On subsets of Alexandroff duplicates

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Abstract. We characterize the subsets of the Alexandroff duplicate which have a G_δ-diagonal and the subsets which are M-spaces in the sense of Morita.

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1. Introduction

All spaces are assumed to be regular T_1, and all mappings to be continuous. We denote all positive integers, real numbers by $\mathbb{N}$, $\mathbb{R}$, respectively.

As it is well known, the Alexandroff duplicate of $\mathbb{R}$ does not have a G_δ-diagonal and the famous Michael line is not an M-space in the sense of Morita, although it is a subspace of the Alexandroff duplicate of $\mathbb{R}$. So, in this paper, we characterize the subspaces of the Alexandroff duplicate $X \times_{ad} (2)$ which have a G_δ-diagonal, where X has a G_δ-diagonal, and also characterize the subspaces of $Y \times_{ad} (2)$ which are M-spaces, where X is a metrizable space. The former gives an answer to the problem posed by S. Watson, [3, Problem 3.1.29], where he asks how to characterize the subsets of $[0,1] \times_{ad} (2)$ which have a G_δ-diagonal.

As for the properties of G_δ-diagonals and M-spaces used here, we refer to Gruenhage [1]. We recall the definition of the Alexandroff duplicate $X \times_{ad} (2)$ of a space X, stated in [3, Definition 3.1.1]. Let (X, τ) be a space. Define the topology on $Z = X \times 2$ by declaring that each $(x, 1)$ is open and that for each open $U \in \tau$, $U \times 2 \setminus \{(x,1)\}$ is open. The space Z so defined is denoted by $X \times_{ad} (2)$, where $_{ad}$ stands for Alexandroff duplicate. In the sequel, we write a subspace of $X \times_{ad} (2)$ in the following form:

$$T(A, B) = A \times \{1\} \cup B \times \{0\},$$

where $A, B \subset X$.

2. On subspaces of Alexandroff duplicates

For a subset A of a space X, we denote by A^d the set of all accumulation points of A in X.

Theorem 2.1. *Assume that a space X has a G_δ-diagonal and $T(A,B) \subset X \times_{ad} (2)$. Then $T(A,B)$ has a G_δ-diagonal if and only if $A \cap B = \bigcup\{C_i : i \in \mathbb{N}\}$ with $(C_i)^d \cap B = \emptyset$ for each i.*

Proof: Only if part: Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a G_δ-diagonal sequence for $T(A,B)$. For each $x \in A \cap B$, there exists $n(x) \in \mathbb{N}$ such that

$$(x,0) \notin S((x,1),\mathcal{U}_{n(x)}).$$

Let

$$C_n = \{x \in A \cap B : n(x) = n\}, \quad n \in \mathbb{N}.$$

Then $A \cap B = \bigcup_n C_n$. Assume that $(C_n)^d \cap B \neq \emptyset$ for some n. For a point $x \in (C_n)^d \cap B$, there exists $U \in \mathcal{U}_n$ such that $(x,0) \in U$. Since x is an accumulation point of C_n, there exists $x' \in C_n$ such that $(x',0),(x',1) \in U$, but this is impossible.

If part: Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be a G_δ-diagonal sequence for $A \cup B$. By the assumption, $A \cap B = \bigcup\{C_n : n \in \mathbb{N}\}$, where $(C_n)^d \cap B = \emptyset$ for each n. Since C_n is discrete in B, there exists a family $\{V(x) : x \in C_n\}$ of open subsets of $A \cup B$ such that for each $x \in C_n$, $V(x) \cap B = \{x\}$ and $x \in V(x) \subset U$ for some $U \in \mathcal{U}_n$. For each $U \in \mathcal{U}_n$, $n \in \mathbb{N}$, let

$$\widehat{U} = (U \setminus \overline{C_n}) \times \{0,1\} \cap T(A,B).$$

For each $x \in C_n$, $n \in \mathbb{N}$, let

$$\widehat{V}(x) = (V(x) \times \{0,1\} \setminus \{(x,1)\}) \cap T(A,B).$$

For each $n \in \mathbb{N}$, define an open cover

$$\mathcal{W}(n) = \{\widehat{U} : U \in \mathcal{U}_n\} \cup \{\widehat{V}(x) : x \in C_n\} \cup \{\{(x,1)\} : x \in A\}.$$

We show that $(\mathcal{W}(n))_{n\in\mathbb{N}}$ is a G_δ-diagonal sequence for $T(A,B)$. To this end, let

$$p = (x,s), \quad q = (y,t)$$

be different points of $T(A,B)$. If $x \neq y$, then there exists $n \in \mathbb{N}$ such that $x \notin S(y,\mathcal{U}_n)$. Then it is easily seen that $p \notin S(q,\mathcal{W}(n))$. If $x = y$, $s = 0$, $t = 1$, then we have $x \in A \cap B$ and $x \in C_n$ for some $n \in \mathbb{N}$. In this case, we easily have

$$p \notin S(q,\mathcal{W}(n)) = \widehat{V}(x).$$

Hence $T(A,B)$ has a G_δ-diagonal. $\square$

We give a remark to some special cases of X:

Remark 2.1. (1) If $X = \mathbb{R}$, $T(A,B)$ has a G_δ-diagonal if and only if $A \cap B$ is countable. It is because any uncountable subset of $\mathbb{R}$ has an accumulation point in $\mathbb{R}$.
(2) If X is metrizable, the above condition for $T(A,B)$ to have a G_δ-diagonal is that for $T(A,B)$ to be submetrizable. This follows from the fact that $T(A,B)$ is paracompact.

Next, we characterize $T(A,B)$ which is an M-space in the sense of Morita. A space X is called an M-*space* if there exists a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of X such that for each n, $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$ and if $x_n \in S(x,\mathcal{U}_n)$, then $\{x_n : n \in \mathbb{N}\}$ clusters in X. Such a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ is called an M-*sequence* for X. On the other hand, in 1963 Arhangel'skiĭ gave the concept of p-spaces. As it is well known, M-spaces and p-spaces are equivalent in the presence of paracompactness [1, Corollary 3.20], and paracompact p-spaces coincide with preimages of a metric space under a perfect mapping [1, Corollary 3.7].

Let (X,d) be a metric space. We denote an open ball with center x and radius r by $B(x,r)$. We note that the projection $\pi : T(A,B) \longrightarrow A \cup B$ is continuous.

In connection with the next theorem, the referee informed us about the interesting fact that E.G. Pytkeev wrote a paper in which he proved that if a space X is a Tychonoff space such that each subspace of X is a paracompact p-space, then the structure of X is very similar to that of the Alexandroff duplicate of a metric space; indeed, then the subspace of all non-isolated points is metrizable.

Theorem 2.2. *Let $T(A,B) \subset X \times_{ad} (2)$, where X is a metric space. Then $T(A,B)$ is an M-space if and only if B is a G_δ-set in $A \cup B$.*

Proof: Only if part: Assume that B were not a G_δ-set. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be an M-space for $T(A,B)$. Since X is a metric space, without loss of generality we can assume that if $(x_n,s_n) \in S((x,s),\mathcal{U}_n)$, $n \in \mathbb{N}$, then $x_n \longrightarrow x$ as $n \longrightarrow \infty$. Let $n \in \mathbb{N}$ be fixed. For each $x \in B$, there exists $U \in \mathcal{U}_n$ such that $(x,0) \in U$. There exists a basic open neighborhood $N(x,r(x))$ of $(x,0)$ in $X \times_{ad} (2)$ such that

$$\begin{aligned} N(x,r(x)) &= B(x,r(x)) \times \{0,1\} \setminus \{(x,1)\}, \\ N(x,r(x)) &\cap T(A,B) \subset U. \end{aligned}$$

Let

$$G_n = \left(\bigcup\{B(x,r(x)) : x \in B\}\right) \cap (A \cup B),$$

which is open in $A \cup B$. By the assumption, there exists $a \in \bigcap_n G_n \setminus B$. Then for each $n \in \mathbb{N}$, there exists a point

$$(x_n,0) \in B \times \{0\} \cap S((a,1),\mathcal{U}_n).$$

Since $(\mathcal{U}_n)$ is an M-sequence and $x_n \longrightarrow a$ as $n \longrightarrow \infty$, $\{(x_n,0) : n \in \mathbb{N}\}$ clusters at $(a,1)$, but this is a contradiction because $\{(a,1)\}$ is open.

If part: Let $B = \bigcap_n G_n,\ G_{n+1} \subset G_n,\ n \in \mathbb{N}$, where each G_n is open in $A \cup B$. Since $A \cup B$ is a metric space, there exists a development $(\mathcal{U}_n)_{n\in\mathbb{N}}$ for $A \cup B$ such that $\mathcal{U}^*_{n+1} < \mathcal{U}_n,\ n \in \mathbb{N}$. We construct a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of open covers of $T(A, B)$ as follows:

$$\mathcal{V}_n = \pi^{-1}(\mathcal{U}_n \mid G_n) \cup \{\{(x, 1)\} \mid x \in A \setminus G_n\}, \quad n \in \mathbb{N}.$$

Then it is easily checked that each $\mathcal{V}_{n+1}$ star-refines $\mathcal{V}_n$. We show that $(\mathcal{V}_n)_{n\in\mathbb{N}}$ is an M-sequence for $T(A, B)$. Let

$$(x_n, r_n) \in S((x, r), \mathcal{V}_n), \quad n \in \mathbb{N}.$$

If $x \in B$, then $(x, 0)$ is a cluster point of $\{(x_n, r_n) \mid n \in \mathbb{N}\}$. If $x \in A \setminus B$, then there exists $k \in \mathbb{N}$ such that $x \notin G_k$. From the construction of $(\mathcal{V}_n)$, it follows that $(x_n, r_n) = (x, 0)$ for $n \geq k$, which means that $(x_n, r_n) \longrightarrow (x, 0)$ as $n \longrightarrow \infty$. $\square$

Corollary 2.1. *Let $T(A, B) \subset X \times_{ad} (2)$, where X is a metric space. Then $T(A, B)$ is metrizable if and only if B is a G_δ-set in $A \cup B$ and $A \cap B = \bigcup_{i\in\mathbb{N}} C_i$, where for each i, $(C_i)^d \cap B = \emptyset$.*

Here, we recall the definition of resolutions of spaces. Let X be a space and for each $x \in X$, let $f_x : X \setminus \{x\} \longrightarrow Y_x$ be a mapping. We topologize

$$Z = \bigcup\{\{x\} \times Y_x : x \in X\}$$

by defining an open set $U \otimes V$ for each $x \in X$ and each open subset U of X with $x \in U$ and open subset V of Y_x as

$$U \otimes V = (\{x\} \times V) \cup \bigcup\{\{p\} \times Y_p : p \in U \cap f_x^{-1}(V)\}.$$

We call Z thus defined the *resolution* of X at each point $x \in X$ into Y_x by f_x [3, Definition 3.1.32], and we denote it by $Z = R(X, f_x, Y_x)$. We note that the projection $\pi : Z \longrightarrow X$ defined by $\pi((x, y)) = x$ for each $(x, y) \in Z$ is continuous.

Example 2.1. There exists a resolution $Z = R(X, f_x, Y_x)$ of a compact space X into paracompact M-spaces Y_x, $x \in X$, such that Z is not an M-space.

Proof: Let $X = \omega_1 + 1$ with the order topology. For each $\alpha < \omega_1$, let Y_α be the copy of $\mathbb{R}$ with the usual topology. Let $f_\alpha : X \setminus \{\alpha\} \longrightarrow Y_\alpha$ be a constant mapping such that $f_\alpha(X \setminus \{\alpha\}) = y_\alpha \in Y_\alpha$. For $\alpha = \omega_1$, $Y_{\omega_1} = \{\omega_1\}$ and let $f_{\omega_1} : X \setminus \{\omega_1\} \longrightarrow Y_{\omega_1}$ be a natural mapping. Let $Z = R(X, f_x, Y_x)$. Assume that there exists an M-sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ for Z. For $p = (\omega_1, \omega_1)$, there exists $\alpha \in \omega_1$ such that

$$\{\alpha\} \times Y_\alpha \subset \bigcap_{n\in\mathbb{N}} S((\omega_1, \omega_1), \mathcal{U}_n).$$

Since Y_α is not countably compact, this is impossible. $\square$

We say that a subset Λ is F_σ-*discrete* in X if $\Lambda = \bigcup_{n\in\mathbb{N}} \Lambda_n$, where each Λ_n is discrete and closed in X. Richardson and Watson showed that if X and each Y_x are metrizable and

$$\Lambda = \{x \in X : |Y_x| > 1\}$$

is F_σ-discrete in X, then $R(X, f_x, Y_x)$ is metrizable [2, Proposition 9]. We recall a characterization of paracompact p-spaces: a space X is a paracompact p-space if and only if there exists a perfect mapping of X onto a metric space.

Theorem 2.3. *Let X be a metric space and each Y_x, $x \in X$, a paracompact p-space. If Λ, defined above, is F_σ-discrete in X, then $Z = R(X, f_x, Y_x)$ is a paracompact p-space.*

Proof: By the above characterization, for each $x \in X$ there exists a perfect mapping $g_x : Y_x \longrightarrow M_x$ with M_x metric. By the condition on Λ, the resolution $Z' = R(X, g_x f_x, M_x)$ is a metric space. So, it suffices to show that the mapping $\Phi : Z \longrightarrow Z'$ defined by

$$\Phi(x, y) = (x, g_x(y)), \quad (x, y) \in Z,$$

is a perfect mapping. It is easily checked that Φ is continuous. To see that Φ is closed, let W be an open set of Z containing $\Phi^{-1}(x, y') = \{x\} \times g_x^{-1}(y')$. There exists a finite open cover $\{U_i \otimes V_i \mid i = 1, \dots, k\}$ of $\Phi^{-1}(x, y')$ in Z such that

$$\Phi^{-1}(x, y') \subset \bigcup_{i=1}^{k} U_i \otimes V_i \subset W,$$

where each U_i is an open neighborhood of x in X. Since $g_x : Y_x \longrightarrow M_x$ is a perfect mapping, there exists an open neighborhood O of y' in M_x such that $g_x^{-1}(O) \subset \bigcup_{i=1}^{k} V_i$. Then we can easily see that $(\bigcap_{i=1}^{k} U_i) \otimes O$ is an open neighborhood of (x, y') in Z' such that $\Phi^{-1}((\bigcap_{i=1}^{k} U_i) \otimes O) \subset W$. Hence Φ is a perfect mapping. $\square$

Since $\pi : R(X, f_x, Y_x) \longrightarrow X$ is a perfect mapping if each Y_x is compact [2, Lemma 6], the following is easy to see:

Theorem 2.4. *Let X be an M-space and let each Y_x be compact. Then $Z = R(X, f_x, Y_x)$ is an M-space.*

References

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