On a selection theorem of Blum and Swaminathan

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Dedicated to Professor Takao Hoshina on his sixtieth birthday.

Abstract. Blum and Swaminathan [Pacific J. Math. 93 (1981), 251–260] introduced the notion of $\mathcal{B}$-fixedness for set-valued mappings, and characterized realcompactness by means of continuous selections for Tychonoff spaces of non-measurable cardinal. Using their method, we obtain another characterization of realcompactness, but without any cardinal assumption. We also characterize Dieudonné completeness and Lindelöf property in similar formulations.

Keywords: set-valued mapping, selection, realcompact, Dieudonné complete, Lindelöf, $\mathcal{B}$-fixed, local intersection property, open lower sections

Classification: 54C60, 54C65, 54D20, 54D60

1. Introduction

Let X be a topological space and Y be a topological vector space. Let us denote by 2^Y the set of all non-empty subsets of Y, and write

$$\mathcal{K}(Y) = \{K \in 2^Y \mid K \text{ is convex}\},$$
$$\mathcal{F}_c(Y) = \{F \in 2^Y \mid F \text{ is closed and convex}\}.$$

A set-valued mapping $\varphi : X \to 2^Y$ is *lower semicontinuous* (l.s.c. for short) if the set

$$\varphi^{-1}(V) = \{x \in X \mid \varphi(x) \cap V \neq \emptyset\}$$

is open in X for every open subset V of Y. For a set-valued mapping $\varphi : X \to 2^Y$, a mapping $f : X \to Y$ is called a *selection* of φ if $f(x) \in \varphi(x)$ for every $x \in X$. A subset S of X is a *zero-set* (respectively a *cozero-set*) if $S = \{x \in X \mid f(x) = 0\}$ (respectively $S = \{x \in X \mid f(x) \neq 0\}$) for some real-valued continuous function f on X. For undefined notations and terminology we refer to [1] or [4].

The following is a well-known selection theorem due to Michael [6, Theorem 3.2″].

Theorem 1.1 (Michael [6]). *A T_1-space X is paracompact if and only if, for every Banach space Y, every l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection.*

This result not only guarantees the existence of a selection but describes paracompactness in terms of continuous selections of l.s.c. set-valued mappings. In addition to this theorem, some topological properties have been characterized by means of continuous selections (see [9]). Among these results, Blum and Swaminathan [2] characterized realcompactness for Tychonoff spaces (that is, completely regular T_1-spaces) of non-measurable cardinal as in Theorem 1.2.

Before stating Theorem 1.2, let us recall some terminology introduced by Blum and Swaminathan [2]. An l.s.c. set-valued mapping $\varphi : X \to 2^Y$ is said to be *of infinite character* if there exists a neighborhood V of the origin of Y such that the open cover $\{\varphi^{-1}(y+V) \mid y \in Y\}$ of X has no finite subcover; and otherwise φ is called *of finite character*. For a family $\mathcal{S}$ of subsets of a space X, a set-valued mapping $\varphi : X \to 2^Y$ is *$\mathcal{S}$-fixed* if $\bigcap\{\varphi(x) \mid x \in S\} \neq \emptyset$ for every $S \in \mathcal{S}$. For a given Tychonoff space X, let $\mathcal{B}$ be the family of subsets of X defined as follows:

$$\mathcal{B} = \{B \subset X \mid B \text{ is a realcompact cozero-set in } X \text{ and } X \setminus B \text{ is not compact}\}.$$

A cardinality τ is called *measurable* if the discrete space of cardinal τ admits a nontrivial $\{0,1\}$-valued countably additive measure.

Theorem 1.2 (Blum and Swaminathan [2]). *For a Tychonoff space X of non-measurable cardinal, the following statements are equivalent:*

(a) *X is realcompact;*
(b) *for every locally convex topological vector space Y, every $\mathcal{B}$-fixed l.s.c. set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ is of finite character;*
(c) *for every locally convex topological vector space Y, every $\mathcal{B}$-fixed l.s.c. set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ of infinite character admits a continuous selection.*

The main purpose of this paper is to obtain another description of realcompactness in terms of $\mathcal{B}$-fixed l.s.c. set-valued mappings as follows. Notice that, in our case, a space X is not assumed to be of non-measurable cardinal.

Theorem 1.3. *A Tychonoff space X is realcompact if and only if, for every Banach space Y, every $\mathcal{B}$-fixed l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection $f : X \to Y$ such that $f(X)$ is separable.*

Let us recall that a Tychonoff space X is *Dieudonné complete* if there exists a complete uniformity on the space X (see [4, 8.5.13]). It is known that every realcompact space is Dieudonné complete. For a Tychonoff space X, Blum and Swaminathan defined the collection $\mathcal{C}$ of subsets of X as follows:

$$\mathcal{C} = \{C \subset X \mid C \text{ is a Dieudonné complete cozero-set in } X \text{ and } X \setminus C \text{ is not compact}\}.$$

Besides they mentioned that several theorems in their paper [2] are valid with substitution of the phrases "Dieudonné complete" for "realcompact", and "$\mathcal{C}$-fixed" for "$\mathcal{B}$-fixed" [2, Remarks (ii)]. Thus it is natural to ask whether Dieudonné completeness can be described by means of $\mathcal{C}$-fixed set-valued mappings in a formulation analogous to Theorem 1.3. In Section 3, we prove the following:

Theorem 1.4. *A Tychonoff space X is Dieudonné complete if and only if, for every Banach space Y, every $\mathcal{C}$-fixed l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection.*

In addition, we will characterize Lindelöf property in Section 3.

Theorem 1.5. *A regular space X is Lindelöf if and only if, for every Banach space Y, every l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection $f : X \to Y$ such that $f(X)$ is separable.*

Yannelis and Prabhakar [13] defined set-valued mappings with open lower sections, and Wu and Shen [12] defined the local intersection property for set-valued mappings. In [12] and [13], selection theorems of such set-valued mappings on paracompact spaces were obtained. Adopting these notions, we will also characterize paracompactness, realcompactness, Dieudonné completeness, and Lindelöf property in Section 4.

2. Proof of Theorem 1.3

Let X be a topological space. For a subset S of X, $\mathrm{cl}_X(S)$ stands for the closure of S in X. Let us denote by $C(X)$ the set of all real-valued continuous functions on X. For $f \in C(X)$, set $Z(f) = \{x \in X \mid f(x) = 0\}$ and $\mathrm{Coz}(f) = \{x \in X \mid f(x) \neq 0\}$. A family $\{p_\lambda \mid \lambda \in \Lambda\}$ of continuous functions $p_\lambda : X \to [0, 1]$ is called a *partition of unity* on X if $\sum_{\lambda\in\Lambda} p_\lambda(x) = 1$ for each $x \in X$. A partition of unity $\{p_\lambda \mid \lambda \in \Lambda\}$ on X is said to be *locally finite* if the cover $\{\mathrm{Coz}(p_\lambda) \mid \lambda \in \Lambda\}$ of X is locally finite. For an open cover $\mathcal{U}$ of X, a partition of unity $\{p_\lambda \mid \lambda \in \Lambda\}$ on X is *subordinated to $\mathcal{U}$* if the cover $\{\mathrm{Coz}(p_\lambda) \mid \lambda \in \Lambda\}$ refines $\mathcal{U}$. Let us denote $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ the set of all positive integers, the set of all rationals, and the set of all reals, respectively. For a set A, $l_1(A)$ means the Banach space of all functions $y : A \to \mathbb{R}$ such that $\sum_{a\in A} |y(a)| < \infty$ with the norm $\|y\| = \sum_{a\in A} |y(a)|$. For $a \in A$, let $\pi_a : l_1(A) \to \mathbb{R}$ be the a-th projection. We will use the following lemma due to Michael [6, p. 369].

Lemma 2.1 (Michael [6])**.** *Let $\mathcal{U}$ be an open cover of a topological space X. Let $\varphi : X \to 2^{l_1(\mathcal{U})}$ be a set-valued mapping defined by*

$$\varphi(x) = \{y \in l_1(\mathcal{U}) \mid \|y\| = 1,\ y(U) \geq 0 \ \text{ for every } \ U \in \mathcal{U}, \\ \text{and } \ y(U) = 0 \ \text{ for all } \ U \in \mathcal{U} \ \text{ with } \ x \notin U\},$$

for $x \in X$. Then φ is l.s.c. and closed-and-convex-valued. Furthermore, if φ has a continuous selection, then there exists a locally finite partition of unity subordinated to $\mathcal{U}$.

A Tychonoff space X is called *realcompact* if every z-ultrafilter (that is, a maximal filter consisting of zero-sets) on X with the countable intersection property has non-empty intersection. For a Tychonoff space X, βX and υX denote the Stone-Čech compactification and the realcompactification of X, respectively.

The following theorem was essentially proved by De Marco and Wilson [3, 4. Theorem] and Tamano [10, Theorem 2.5].

Lemma 2.2 (De Marco and Wilson [3], Tamano [10]). *For a Tychonoff space X and a point $a \in \beta X$, $a \in \beta X \setminus \upsilon X$ if and only if there exists a (locally finite) countable partition of unity $\{p_i \mid i \in \mathbb{N}\}$ on X such that $a \in \mathrm{cl}_{\beta X}(Z(p_i))$ for each $i \in \mathbb{N}$.*

Using this lemma, we prove the following:

Lemma 2.3. *Let X be a non-compact realcompact space and Y be a topological vector space. Then every $\mathcal{B}$-fixed set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ has a continuous selection $f : X \to Y$ such that $f(X)$ is separable.*

Proof: Since X is a non-compact realcompact space, that is, $\upsilon X = X \subsetneq \beta X$, we may choose $a \in \beta X \setminus \upsilon X$. By Lemma 2.2, there exists a locally finite countable partition of unity $\{p_i \mid i \in \mathbb{N}\}$ on X such that $a \in \mathrm{cl}_{\beta X}(Z(p_i))$ for each $i \in \mathbb{N}$. Then $X \setminus \mathrm{Coz}(p_i) = Z(p_i)$ is not compact. By virtue of [5, 8.14. Theorem], $\mathrm{Coz}(p_i)$ is realcompact. Thus we have $\{\mathrm{Coz}(p_i) \mid i \in \mathbb{N}\} \subset \mathcal{B}$. Since φ is $\mathcal{B}$-fixed, we can take $y_i \in \bigcap\{\varphi(x) \mid x \in \mathrm{Coz}(p_i)\}$ for each $i \in \mathbb{N}$. Define a mapping $f : X \to Y$ by the formula $f(x) = \sum_{i\in\mathbb{N}} p_i(x)y_i$ for each $x \in X$. Then f is a continuous selection of φ since $\{p_i \mid i \in \mathbb{N}\}$ is locally finite and φ is convex-valued. It remains to prove the separability of $f(X)$. For $n \in \mathbb{N}$, put $\Lambda_n = \{((q_i)_{i=1}^n, k) \in \mathbb{Q}^n \times \mathbb{N} \mid \sum_{i=1}^n r_i y_i \in f(X) \text{ and } |q_i - r_i| < 1/k \text{ for some } (r_i)_{i=1}^n \in \mathbb{R}^n\}$. For $n \in \mathbb{N}$, $\lambda = ((q_i)_{i=1}^n, k) \in \Lambda_n$, and $j \in \{1, 2, \ldots, n\}$, choose $r_j(\lambda) \in \mathbb{R}$ so that $\sum_{j=1}^n r_j(\lambda)y_j \in f(X)$ and $|q_j - r_j(\lambda)| < 1/k$ for every $j \in \{1, 2, \ldots, n\}$. Then the set $A = \{\sum_{i=1}^n r_i(\lambda)y_i \mid \lambda \in \Lambda_n,\ n \in \mathbb{N}\}$ is a countable dense subset of $f(X)$. □

Proof of Theorem 1.3: If X is compact, the "only if" part follows from Theorem 1.1; otherwise from Lemma 2.3.

To see the "if" part, let X be a Tychonoff space satisfying that, for every Banach space Y, every $\mathcal{B}$-fixed l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}_c(Y)$ admits a continuous selection $f : X \to Y$ such that $f(X)$ is separable. Assume that X is not realcompact and take $a_0 \in \upsilon X \setminus X$. We will deduce a contradiction. Put $\mathcal{U} = \{\mathrm{Coz}(p) \mid p \in C(X),\ a_0 \in \mathrm{cl}_{\beta X}(Z(p))\}$. Then $\mathcal{U}$ is an open cover of X. Put $Y = l_1(\mathcal{U})$ and define a set-valued mapping $\varphi : X \to 2^Y$ as in Lemma 2.1. Then φ is l.s.c. and $\varphi(x) \in \mathcal{F}_c(Y)$ for each $x \in X$.

We claim that φ is $\mathcal{B}$-fixed. To prove this, let $B \in \mathcal{B}$. Then $B = \mathrm{Coz}(h)$ for some $h \in C(X)$ as B is a cozero-set. Since $\mathrm{Coz}(h)$ is realcompact and $\mathrm{cl}_{\beta X}(Z(h))$ is compact, $\mathrm{Coz}(h) \cup \mathrm{cl}_{\beta X}(Z(h))$ is realcompact ([5, 8.16. Theorem]) and contains X. Thus we have $\upsilon X \subset \mathrm{Coz}(h) \cup \mathrm{cl}_{\beta X}(Z(h))$, and hence $a_0 \in \upsilon X \setminus X \subset \mathrm{cl}_{\beta X}(Z(h))$. Thus $B = \mathrm{Coz}(h) \in \mathcal{U}$. Let $y \in l_1(\mathcal{U})$ be the element defined by

$$y(U) = \begin{cases} 1, & \text{if } U = B, \\ 0, & \text{if } U \neq B, \end{cases}$$

for each $U \in \mathcal{U}$. Then $y \in \bigcap\{\varphi(x) \mid x \in B\}$. Therefore φ is $\mathcal{B}$-fixed.

By hypothesis, φ admits a continuous selection $f : X \to Y$ such that $f(X)$ is separable. Then there exists a countable subset $\{x_n \mid n \in \mathbb{N}\}$ of Y whose closure contains $f(X)$. Put $\mathcal{U}' = \{U \in \mathcal{U} \mid x_n(U) \neq 0 \text{ for some } n \in \mathbb{N}\}$. Then $\mathcal{U}'$ is a countable subset of $\mathcal{U}$. We may regard $l_1(\mathcal{U}')$ as a linear subspace of $l_1(\mathcal{U})$ by canonical identification. Since $l_1(\mathcal{U}')$ is a closed subspace of $l_1(\mathcal{U})$, $f(X) \subset \mathrm{cl}_{l_1(\mathcal{U})}(\{x_n \mid n \in \mathbb{N}\}) \subset l_1(\mathcal{U}')$, so that $\pi_U(f(X)) = \{0\}$ for each $U \in \mathcal{U} \setminus \mathcal{U}'$. Let us denote $\mathcal{U}' = \{U_i \mid i \in \mathbb{N}\}$, and put $p_i = \pi_{U_i} \circ f$ for $i \in \mathbb{N}$. Then $\{p_i \mid i \in \mathbb{N}\}$ is a countable partition of unity on X such that $\mathrm{Coz}(p_i) \subset U_i$ for each $i \in \mathbb{N}$. Then $a_0 \in \mathrm{cl}_{\beta X}(Z(p_i))$ for each $i \in \mathbb{N}$. Thus $a_0 \in \beta X \setminus \upsilon X$ due to Lemma 2.2, that contradicts the choice of a_0. Hence X is realcompact. $\square$

Remark 2.4. Note that in [2, Theorem 2], the implication (a)$\Rightarrow$(b) of Theorem 1.2 was shown without assuming that X is of non-measurable cardinal. Here we show that the other implication (b)$\Rightarrow$(a) also holds for a Tychonoff space X of any cardinal. The set-valued mapping defined in the proof of the "if" part of Theorem 1.3 can be shown to be of infinite character as in the proof of [2, Theorem 8]. Thus for a Tychonoff space X, the realcompactness of X is equivalent to the following statement:

For every Banach space Y, every $\mathcal{B}$-fixed and l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}_c(Y)$ of infinite character admits a continuous selection $f : X \to Y$ such that $f(X)$ is separable.

Since the statement (b) of Theorem 1.2 implies the above statement, the implication (b)$\Rightarrow$(a) in Theorem 1.2 is valid for every Tychonoff space X of any cardinal.

On the other hand, the implication (c)$\Rightarrow$(a) of Theorem 1.2 need not be true for Tychonoff spaces of any cardinal. Indeed, a discrete space D of measurable cardinal satisfies condition (c) of Theorem 1.2 since every set-valued mapping on D has a continuous selection. But D is not realcompact (see [4, 3.11.D]).

3. Characterizations of Dieudonné completeness and Lindelöf property

For a Tychonoff space X, let Φ_X be the set of all normal open covers of X.

Then Φ_X forms the finest uniformity on X. Let us denote μX the Dieudonné completion of X (that is, the completion with respect to Φ_X). We will use the fact that a cozero-set of a Dieudonné complete space is Dieudonné complete [4, 8.5.13 (f)] as a subspace.

For the proof of Theorem 1.4, we use the following lemma which is essentially proved in [10, Theorem 2.6] (see also [4, 8.5.13(b)]).

Lemma 3.1 (Tamano [10]). *For a Tychonoff space X and a point $a \in \beta X$, $a \in \beta X \setminus \mu X$ if and only if there exists a (locally finite) partition of unity $\{p_\lambda \mid \lambda \in \Lambda\}$ on X such that $a \in \mathrm{cl}_{\beta X}(Z(p_\lambda))$ for each $\lambda \in \Lambda$.*

With this lemma, the following can be shown as in the proof of Lemma 2.3.

Lemma 3.2. *For a non-compact Dieudonné complete space X and a topological vector space Y, every $\mathcal{C}$-fixed set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ has a continuous selection.*

To prove Theorem 1.4, we also need the following:

Proposition 3.3. *Let X be a Tychonoff space. If X is the union of a compact subspace K and a Dieudonné complete subspace S, then X is Dieudonné complete.*

Proof: Let $\mathcal{F}$ be a Cauchy filter basis with respect to Φ_X. In case $\mathrm{cl}_X(F) \cap K \neq \emptyset$ for each $F \in \mathcal{F}$, $\mathcal{F}$ converges to a point of K since K is compact. Otherwise, suppose that $\mathrm{cl}_X(F_0) \cap K = \emptyset$ for some $F_0 \in \mathcal{F}$. Since K is compact, there exist a zero-set Z and a cozero-set C in X such that $\mathrm{cl}_X(F_0) \subset Z \subset C \subset X \setminus K \subset S$. Hence C is a cozero-set of the Dieudonné complete space S, so that the subspace C of S is Dieudonné complete. Put $\mathcal{F}' = \{F \cap C \mid F \in \mathcal{F}\}$. As $F_0 \subset C$, $\mathcal{F}'$ is a filter basis on C.

We claim that $\mathcal{F}'$ is a Cauchy filter basis with respect to Φ_C. To prove this, let $\mathcal{U} \in \Phi_C$. Since $\mathcal{U}$ is a normal open cover of C, there exists a locally finite (in C) cozero-set cover $\mathcal{U}'$ of C which refines $\mathcal{U}$. Then there exists a countable collection $\{\mathcal{V}_i \mid i \in \mathbb{N}\}$ of locally finite (in X) families $\mathcal{V}_i$ consisting of cozero-sets of X such that each $\mathcal{V}_i$ refines $\mathcal{U}'$ and $C = \bigcup\bigcup_{i\in\mathbb{N}} \mathcal{V}_i$. Indeed, since C is a cozero-set of X, there exists a continuous function $f : X \to [0,1]$ such that $C = \mathrm{Coz}(f)$. Putting $Z_i = f^{-1}([1/i, 1])$ and $U_i = f^{-1}((1/i, 1])$ for each $i \in \mathbb{N}$, we obtain zero-sets Z_i and cozero-sets U_i of X satisfying $U_i \subset Z_i \subset U_{i+1}$ and $C = \bigcup_{i\in\mathbb{N}} U_i$. Set $\mathcal{V}_i = \{U \cap U_i \mid U \in \mathcal{U}'\}$ for each $i \in \mathbb{N}$. Then $\{\mathcal{V}_i \mid i \in \mathbb{N}\}$ is the desired collection.

Finally, put $\mathcal{W} = \bigcup_{i\in\mathbb{N}} \mathcal{V}_i \cup \{X \setminus Z\}$. Then $\mathcal{W}$ is a σ-locally finite (in X) cozero-set cover of X. Hence $\mathcal{W}$ is a normal open cover of X, and $\mathcal{W} \in \Phi_X$. Because $\mathcal{F}$ is a Cauchy filter basis with respect to Φ_X, it follows that $F \subset W$ for some $F \in \mathcal{F}$ and some $W \in \mathcal{W}$. As $F_0 \cap F \neq \emptyset$, we have $W \neq X \setminus Z$, so that $W \in \mathcal{V}_i$ for some $i \in \mathbb{N}$. Since $\mathcal{V}_i$ refines $\mathcal{U}$, we may take $U \in \mathcal{U}$ so that $W \subset U$. Thus $F \cap C \subset U$. Therefore $\mathcal{F}'$ is a Cauchy filter basis with respect to Φ_C.

Since C is Dieudonné complete, $\mathcal{F}'$ converges to some x in C. As $F_0 \subset C$, $\mathcal{F}$ converges to x in X. Therefore X is Dieudonné complete. □

Proof of Theorem 1.4: The proof is quite similar to that of Theorem 1.3. If X is compact, the "only if" part follows from Theorem 1.1; otherwise from Lemma 3.2. Proof of the "if" part is obtained by replacing "realcompact", "υX", and "$\mathcal{B}$" in that of Theorem 1.3 with "Dieudonné complete", "μX", and "$\mathcal{C}$", respectively. $\square$

Next, we show Theorem 1.5.

Proof of Theorem 1.5: The "only if" part follows from Theorem 1.1 and the fact that the continuous image of a Lindelöf space is Lindelöf. To prove the "if" part, given an open cover $\mathcal{U}$ of X, apply the same argument as in the proof of the "if" part of Theorem 1.3. Then we can obtain some countable partition of unity on X subordinated to $\mathcal{U}$. $\square$

Lindelöf property is also characterized by set-valued selections. Let us recall some notations and terminology. For a topological space Y, set

$$\mathcal{F}(Y) = \{F \in 2^Y \mid F \text{ is closed}\},$$
$$\mathcal{C}(Y) = \{C \in 2^Y \mid C \text{ is compact}\}.$$

A set-valued mapping $\varphi : X \to 2^Y$ is *upper semicontinuous* (u.s.c. for short) if the set

$$\varphi^{\#}(V) = \{x \in X \mid \varphi(x) \subset V\}$$

is open in X for every open subset V of Y.

Proposition 3.4. *A regular space X is Lindelöf if and only if, for every completely metrizable space Y and every l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, there exist a u.s.c. set-valued mapping $\psi : X \to \mathcal{C}(Y)$ and an l.s.c. set-valued mapping $\theta : X \to \mathcal{C}(Y)$ such that $\theta(x) \subset \psi(x) \subset \varphi(x)$ for every $x \in X$ and $\bigcup_{x\in X} \psi(x)$ is separable.*

Proof: To prove the "only if" part, let X be a Lindelöf space, Y a completely metrizable space, and $\varphi : X \to \mathcal{F}(Y)$ an l.s.c. set-valued mapping. Due to Michael's compact-valued selection theorem [7, Theorem 1.1], there exist a u.s.c. set-valued mapping $\psi : X \to \mathcal{C}(Y)$ and an l.s.c. set-valued mapping $\theta : X \to \mathcal{C}(Y)$ such that $\theta(x) \subset \psi(x) \subset \varphi(x)$ for every $x \in X$ (see also [8, p. 305, Theorem 3]). Since Y is metrizable, it suffices to show that $\bigcup_{x\in X} \psi(x)$ is Lindelöf. Let $\mathcal{V}$ be a family of open sets of Y covering $\bigcup_{x\in X} \psi(x)$. For $x \in X$, $\psi(x)$ is compact, hence $\psi(x)$ is covered with some finite subset $\mathcal{V}_x$ of $\mathcal{V}$. Then $\{\psi^{\#}(\bigcup \mathcal{V}_x) \mid x \in X\}$ is an open cover of X. Since X is Lindelöf, $X = \bigcup_{i\in\mathbb{N}} \psi^{\#}(\bigcup \mathcal{V}_{x_i})$ for some countable set $\{x_i \mid i \in \mathbb{N}\}$ of X. Then $\bigcup_{i\in\mathbb{N}} \mathcal{V}_{x_i}$ is a countable subfamily of $\mathcal{V}$ that covers $\bigcup_{x\in X} \psi(x)$.

To prove the "if" part, let $\mathcal{U}$ be an open cover of X. Topologize $\mathcal{U}$ by the discrete topology. Note that $\mathcal{U}$ is completely metrizable. Define a set-valued mapping $\varphi : X \to 2^{\mathcal{U}}$ by $\varphi(x) = \{U \in \mathcal{U} \mid x \in U\}$ for $x \in X$. Then φ is closed-valued and l.s.c. By hypothesis, there exists a set-valued mapping $\psi : X \to 2^{\mathcal{U}}$ such that $\psi(x) \subset \varphi(x)$ for every $x \in X$ and $\bigcup_{x\in X} \psi(x)$ is separable. Then $\bigcup_{x\in X} \psi(x)$ is a countable subcover of $\mathcal{U}$. $\square$

4. Characterizations in terms of convex-valued mappings with the local intersection property or with open lower sections

Let $\varphi : X \to 2^Y$ be a set-valued mapping. Then φ is said to have *open lower sections* if $\varphi^{-1}(\{y\})$ is open in X for every $y \in Y$ ([13]). We say that φ has *the local intersection property* if each $x \in X$ has a neighborhood U with $\bigcap\{\varphi(z) \mid z \in U\} \neq \emptyset$ ([12]). Note that a set-valued mapping having open lower sections is l.s.c. and has the local intersection property. But lower semicontinuity need not imply the local intersection property and vice versa.

Yannelis and Prabhakar [13] showed that if X is paracompact, then every convex-valued mapping with open lower sections from X into a topological vector space admits a continuous selection. Later, Wu and Shen [12] improved this result by establishing that if X is paracompact, then every convex-valued mapping with the local intersection property from X into a topological vector space admits a continuous selection. By the following theorem we show that the converse of Yannelis and Prabhakar's result (and hence, one of Wu and Shen's result above) is also true.

Theorem 4.1. *For a T_1-space X, the following statements are equivalent:*

(a) *X is paracompact;*
(b) *for every topological vector space Y, a set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having the local intersection property admits a continuous selection;*
(c) *for every topological vector space Y, a set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having open lower sections admits a continuous selection.*

Proof: It suffices to show (c)$\Rightarrow$(a). Let X be a T_1-space satisfying (c) and $\mathcal{U}$ an open cover of X. For $x \in X$, let $\varphi(x)$ be the set of elements $y \in l_1(\mathcal{U})$ such that $\|y\| = 1$, $y(U) \geq 0$ for every $U \in \mathcal{U}$, $y(U) = 0$ for all but finitely many $U \in \mathcal{U}$, and $y(U) = 0$ for all $U \in \mathcal{U}$ with $x \notin U$. Then the resulting mapping $\varphi : X \to 2^{l_1(\mathcal{U})}$ is convex-valued. To see that φ has open lower sections, let $y \in Y$ with $\varphi^{-1}(\{y\}) \neq \emptyset$ and take $x \in \varphi^{-1}(\{y\})$. Choose $U_1, U_2, \ldots, U_n \in \mathcal{U}$ so that $\{U_1, U_2, \ldots, U_n\} = \{U \in \mathcal{U} \mid y(U) > 0\}$. Then $x \in \bigcap_{i=1}^n U_i \subset \varphi^{-1}(\{y\})$. Thus $\varphi^{-1}(\{y\})$ is open in X.

By hypothesis, there exists a continuous selection $f : X \to l_1(\mathcal{U})$ of φ. Putting $p_U = \pi_U \circ f$ for $U \in \mathcal{U}$, we obtain a partition of unity $\{p_U \mid U \in \mathcal{U}\}$ subordinated to $\mathcal{U}$. Therefore X is paracompact. $\square$

In formulations similar to Theorem 4.1, we can characterize realcompactness, Dieudonné completeness, and Lindelöf property. A topological space satisfies *the discrete countable chain condition* (DCCC for short) if every discrete collection of non-empty open sets is countable. Every Lindelöf T_1-space and every separable space satisfy the DCCC. We also note that every metrizable space satisfying the DCCC is second countable (see [11]).

Theorem 4.2. *For a Tychonoff space X, the following statements are equivalent:*

(a) *X is realcompact;*
(b) *for every Hausdorff topological vector space Y, every $\mathcal{B}$-fixed set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having the local intersection property admits a continuous selection $f : X \to Y$ such that $f(X)$ satisfies the DCCC;*
(c) *for every Hausdorff topological vector space Y, every $\mathcal{B}$-fixed set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having open lower sections admits a continuous selection $f : X \to Y$ such that $f(X)$ satisfies the DCCC.*

Proof: If X is compact, (a)$\Rightarrow$(b) follows from Theorem 4.1 and the fact that the continuous image of compact space is compact; otherwise from Lemma 2.3. The implication (b)$\Rightarrow$(c) is trivial. To prove (c)$\Rightarrow$(a), suppose that X satisfies the statement (c). Assume that X is not realcompact and take $a_0 \in \upsilon X \setminus X$. We derive a contradiction. Set $\mathcal{U} = \{\mathrm{Coz}(p) \mid p \in C(X),\ a_0 \in \mathrm{cl}_{\beta X}(Z(p))\}$. Then $\mathcal{U}$ is an open cover of X. For $x \in X$, let $\varphi(x)$ be the set of elements y of $l_1(\mathcal{U})$ such that $\|y\| = 1$, $y(U) \geq 0$ for every $U \in \mathcal{U}$, $y(U) = 0$ for all but finitely many $U \in \mathcal{U}$, and $y(U) = 0$ for all $U \in \mathcal{U}$ with $x \notin U$. By referring to the proof of the "if" part of Theorem 1.3 and the proof of (c)$\Rightarrow$(a) of Theorem 4.1, we can verify that the resulting mapping $\varphi : X \to 2^{l_1(\mathcal{U})}$ is a $\mathcal{B}$-fixed and convex-valued mapping having open lower sections.

By hypothesis, φ admits a continuous selection $f : X \to l_1(\mathcal{U})$ such that $f(X)$ satisfies the DCCC. Since $f(X)$ is metrizable, $f(X)$ is separable. As in the proof of the "if" part of Theorem 1.3, there exists a countable partition of unity $\{p_i \mid i \in \mathbb{N}\}$ subordinated to $\mathcal{U}$. Then $a_0 \in \mathrm{cl}_{\beta X}(Z(p_i))$ for each $i \in \mathbb{N}$, so that $a_0 \in \beta X \setminus \upsilon X$ due to Lemma 2.2, but that contradicts the choice of a_0. Hence X is realcompact. □

Remark 4.3. By the proof of Theorem 4.2, it also holds that a Tychonoff space is realcompact if and only if, for every normed space Y, every $\mathcal{B}$-fixed set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having the local intersection property admits a continuous selection $f : X \to Y$ such that $f(X)$ is separable.

Theorem 4.4. *For a Tychonoff space X, the following statements are equivalent:*

(a) *X is Dieudonné complete;*
(b) *for every topological vector space Y, every $\mathcal{C}$-fixed set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having the local intersection property admits a continuous selection;*

(c) *for every topological vector space Y, every $\mathcal{C}$-fixed set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having open lower sections admits a continuous selection.*

Proof: If X is compact, the (a)⇒(b) follows from Theorem 4.1; otherwise from Lemma 3.2. The implication (b)⇒(c) is clear. The implication (c)⇒(a) is obtained by replacing "realcompact", "υX", and "$\mathcal{B}$" in the proof of Theorem 4.2 with "Dieudonné complete", "μX", and "$\mathcal{C}$", respectively. □

Theorem 4.5. *For a regular space X, the following statements are equivalent:*

(a) *X is Lindelöf;*
(b) *for every topological vector space Y, every set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having the local intersection property admits a continuous selection $f : X \to Y$ such that $f(X)$ is Lindelöf;*
(c) *for every topological vector space Y, every set-valued mapping $\varphi : X \to \mathcal{K}(Y)$ having open lower sections admits a continuous selection $f : X \to Y$ such that $f(X)$ is Lindelöf.*

Proof: The implication (a)⇒(b) follows from Theorem 4.1 and the fact that the continuous image of a Lindelöf space is Lindelöf. The implication (b)⇒(c) is clear. To prove (c)⇒(a), let $\mathcal{U}$ be an open cover of X. By the same argument as in the proof of (c)⇒(a) of Theorem 4.2, there exists a countable partition of unity on X subordinated to $\mathcal{U}$. □

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