right| \le C'_{kr}\, 2^{nN_L} \cdot \delta_{kn}\,. \tag{39}$$

(3°) R'_{krn} is $\mathscr{C}^\infty$ with respect to the first variable on its domain $\mathcal{A}_0(K_{r-n})$ and $\forall\, \varphi' \in \omega' + \mathcal{U}'$, $\psi'_\ell \in \mathcal{U}'$ $\quad (\ell = 1, \dots, L)$, it is

$$\left|\mathrm{d}^L_{\psi'_1,\dots,\psi'_L} R'_{krn}(S_{2^{-n}}\varphi', x)\right| \le C'_{kr}\, 2^{nN_1} \cdot \delta_{kn}^{-L}.$$

Proof of (38): (1°) follows easily from the fact that Λ_r is identical on $\mathcal{A}_0(K_{r-1})$. The other properties will be proved by induction. For $r = 1$ this is affirmed by §6, Properties of R_{krn}. Assuming that (38) is satisfied for un certain r, we have to prove it for $r + 1$. So we have to prove that every one of the three terms on the right-hand side of the defining equality (37) satisfies (38). This is clear for $R_{k,r+1,n}(S_{2^{-n}}\varphi, x)$ due to Properties of R_{krn}, (33). We are going to prove it for $R'_{krn}(S_{2^{-n}}(\Lambda_r\varphi), x)$. Let, by the hypothesis, $\omega \in \mathcal{A}_0(K_{r+1})$. Then $\omega' := \Lambda_r(\omega) \in \mathcal{A}_0(K_r)$ and by the induction assumption we have $\mathcal{U}' \subset \mathcal{A}(K_r)$ and $C'_{kr} > 0$, both independent of n, fulfilling (2°). Now, $\mathcal{U} := \Lambda_r^{-1}\mathcal{U}'$ is a neighbourhood of zero in $\mathcal{A}(K_{r+1})$. For $\psi_j \in \mathcal{U}$ $\quad (j = 1, \dots, \ell)$ and $\varphi \in \omega + \mathcal{U}$ we have $\psi'_j := \Lambda_r\, \psi_j \in \mathcal{U}'$, $\varphi' := \Lambda_r\varphi \in \omega' + \mathcal{U}'$, hence (chain rule with the inner function Λ_r linear)

$$\left|\mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} \partial^\alpha R'_{krn}(S_{2^{-n}}(\Lambda_r\varphi),\, x)\right| = \left|\mathrm{d}^\ell_{\psi'_1,\dots,\psi'_\ell} \partial^\alpha R'_{krn}(S_{2^{-n}}\varphi', x)\right| \le C'_{kr}\, 2^{nN_L}.$$

Exactly by the same way we deduce the same estimation for the last term in (37), i.e.

$$\left|\mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} \partial^\alpha R_{k,r+1,n}(S_{2^{-n}}(\Lambda_r\varphi),\, x)\right| \le C\, 2^{nN_L},$$

only we have to start with (33) instead of the induction assumption. Thus (38) is proved by induction. □

Proof of (39): Taking (39) as the induction assumption, we get by the recurrent definition (37) of R'_{krn}:

$$\begin{aligned}
&\Big|\mathrm{d}^{\ell}_{\psi_1,\ldots,\psi_\ell}\partial^\alpha\big(R'_{k,r+1,n}(S_{2^{-n}}\varphi,x)-R(S_{2^{-n}}\varphi,x)\big)\Big|\\
&\le \Big|\mathrm{d}^{\ell}_{\psi_1,\ldots,\psi_\ell}\partial^\alpha\big(R_{k,r+1,n}(S_{2^{-n}}\varphi,x)-R(S_{2^{-n}}\varphi,x)\big)\Big|\\
&\quad+\Big|\mathrm{d}^{\ell}_{\psi_1,\ldots,\psi_\ell}\partial^\alpha\big[R'_{k,r,n}(S_{2^{-n}}(\Lambda_r\varphi),x)-R(S_{2^{-n}}(\Lambda_r\varphi),x)\big]\\
&\quad-\mathrm{d}^{\ell}_{\psi_1,\ldots,\psi_\ell}\partial^\alpha\big[R_{k,r+1,n}(S_{2^{-n}}(\Lambda_r\varphi),x)-R(S_{2^{-n}}(\Lambda_r\varphi),x)\big]\Big|.
\end{aligned}$$

As above, we estimate every one of these three terms using (34) and the induction assumption. □

Proof of (3°): by §6 (3°) can be the same as the proof of (38). □

§8. Definition of R'_{kn}. We define

$$R'_{kn}(S_{2^{-n}}\varphi,x)=\lim_{r\to\infty}R'_{krn}(S_{2^{-n}}\varphi,x)$$

for $\varphi\in\mathcal{A}_0(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$. Every φ belongs to $\mathcal{A}_0(K_{r-1})$ for some $r\in\mathbb{N}$; up from this r the sequence $\{R'_{krn}\}_r$ is constant thanks to §7, Property 1°, so it is $R'_{kn}(S_{2^{-n}}\varphi,x)=R'_{krn}(S_{2^{-n}}\varphi,x)$.

Properties of R'_{kn}. (1°). For every $k,n\in\mathbb{N}$ the function R'_{kn} is defined on $\mathcal{A}_0(\mathbb{R}^d)\times\mathbb{R}^d$.

Moreover, if $B\Subset\mathbb{R}^d$, $\mathscr{B}$ a bounded set $\subset\mathcal{A}_0(B)$ and $L\in\mathbb{N}_0$, then there is an absolutely convex open neighbourhood of zero $\mathcal{U}'=\mathcal{U}'_k\subset\mathcal{A}(B)$ and a constant $C'_k>0$, both independent of n, such that for every $n\in\mathbb{N}$, $x\in\mathbb{R}^d$ the following hold.

(2°) If $1\le L\le k$, $\ell=0,1,\ldots,L-1$, $\alpha\in\mathbb{N}_0^d$, $|\alpha|\le L$, $\varphi\in\mathscr{B}$, $2^{-nN_L-1}>V_{N_L}(\varphi)$, $\psi_1,\ldots,\psi_\ell\in\mathcal{U}'$, then $\varphi,x\mapsto\mathrm{d}^{\ell}_{\psi'_1,\ldots,\psi'_\ell}\,\partial^\alpha\,R'_{kn}(S_{2^{-n}}\varphi,x)$ is continuous and

$$\Big|\mathrm{d}^{\ell}_{\psi_1,\ldots,\psi_\ell}\,\partial^\alpha\,R'_{kn}(S_{2^{-n}}\varphi,x)\Big|\le C'_k\,2^{nN_L},\tag{40}$$

$$\Big|\partial^\alpha\mathrm{d}^{\ell}_{\psi_1,\ldots,\psi_\ell}\,\big(R'_{kn}(S_{2^{-n}}\varphi,x)-R(S_{2^{-n}}\varphi,x)\big)\Big|\le C'_k\,2^{nN_L}\cdot\delta_{kn}\,.\tag{41}$$

(3°) R'_{kn} is $\mathscr{C}^\infty$ with respect to the first variable and if $\varphi\in\mathscr{B}$, $\psi_\ell\in\mathcal{U}'$ ($\ell=1,\ldots,L$), then

$$\big|\mathrm{d}^{L}_{\psi_1,\ldots,\psi_L}\,R'_{kn}(S_{2^{-n}}\varphi,x)\big|\le C'_k\,2^{nN_1}\cdot\delta_{kn}^{-L}.\tag{42}$$

Proof: We have $R'_{kn}(S_{2^{-n}}\varphi, x) = R'_{krn}(S_{2^{-n}}\varphi, x)$ for $\varphi \in \mathcal{A}_0(K_{r-1})$ so by §7, Property (3°), R'_{kn} is smooth with respect to the first variable on every $\mathcal{A}_0(K_{r-1})$. As smoothness depends only on the behaviour of R'_{kn} on bounded sets, R'_{kn} is smooth on $\mathcal{A}_0(\mathbb{R}^d)$. For proving the estimations, we can assume without loss of generality that $B = K_{r-1}$ for some $r \in \mathbb{N}$. It is known that the bounded sets in $\mathscr{D}$ are relatively compact; thus, for a given L, the set $\mathscr{B}$ can be covered with a finite number of sets $\omega'_m + \mathcal{U}'_m$ where $\mathcal{U}'_m$ is assigned to ω'_m by §7, Properties of R'_{krn}. Putting $\mathcal{U}' = \bigcap \mathcal{U}'_m$, we get the properties of R'_{kn} from Properties of R'_{krn}. □

§**9.** Up to now, we have constructed functions that were $\mathscr{C}^\infty$ with respect to the first variable. Now we are going to regularize the function R'_{kn} by convolution with respect to the second variable to obtain a simultaneously $\mathscr{C}^\infty$ function.

Notation. The function ρ_δ is introduced in §2, Notation. If k, n are chosen, we have still $\delta = \delta_{kn} = 2^{-n(k+1)N_k}$. Denote furthermore

$$\rho^{\otimes d}(x) := \rho(x_1) \cdot \ldots \cdot \rho(x_d), \quad \rho_\delta^{\otimes d}(x) := \rho_\delta(x_1) \cdot \ldots \cdot \rho_\delta(x_d)$$
$$\text{for} \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d .$$

Definition. We define a function $\widetilde{R}_{kn}$ on $\mathcal{A}_0(\mathbb{R}^d) \times \mathbb{R}^d$ by convolution as follows.

$$\widetilde{R}_{kn}(\varphi, x) := R'_{kn}(\varphi, x) * \rho_\delta^{\otimes d}(x) = \int R'_{kn}(\varphi, y)\, \rho_\delta^{\otimes d}(x - y)\, \mathrm{d}y.$$

Properties of $\widetilde{R}_{kn}$. For every $k, n \in \mathbb{N}$, $\widetilde{R}_{kn}$ is a $\mathscr{C}^\infty$ function on $\mathcal{A}_0(\mathbb{R}^d) \times \mathbb{R}^d$. That is: $\widetilde{R}_{kn} \in \mathcal{E}(\mathbb{R}^d)$. Moreover, if $B \Subset \mathbb{R}^d$, $\mathscr{B}$ a bounded set $\subset \mathcal{A}_0(B)$ and $L \in \mathbb{N}$, then there is an absolutely convex open neighbourhood of zero $\mathcal{U} = \mathcal{U}_k \subset \mathcal{A}(B)$ and a constant $\widetilde{C}_k > 0$, both independent of n, such that for every $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, $\ell = 0, 1, \ldots, L$, $\psi_1, \ldots, \psi_\ell \in \mathcal{U}$, $\varphi \in \mathscr{B}$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le L$, we have

$$\left| \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} \partial^\alpha \widetilde{R}_{kn}(S_{2^{-n}}\varphi, x) \right| \le \widetilde{C}_k\, 2^{nN_1} \cdot \delta_{kn}^{-2L}. \tag{43}$$

If in addition $L \le k$, $2^{-nN_L - 1} > V_{N_L}(\varphi)$ and $\ell \le L - 1$, then

$$\left| \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell}\, \partial^\alpha \widetilde{R}_{kn}(S_{2^{-n}}\varphi, x) \right| \le \widetilde{C}_k\, 2^{nN_L}. \tag{44}$$

If in addition $|\alpha| \le L - 1$, then

$$\left| \partial^\alpha \mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell} \left(\widetilde{R}_{kn}(S_{2^{-n}}\varphi, x) - R(S_{2^{-n}}\varphi, x) \right) \right| \le \widetilde{C}_k\, 2^{nN_L} \cdot \delta_{kn} . \tag{45}$$

Proof of (43):

$$\left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} \widetilde{R}_{kn}(S_{2^{-n}}\varphi, x)\right| = \left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell}\big(R'_{kn}(S_{2^{-n}}\varphi, x) * \rho_\delta^{\otimes d}(x)\big)\right|$$
$$= \left|\big(\mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R'_{kn}(S_{2^{-n}}\varphi, x)\big) * \partial^\alpha \rho_\delta^{\otimes d}(x)\right|.$$

By (42), this is

$$\le C'_k\, 2^{nN_1} \cdot \delta^{-L} \left\|\partial^\alpha \rho_\delta^{\otimes d}\right\|_{\mathscr{L}^1} = C'_k\, 2^{nN_1} \cdot \delta^{-L}\, \delta^{-|\alpha|} \left\|\partial^\alpha \rho^{\otimes d}\right\|_{\mathscr{L}^1}.$$

As $|\alpha| \le L$, we obtain (43).

We see that for given k, n the derivatives $\left|\mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} \partial^\alpha \widetilde{R}_{kn}(S_{2^{-n}}\varphi, x)\right|$ are equibounded if $\varphi \in \mathscr{B}$, $x \in \mathbb{R}^d$, hence they are continuous on $\mathscr{B} \times \mathbb{R}^d$ for any bounded $\mathscr{B} \subset \mathcal{A}_0(B) \times \mathbb{R}^d$ $\quad (B \Subset \mathbb{R}^d)$. They are continuous on $\mathcal{A}_0(B) \times \mathbb{R}^d$ because they are continuous on convergent sequences in a metric space. Thus the order of taking derivatives does not matter and $\widetilde{R}_{kn}$ is smooth ([13, 1.11.5.(2°)]). □

Proof of (44):

$$\left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} \widetilde{R}_{kn}(S_{2^{-n}}\varphi, x)\right| = \left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell}\big(R'_{kn}(S_{2^{-n}}\varphi, x) * \rho_\delta^{\otimes d}(x)\big)\right|$$
$$= \left|\big(\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R'_{kn}(S_{2^{-n}}\varphi, x)\big) * \rho_\delta^{\otimes d}(x)\right|.$$

Then we deduce easily (44) from (40). □

Proof of (45): We first estimate

$$\begin{aligned}
&\left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} \widetilde{R}_{kn}(S_{2^{-n}}\varphi, x) - \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R'_{kn}(S_{2^{-n}}\varphi, x)\right| \\
&= \left|\int \Big(\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R'_{kn}(S_{2^{-n}}\varphi, y) - \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R'_{kn}(S_{2^{-n}}\varphi, x)\Big) \rho_\delta^{\otimes d}(x-y)\,\mathrm{d}y\right| \\
&\le \sup\Big\{\left|\partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R'_{kn}(S_{2^{-n}}\varphi, y) - \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R'_{kn}(S_{2^{-n}}\varphi, x)\right|; \\
&\qquad\qquad |x_1 - y_1| \le \delta, \dots, |x_d - y_d| \le \delta\Big\} \le C'_k\, 2^{nN_L} \cdot d\,\delta
\end{aligned}$$

because the function $x \mapsto \partial^\alpha \mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell} R'_{kn}(S_{2^{-n}}\varphi, x)$ has its derivatives of order 1 estimated by (40). Then (45) follows from (41). □

§**10. Lemma.** *Let $\mathscr{B}$ be a bounded set in $\mathscr{D}(\mathbb{R}^d)$. Then there is a natural number k such that $V'_{N_k}(\varphi) \ge 2^{-k}$ for all $\varphi \in \mathscr{B}$ (V'_N defined by §2, Notation).*

As V'_N is non-decreasing, this inequality holds for all sufficiently large k.

Proof: If not, there would be a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset \mathscr{B}$ such that $V'_{N_k}(\varphi_k) < 2^{-k}$ $(\forall\, k)$. As $\overline{\mathscr{B}}$ is a metrizable compact, a subsequence $\{\varphi_{k_n}\}_{n=1}^{\infty}$ is convergent in $\mathcal{A}_0$, $\lim \varphi_{k_n} = \varphi \in \mathcal{A}_0$. We have $V'_N(\varphi_k) \le V'_{N_k}(\varphi_k) < 2^{-k}$ for $N \le N_k$. As $N_k \nearrow \infty$ (see §3, Properties of R), we have $0 \le V'_N(\varphi) \le \lim 2^{-k} = 0$ for all $N \in \mathbb{N}$, that is impossible. (Proof: If φ has all moments of order ≥ 1 equal to 0, then its Fourier transform has zero derivatives at origin; being holomorphic, it must be constant). □

Notation. Denote $\vartheta_{kn}(\varphi) := \vartheta(2^{nN_k+k+1} \cdot V'_{N_k}(\varphi))$ (ϑ by §2, Notation). ϑ_{kn} is $\mathscr{C}^\infty$ on $\mathcal{A}_0$ and

$$\begin{aligned} \vartheta_{kn}(\varphi) &= 0 \qquad \text{if} \quad 2^{nN_k+k+1} \cdot V'_{N_k}(\varphi) \ge 2, \\ \vartheta_{kn}(\varphi) &= 1 \qquad \text{if} \quad 2^{nN_k+k+1} \cdot V'_{N_k}(\varphi) \le 1. \end{aligned}$$

Definition. We define

$$\widetilde{R}_n(\varphi, x) := \sum_{k=1}^{\infty} \big(\vartheta_{kn}(\varphi) - \vartheta_{k+1,n}(\varphi)\big) \cdot \widetilde{R}_{kn}(\varphi, x) \tag{46}$$

$$= \sum_{k=1}^{\infty} \vartheta_{kn}(\varphi) \cdot \Big(\widetilde{R}_{kn}(\varphi, x) - \widetilde{R}_{k-1,n}(\varphi, x)\Big) \tag{47}$$

if we set $\widetilde{R}_{0,n} = 0$.

Remark. For a given n and φ, at most two terms of the sum (46) are nonzero. Only a finite number of terms of the sum (47) are nonzero: if k satisfies the above lemma, then $\vartheta_{kn}(\varphi) = 0$.

$$\sum_{k=1}^{\infty} (\vartheta_{kn}(\varphi) - \vartheta_{k+1,n}(\varphi)) = 1 \tag{48}$$

is a smooth partition of unity on $\Big\{\varphi \in \mathcal{A}_0;\ V'_{N_1}(\varphi) < 2^{-nN_1-2}\Big\}$. Indeed, the sequence $\Big\{2^{nN_k+k+1} V'_{N_k}(\varphi)\Big\}_k$ is non-decreasing and its first member is $2^{nN_1+2} V'_{N_1}(\varphi) < 1$. Thanks to Lemma, there is the greatest index k' for which $2^{nN_{k'}+k'+1} V'_{N_{k'}}(\varphi) \le 1$. Then $2^{nN_{k'+1}+k'+2} V'_{N_{k'+1}}(\varphi) > 1$, $2^{nN_{k'+2}+k'+3} V'_{N_{k'+2}}(\varphi) > 2$, hence

$$(\vartheta_{k',n}(\varphi) - \vartheta_{k'+1,n}(\varphi)) + (\vartheta_{k'+1,n}(\varphi) - \vartheta_{k'+2,n}(\varphi)) = 1$$

and the other terms of (48) are zero.

Properties of $\widetilde{R}_n$. $\widetilde{R}_n \in \mathcal{E}(\mathbb{R}^d)$. If $\widetilde{B} \Subset \mathbb{R}^d$, $\widetilde{\mathscr{B}} \subset \mathcal{A}_0(\widetilde{B})$ is a bounded set and $L \in \mathbb{N}$, then there is an absolutely convex open neighbourhood of zero $\widetilde{\mathcal{U}} \subset \mathcal{A}(\widetilde{B})$ and a constant $\widetilde{C} > 0$, both independent of n, such that for every $n \in \mathbb{N}$, $x \in \mathbb{R}^d$, the following hold:
If $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le L-1$, $\ell \in \mathbb{N}_0$, $\ell \le L-1$, $\widetilde{\varphi} \in \widetilde{\mathscr{B}}$ with $\|\widetilde{\varphi}\|_{\mathscr{L}^2} \ge 1$, $\widetilde{\psi}_1, \dots, \widetilde{\psi}_\ell \in \widetilde{\mathcal{U}}$, then

$$\big|\mathrm{d}^{\ell}_{\widetilde{\psi}_1,\dots,\widetilde{\psi}_\ell} \partial^\alpha \widetilde{R}_n(S_{2^{-n}}\widetilde{\varphi}, x)\big| \le \widetilde{C}\, 2^{nN_L(L+1)(2L+1)}. \tag{49}$$

If $(\forall\, q \in \mathbb{N})$ $\widetilde{\varphi} \in \widetilde{\mathscr{B}} \cap \mathcal{A}_{N_q}$ with $\|\widetilde{\varphi}\|_{\mathscr{L}^2} \ge 1$ $q \in \mathbb{N}$, then

$$\Big|\widetilde{R}_n(S_{2^{-n}}\widetilde{\varphi}, x) - R(S_{2^{-n}}\widetilde{\varphi}, x)\Big| \le \widetilde{C}\, 2^{-nq}. \tag{50}$$

Proof of (49): For a nonzero term of (46) or (47), we have $2^{nN_k+k+1} \cdot V'_{N_k}(\widetilde{\varphi}) < 2$, i.e. $V'_{N_k}(\widetilde{\varphi}) < 2^{-nN_k-k}$. If $\|\widetilde{\varphi}\|_{\mathscr{L}^2} > 1$, then (§2, Notation) $V_{N_k}(\widetilde{\varphi}) \le V'_{N_k}(\widetilde{\varphi}) < 2^{-nN_k-k}$, so the hypothesis $V_{N_L}(\widetilde{\varphi}) < 2^{-nN_L-1}$ $(L \le k)$ in §9, Properties of $\widetilde{R}_{kn}$ for (44) and (45) is always satisfied. By the Leibniz rule (formulated in the proof of §6, Property (2°)) applied to the definition (47) of $\widetilde{R}_n$ (recall that $V'_N(\widetilde{\varphi}) = V'_N(S_{2^{-n}}\widetilde{\varphi})$), we have

$$\begin{aligned}(51)\quad & \mathrm{d}^{\ell}_{\widetilde{\psi}_1,\dots,\widetilde{\psi}_\ell} \partial^\alpha \widetilde{R}_n(S_{2^{-n}}\widetilde{\varphi}, x) \\ &= \sum_{k=1}^{\infty} \sum_{\substack{I_1 \cup I_2 = \{1,\dots,\ell\} \\ \text{disjoint}}} \mathrm{d}^{\#I_1}_{\widetilde{\boldsymbol{\psi}}_{I_1}} \vartheta_{kn}(\widetilde{\varphi}) \cdot \mathrm{d}^{\#I_2}_{\widetilde{\boldsymbol{\psi}}_{I_2}} \partial^\alpha \Big(\widetilde{R}_{kn}(S_{2^{-n}}\widetilde{\varphi}, x) - \widetilde{R}_{k-1,n}(S_{2^{-n}}\widetilde{\varphi}, x)\Big)\end{aligned}$$

(If $I = (i_1, \dots, i_{\#I})$, then $\widetilde{\boldsymbol{\psi}}_I$ denotes $(\widetilde{\psi}_{i_1}, \dots, \widetilde{\psi}_{i_{\#I}})$). Due to Lemma, the sum $\sum\limits_{k=1}^{\infty}$ can be replaced with $\sum\limits_{k=1}^{k_0}$ with a number k_0 depending on $\widetilde{\mathscr{B}}$ but not on $\widetilde{\varphi} \in \widetilde{\mathscr{B}}$.

First we estimate

$$\begin{aligned}\mathrm{d}^{\#I}_{\widetilde{\boldsymbol{\psi}}_I}\big(\vartheta_{kn}(\widetilde{\varphi})\big) &= \mathrm{d}^{\#I}_{\widetilde{\boldsymbol{\psi}}_I} \vartheta\big(2^{nN_k+k+1} \cdot V'_{N_k}(\widetilde{\varphi})\big) \\ &= \sum_{M=1}^{\#I} \sum_{\substack{I = I_1 \cup \dots \cup I_M \\ \ne \emptyset,\ \text{disjoint}}} (\partial^M \vartheta)\big(2^{nN_k+k+1} \cdot V'_{N_k}(\widetilde{\varphi})\big) \prod_{m=1}^{M} \mathrm{d}^{\#I_m}_{\widetilde{\boldsymbol{\psi}}_{I_m}}\big(2^{nN_k+k+1} \cdot V'_{N_k}(\widetilde{\varphi})\big)\end{aligned}$$

(chain rule [13, 1.8.3] or [8, Theorem 12], the summation is extended over all decompositions $I = I_1 \cup \dots \cup I_M$ on non-empty disjoint parts). Let us choose (by [9, §2, Proposition]) an absolutely convex open neighbourhood of zero $\widetilde{\mathcal{U}} \subset \mathcal{A}(\widetilde{B})$ such that for every $k = 1, \ldots, k_0$, $\ell = 1, \ldots, L$, $\widetilde{\varphi} \in \mathscr{B}$, $\widetilde{\psi}_1, \ldots, \widetilde{\psi}_\ell \in \widetilde{\mathcal{U}}$, we have $|\mathrm{d}^\ell_{\widetilde{\psi}_1,\ldots,\widetilde{\psi}_\ell} V'_{N_k}(\widetilde{\varphi})| \leq 1$. We obtain:

$$\left|\mathrm{d}^{\#I}_{\widetilde{\boldsymbol{\psi}}_I}\big(\vartheta_{kn}(\widetilde{\varphi})\big)\right| \leq C_k\, 2^{nN_k\cdot \#I} \leq C_k\, 2^{nN_k\, L} \tag{52}$$

with a constant C_k depending only on ϑ, L and k, not on n.

We are going to estimate the second term in (51)

$$\mathrm{d}^{\#I_2}_{\widetilde{\boldsymbol{\psi}}_{I_2}}\partial^\alpha\Big(\widetilde{R}_{kn}(S_{2^{-n}}\widetilde{\varphi},x) - \widetilde{R}_{k-1,n}(S_{2^{-n}}\widetilde{\varphi},x)\Big).$$

We distinguish two cases. If $L \leq k-1$ then $\#I_2 \leq k-1$ and we can use the estimation (45) as follows:

$$\begin{aligned}
&\left|\mathrm{d}^{\#I_2}_{\widetilde{\boldsymbol{\psi}}_{I_2}}\partial^\alpha\Big(\widetilde{R}_{kn}(S_{2^{-n}}\widetilde{\varphi},x) - \widetilde{R}_{k-1,n}(S_{2^{-n}}\widetilde{\varphi},x)\Big)\right| \\
&= \left|\mathrm{d}^{\#I_2}_{\widetilde{\boldsymbol{\psi}}_{I_2}}\partial^\alpha\bigg(\Big(\widetilde{R}_{kn}(S_{2^{-n}}\widetilde{\varphi},x) - R(S_{2^{-n}}\widetilde{\varphi},x)\Big)\right. \\
&\quad \left. - \Big(\widetilde{R}_{k-1,n}(S_{2^{-n}}\widetilde{\varphi},x) - R(S_{2^{-n}}\widetilde{\varphi},x)\Big)\bigg)\right| \\
&\leq 2^{nN_L}\cdot(\widetilde{C}_k\delta_{kn} + \widetilde{C}_{k-1}\delta_{k-1,n}) \leq 2^{nN_L}\cdot(\widetilde{C}_k + \widetilde{C}_{k-1})\,\delta_{k-1,n}.
\end{aligned}$$

Together with (52), a term of the sum in (51) for $L \leq k-1$ fulfills

$$\begin{aligned}
&\left|\mathrm{d}^{\#I_1}_{\widetilde{\boldsymbol{\psi}}_{I_1}}\vartheta_{kn}(\widetilde{\varphi})\cdot \mathrm{d}^{\#I_2}_{\widetilde{\boldsymbol{\psi}}_{I_2}}\partial^\alpha\Big(\widetilde{R}_{kn}(S_{2^{-n}}\widetilde{\varphi},x) - \widetilde{R}_{k-1,n}(S_{2^{-n}}\widetilde{\varphi},x)\Big)\right| \\
&\leq C_k\, 2^{nN_k\,L}\cdot 2^{nN_L}\,(\widetilde{C}_k + \widetilde{C}_{k-1})\,\delta_{k-1,n} \leq C_k\,(\widetilde{C}_k + \widetilde{C}_{k-1})
\end{aligned}$$

(δ is defined in §9, Notation), that is a constant independent of n, however it depends on $\mathscr{B}$ and the number of nonzero terms of the sum in (51) depends on $\mathscr{B}$, too.

If $L \geq k$, we use the estimations (43) valid for all L and we obtain:

$$\mathrm{d}^{\#I_2}_{\widetilde{\boldsymbol{\psi}}_{I_2}}\partial^\alpha\widetilde{R}_{kn}\big(S_{2^{-n}}\widetilde{\varphi},x\big) \leq \widetilde{C}_k\, 2^{nN_1}\cdot\delta_{kn}^{-2L} \leq \widetilde{C}_k\, 2^{nN_1}\cdot\delta_{Ln}^{-2L}.$$

Together with (52), a term of the sum in (51) for $L \geq k$ fulfills

$$\begin{aligned}
&\left|\mathrm{d}^{\#I_1}_{\widetilde{\boldsymbol{\psi}}_{I_1}}\vartheta_{kn}(\widetilde{\varphi})\cdot \mathrm{d}^{\#I_2}_{\widetilde{\boldsymbol{\psi}}_{I_2}}\partial^\alpha\Big(\widetilde{R}_{kn}(S_{2^{-n}}\widetilde{\varphi},x) - \widetilde{R}_{k-1,n}(S_{2^{-n}}\widetilde{\varphi},x)\Big)\right| \\
&\leq C_k\, 2^{nN_k\,L}(\widetilde{C}_k + \widetilde{C}_{k-1})\, 2^{nN_1}\cdot\delta_{L,n}^{-2L} \leq C_k\,(\widetilde{C}_k + \widetilde{C}_{k-1})2^{nN_{L}(2L+1)(L+1)}.
\end{aligned}$$

So in both cases we can use the last estimation and (49) follows. □

Proof of (50): If $\widetilde{\varphi} \in \mathcal{A}_{N_q}$ then $V'_{N_q}(\widetilde{\varphi}) = 0$ and $k \geq q$ for all nonzero terms of the sum in (46). In that case, it follows from the definition and (48)

$$\begin{aligned}
&\left|\widetilde{R}_n(S_\varepsilon\widetilde{\varphi}, x) - R(S_\varepsilon\widetilde{\varphi}, x)\right| \\
&= \left|\sum_{k=q}^{k_0}(\vartheta_{kn}(\widetilde{\varphi}) - \vartheta_{k+1,n}(\widetilde{\varphi})) \cdot \widetilde{R}_{kn}(S_\varepsilon\widetilde{\varphi}, x) - \sum_{k=q}^{k_0}(\vartheta_{kn}(\widetilde{\varphi}) - \vartheta_{k+1,n}(\widetilde{\varphi})) \cdot R(S_\varepsilon\widetilde{\varphi}, x)\right| \\
&\leq \sum_{k=q}^{k_0}(\vartheta_{kn}(\widetilde{\varphi}) - \vartheta_{k+1,n}(\widetilde{\varphi})) \cdot \left|\widetilde{R}_{kn}(S_\varepsilon\widetilde{\varphi}, x) - R(S_\varepsilon\widetilde{\varphi}, x)\right| \\
&\leq \sum_{k=q}^{k_0} \widetilde{C}_k\, 2^{nN_1}\delta_{kn} = \sum_{k=q}^{k_0} \widetilde{C}_k\, 2^{nN_1 - n(k+1)N_k} \leq \widetilde{C}\, 2^{-nq}
\end{aligned}$$

by (45), as $V_{N_1}(\widetilde{\varphi}) = 0$. □

§**11.** Now we have all tools for defining the desired representative $\widetilde{R}$.

Definition of $\widetilde{R}$. We define

$$\widetilde{R}(\varphi, x) := \sum_{n=1}^{\infty}\left(\vartheta_{n+1}\left(\|\varphi\|_{\mathscr{L}^2}^{2/d}\right) - \vartheta_n\left(\|\varphi\|_{\mathscr{L}^2}^{2/d}\right)\right)\widetilde{R}_n(\varphi, x)$$

for $\varphi \in \mathcal{A}_0(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ (ϑ by §2, Notation).

Remark. Note that $\vartheta_{n+1}\left(\|\varphi\|_{\mathscr{L}^2}^{2/d}\right) - \vartheta_n\left(\|\varphi\|_{\mathscr{L}^2}^{2/d}\right) \neq 0$ iff $2^n < \|\varphi\|_{\mathscr{L}^2}^{2/d} < 2^{n+2}$ and, for a given φ, at most 2 terms of this sum are $\neq 0$.

$$\sum_{n=1}^{\infty}\left(\vartheta_{n+1}\left(\|\varphi\|_{\mathscr{L}^2}^{2/d}\right) - \vartheta_n\left(\|\varphi\|_{\mathscr{L}^2}^{2/d}\right)\right) = 1$$

is a smooth partition of unity on $\{\varphi \in \mathscr{D};\, \|\varphi\|_{\mathscr{L}^2} > 4\}$.

Properties of $\widetilde{R}$. $\widetilde{R} \in \mathcal{E}(\mathbb{R}^d)$. If $B \Subset \mathbb{R}^d$, $\mathscr{B}$ a bounded set $\subset \mathcal{A}_0(B)$ and $L \in \mathbb{N}$, then there is an absolutely convex open neighbourhood of zero $\mathcal{U} \subset \mathcal{A}(B)$ and a constant $C > 0$, such that for every $x \in \mathbb{R}^d$ the following hold.

(1°) If $\varepsilon \in\,]0,1]$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq L-1$, $\ell \in \mathbb{N}_0$, $\ell \leq L-1$, $\varphi \in \mathscr{B}$, $\psi_1, \ldots, \psi_\ell \in \mathcal{U}$, then

$$\left|\mathrm{d}^\ell_{\psi_1,\ldots,\psi_\ell}\partial^\alpha\, \widetilde{R}(S_\varepsilon\varphi, x)\right| \leq C\,\varepsilon^{-N_L(2L+1)(L+1)}. \tag{53}$$

Consequently, $\widetilde{R} \in \mathcal{E}_M^{\mathrm{d}}$.

(2°) $\forall\, q \in \mathbb{N}$ it holds: If $\varphi \in \mathscr{B} \cap \mathcal{A}_{N_q}$ and $0 < \varepsilon < \min\left\{1,\ \frac{1}{4}\|\chi_B\|_{\mathscr{L}^2}^{-2/d}\right\}$, then

$$\left|\widetilde{R}(S_\varepsilon\varphi, x) - R(S_\varepsilon\varphi, x)\right| \le C \cdot \varepsilon^q. \tag{54}$$

Due to [9, §8, (0°)], we have $\widetilde{R} - R \in \mathcal{N}$.

Proof of (53): We write simply $\|\bullet\|$ instead of $\|\bullet\|_{\mathscr{L}^2}$ and we assume B to be convex and balanced. For proving (53), we calculate (see (2)):

$$\begin{aligned}\widetilde{R}(S_\varepsilon\varphi, x) &= \sum_{n=1}^{\infty} \Big(\vartheta_{n+1}\big(\|S_\varepsilon\varphi\|^{2/d}\big) - \vartheta_n\big(\|S_\varepsilon\varphi\|^{2/d}\big)\Big)\widetilde{R}_n(S_\varepsilon\varphi, x)\\ &= \sum_{n=1}^{\infty} \Big(\vartheta_1\big(2^{-n}\|S_\varepsilon\varphi\|^{2/d}\big) - \vartheta_0\big(2^{-n}\|S_\varepsilon\varphi\|^{2/d}\big)\Big)\widetilde{R}_n\big(S_{2^{-n}}(S_{2^n\varepsilon}\varphi), x\big)\\ &= \sum_{n=1}^{\infty} \Big(\vartheta_1\big(\|S_{2^n\varepsilon}\varphi\|^{2/d}\big) - \vartheta_0\big(\|S_{2^n\varepsilon}\varphi\|^{2/d}\big)\Big)\widetilde{R}_n\big(S_{2^{-n}}(S_{2^n\varepsilon}\varphi), x\big).\end{aligned} \tag{55}$$

By the definition of ϑ, a term of this sum can be nonzero only if

$$1 < \|S_{2^n\varepsilon}\varphi\|^{2/d} < 4, \qquad \text{i.e.} \quad 1 < \frac{1}{2^n\,\varepsilon}\|\varphi\|^{2/d} < 4. \tag{56}$$

$\overline{\mathscr{B}} \Subset \mathcal{A}_0(B)$, hence there are constants $c_1, c_2 > 0$ such that

$$c_2 \le \|\varphi\|^{2/d} \le c_1 \qquad (\forall\, \varphi \in \mathscr{B}).$$

Due to (56), it follows that for nonzero terms of the sum in (55), we have

$$\tfrac{1}{4}c_2 < 2^n\varepsilon < c_1. \tag{57}$$

By the Leibniz rule (formulated in the proof of §6, Property 2°) applied to (55), we have

$$\begin{aligned}&\mathrm{d}^\ell_{\psi_1,\dots,\psi_\ell}\partial^\alpha\,\widetilde{R}(S_\varepsilon\varphi, x)\\ &\quad= \sum_{n=1}^{\infty}\ \sum_{\substack{I_1\cup I_2=\{1,\dots,\ell\}\\ \text{disjoint}}} \mathrm{d}^{\#I_1}_{\boldsymbol{\psi}_{I_1}}\Big(\vartheta_1\big(\|S_{2^n\varepsilon}\varphi\|^{2/d}\big) - \vartheta_0\big(\|S_{2^n\varepsilon}\varphi\|^{2/d}\big)\Big)\\ &\qquad\qquad \cdot\,\mathrm{d}^{\#I_2}_{\boldsymbol{\psi}_{I_2}}\partial^\alpha\widetilde{R}_n\big(S_{2^{-n}}(S_{2^n\varepsilon}\varphi), x\big).\end{aligned} \tag{58}$$

The function $\varphi \mapsto \vartheta_1\big(\|S_{2^n\varepsilon}\varphi\|^{2/d}\big) - \vartheta_0\big(\|S_{2^n\varepsilon}\varphi\|^{2/d}\big)$ is composed of functions

$$\big[t \mapsto \vartheta_1\big(t^{1/d}\big) - \vartheta_0\big(t^{1/d}\big)\big] \;\in\; \mathscr{D}\big(\big[(1, 4^d\big]\big)$$

and

$$\varphi \mapsto \|S_{2^n\varepsilon}\varphi\|^2.$$

Hence, for proving that the term $\mathrm{d}^{\#I_1}_{\boldsymbol{\psi}_{I_1}}\Big(\vartheta_1\big(\|S_{2^n\varepsilon}\varphi\|^{2/d}\big) - \vartheta_0\big(\|S_{2^n\varepsilon}\varphi\|^{2/d}\big)\Big)$ in (58) is equi-bounded under the hypotheses (1°) (for a fixed L), it is sufficient to prove the same for derivatives of $\|S_{2^n\varepsilon}\varphi\|^2$ up to a certain order. We have

$$\mathrm{d}_\psi\|S_{2^n\varepsilon}\varphi\|^2 = \mathrm{d}_\psi(S_{2^n\varepsilon}\varphi, S_{2^n\varepsilon}\varphi) = 2\Re(S_{2^n\varepsilon}\varphi, S_{2^n\varepsilon}\psi),$$
$$\mathrm{d}^2_{\psi_1,\psi_2}\|S_{2^n\varepsilon}\varphi\|^2 = 2\Re(S_{2^n\varepsilon}\psi_1, S_{2^n\varepsilon}\psi_2)$$

and the higher derivatives are zero. So we have using the Hölder inequality

$$\Big|\mathrm{d}_\psi\|S_{2^n\varepsilon}\varphi\|^2\Big| = \Big|2\Re\int S_{2^n\varepsilon}\varphi\cdot S_{2^n\varepsilon}\overline{\psi}\Big| \le \|S_{2^n\varepsilon}\varphi\|\,\|S_{2^n\varepsilon}\psi\| = \frac{1}{(2^n\varepsilon)^d}\|\varphi\|\|\psi\|.$$

Thanks to (57), this will be equi-bounded if $\mathcal{U} \subset \{\psi;\, \|\psi\| < 1\}$, $\psi \in \mathcal{U}$, $\varphi \in \mathscr{B}$, as the bounded set $\mathscr{B}$ is absorbed by $\mathcal{U}$. The same can be deduced for the second derivative.

Now, we have to estimate the term $\mathrm{d}^{\#I_2}_{\boldsymbol{\psi}_{I_2}}\partial^\alpha\,\widetilde{R}_n\big(S_{2^{-n}}(S_{2^n\varepsilon}\varphi), x\big)$ in (58). We apply §10, Properties of $\widetilde{R}_n$, namely the estimation (49), to the bounded set

$$\widetilde{\mathscr{B}} := \big\{S_\eta\,\varphi;\ \varphi \in \mathscr{B},\ \tfrac14 c_2 \le \eta \le c_1\big\}\,.$$

The supports of the functions $\widetilde{\varphi} \in \widetilde{\mathscr{B}}$ are contained in $\widetilde{B} := c_1\,B$ (for B convex and balanced). Thanks to (57) we have $S_{2^n\varepsilon}\varphi \in \widetilde{\mathscr{B}}$ for $\varphi \in \mathscr{B}$. Thus we get $\widetilde{\mathcal{U}}$ and $\widetilde{C}$ by §10, Properties of $\widetilde{R}_n$. Let

$$\mathcal{U} := \Big\{\varphi \in \mathcal{A}(B);\ S_\eta\,\varphi \in \widetilde{\mathcal{U}}\quad \forall\,\eta\quad\text{with}\quad \tfrac14 c_2 \le \eta \le c_1\Big\}\,.$$

$\mathcal{U}$ is a neighbourhood of zero in $\mathcal{A}(B)$ because it absorbs bounded sets in a metric vector space. If $\varphi \in \mathscr{B}$, $\psi_1,\dots,\psi_\ell \in \mathcal{U}$, then $\widetilde{\varphi} := S_{2^n\varepsilon}\varphi \in \widetilde{\mathscr{B}}$ and $\widetilde{\psi}_j := S_{2^n\varepsilon}\psi_j \in \widetilde{\mathcal{U}}$ $(j = 1,\dots,\ell)$ due to (57). If $\|\widetilde{\varphi}\| = \|S_{2^n\varepsilon}\varphi\| < 1$, then (by (56)) the term $\mathrm{d}^{\#I_2}_{\boldsymbol{\psi}_{I_2}}\partial^\alpha\widetilde{R}_n\big(S_{2^{-n}}(S_{2^n\varepsilon}\varphi), x\big)$ in (58) is multiplied by zero, so we have to estimate this term only if $\|\widetilde{\varphi}\| \ge 1$, hence we can use (49). It follows using the chain rule for the inner function $S_{2^n\varepsilon}$ linear:

$$\big|\mathrm{d}^{\#I_2}_{\boldsymbol{\psi}_{I_2}}\partial^\alpha\widetilde{R}_n\big(S_{2^{-n}}(S_{2^n\varepsilon}\varphi), x\big)\big| = \big|\mathrm{d}^{\#I_2}_{\widetilde{\boldsymbol{\psi}}_{I_2}}\partial^\alpha\widetilde{R}_n\big(S_{2^{-n}}\widetilde{\varphi}, x\big)\big|$$
$$\le \widetilde{C}\cdot 2^{nN_L(2L+1)(L+1)} \le \widetilde{C}\,c_1^{N_L(2L+1)(L+1)}\cdot\varepsilon^{-N_L(2L+1)(L+1)}$$

due to (57). From (58) we deduce (53). □

Proof of (54): By (9), $\|\varphi\| \geq \|\chi_B\|^{-1}$ (norms in $\mathscr{L}^2$). If $\varepsilon < \frac{1}{4}\|\chi_B\|^{-2/d}$, then

$$\|S_\varepsilon\varphi\|^{2/d} = \|\varphi\|^{2/d} \cdot \varepsilon^{-1} > \|\varphi\|^{2/d} \cdot 4\,\|\chi_B\|^{2/d} \geq 4.$$

Under this hypothesis we have by §11, Remark similarly to (55):

$$\begin{aligned} R(S_\varepsilon\,\varphi, x) &= \sum_{n=1}^{\infty}\Big(\vartheta_{n+1}\Big(\|S_\varepsilon\varphi\|^{2/d}\Big) - \vartheta_n\Big(\|S_\varepsilon\varphi\|^{2/d}\Big)\Big)R(S_\varepsilon\,\varphi, x) \\ &= \sum_{n=1}^{\infty}\Big(\vartheta_1\big(\|S_{2^n\varepsilon}\,\varphi\|^{2/d}\big) - \vartheta_0\big(\|S_{2^n\varepsilon}\,\varphi\|^{2/d}\big)\Big)R\big(S_{2^{-n}}(S_{2^n\varepsilon}\,\varphi), x\big). \end{aligned}$$

With (55) it gives

$$\begin{aligned} &\Big|\widetilde{R}(S_\varepsilon\varphi, x) - R(S_\varepsilon\varphi, x)\Big| \\ &\leq \sum_{n=1}^{\infty}\Big(\vartheta_1\big(\|S_{2^n\varepsilon}\,\varphi\|^{2/d}\big) - \vartheta_0\big(\|S_{2^n\varepsilon}\,\varphi\|^{2/d}\big)\Big) \\ &\quad\cdot\Big|\widetilde{R}_n\big(S_{2^{-n}}(S_{2^n\varepsilon}\,\varphi), x\big) - R\big(S_{2^{-n}}(S_{2^n\varepsilon}\,\varphi), x\big)\Big|. \end{aligned}$$

If $\varphi \in \mathscr{B} \cap \mathcal{A}_{N_q}$ then $\widetilde{\varphi} := S_{2^n\varepsilon}\,\varphi) \in \widetilde{\mathscr{B}} \cap \mathcal{A}_{N_q}$. By (50) this is $\leq \widetilde{C} \cdot 2^{-nq}$ and by (57) this is $\leq \widetilde{C} \cdot \big(\frac{4}{c_2}\big)^q\,\varepsilon^q$. By the above lemma, $\mathscr{B} \cap \mathcal{A}_{N_q} = \emptyset$ for sufficiently large q, so the constant in our estimation can be independent of q. Hence, (54) is proved. □

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