A note on the centralizer of topological isometric extensions

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Abstract. The centralizer of a semisimple isometric extension of a minimal flow is described.

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If $S : X \longrightarrow X$ is a homeomorphism of a compact metric space X then a pair (X, S) is called a *compact metric flow*. Such a homeomorphism "generates" a continuous action of the group $\mathbb{Z}$ on X:

$$X \times \mathbb{Z} \ni (x, n) \longmapsto S^n(x) \in X.$$

A flow (X, S) is called *minimal* if for every closed subset $F \subset X$ with $S(F) = F$ either $F = \emptyset$ or $F = X$. Equivalently, (X, S) is minimal iff every orbit is dense in X: $\mathrm{cl}\{S^n(x) : n \in \mathbb{Z}\} = X$ for each $x \in X$.

Let us consider the following commutative diagram

(1)
$$\begin{tikzcd} \mathcal{X} \arrow[d, "\psi"'] \arrow[dr, "\varphi"] & \\ \mathcal{Y} \arrow[r, "\pi"'] & \mathcal{Z} \end{tikzcd}$$

where $\mathcal{X} = (X, S)$, $\mathcal{Y} = (Y, T)$, $\mathcal{Z} = (Z, U)$ are compact metric flows and φ is a group (say G-) extension. In a case of $\mathcal{Z}$ being minimal, $\mathcal{Y}$ is called an *isometric extension* of $\mathcal{Z}$. As a straightforward corollary of Theorem 3.1 of [3] we have the following.

Proposition 1. *Assume that in* (1) *$\mathcal{Z}$ is minimal. Then every minimal subflow of $\mathcal{Y}$ is an isometric extension of $\mathcal{Z}$.*

Proof: Let $N \subset Y$ be a minimal subset. Take a minimal subset $M \subset \psi^{-1}(N)$. By Theorem 3.1 of [3], there exists a closed subgroup $H < G$ and a homomorphism

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$\phi : (M, S) \longrightarrow \mathcal{Z}$ such that ϕ is an H-extension. Since for every $x \in M$, $\phi(x) = \varphi(x)$ and obviously $\psi(M) = N$, we have the following commutative diagram

$$\begin{tikzcd}
(M,S) \arrow[d, "\psi|_M"'] \arrow[dr, "\phi"] & \\
(N,T) \arrow[r, "\pi|_N"'] & \mathcal{Z}
\end{tikzcd}$$

thus (N, T) is an isometric extension of $\mathcal{Z}$. □

Remark 1. In Theorem 3.1 of [3] $\mathcal{Z}$ is assumed to be a minimal rotation. Nevertheless, the arguments are valid for an arbitrary minimal flow.

Remark 2. If we assume that in (1) $\mathcal{Y}$ is semisimple, the above proposition "allows" us to consider only isometric extensions of $\mathcal{Z}$.

It is well-known that if in (1) $\mathcal{Y}$ is a minimal isometric extension of $\mathcal{Z}$, then $\mathcal{X}$ can be assumed to be minimal and one may easily verify that $\mathcal{X}$ is a $\mathcal{K}$-extension of $\mathcal{Y}$, where $\mathcal{K} = \{g \in G : \forall_{x\in X}\ \psi(gx) = \psi(x)\}$ and that $\pi^{-1}(z)$ is homeomorphic to a homogeneous space $G/\mathcal{K}$ for every $z \in Z$.

We will be interested in special kinds of isometric extensions, which are factors of "cocycle" extensions of minimal flows. Let $\mathcal{Z} = (Z, U)$ be a minimal flow and $\alpha : Z \longrightarrow G$ be a continuous function (called a *topological cocycle*) with values in a compact topological group. We define a homeomorphism $U_\alpha : Z \times G \longrightarrow Z \times G$ by $U_\alpha(z, g) = (Uz, \alpha(z)g)$. Of course $(Z \times G, U_\alpha)$ is a G-extension of $\mathcal{Z}$. Now let $K < G$ be a closed subgroup. We define a homeomorphism $U_{G,K,\alpha} : Z\times G/K \longrightarrow Z\times G/K$ by $U_{G,K,\alpha}(z, gK) = (Uz, \alpha(z)gK)$. Then, obviously, $(Z\times G/K, U_{G,K,\alpha})$ is an isometric extension of $\mathcal{Z}$.

Remark 3. We can realize a group extension $\varphi : \mathcal{X} \longrightarrow \mathcal{Z}$ as a topological cocycle if and only if there is a continuous selector of the homomorphism φ (i.e. there exists a continuous map $j : Z \longrightarrow X$ such that $\varphi \circ j = \mathrm{Id}_Z$).

Let us now consider the situation described by the following commutative diagram:

$$\begin{tikzcd}
(Z\times G', U_{\alpha'}) \arrow[d, "\psi'"'] \arrow[drr, "\varphi'"] & & & & (Z\times G, U_\alpha) \arrow[d, "\psi"] \arrow[dll, "\varphi"'] \\
(Z\times G'/K', U_{G',K',\alpha'}) \arrow[rr, "\pi'"'] & & (Z,U) & & (Z\times G/K, U_{G,K,\alpha}) \arrow[ll, "\pi"]
\end{tikzcd}$$

where φ, φ', π and π' are projections onto the first coordinate, while ψ and ψ' are natural projections along the second coordinate.

Our first aim is to describe those homomorphisms

$$t : (Z \times G'/K', U_{G',K',\alpha'}) \longrightarrow (Z \times G/K, U_{G,K,\alpha}),$$

which are lifting of some element of

$$\mathrm{Aut}(\mathcal{Z}) = \{S : Z \longrightarrow Z |\ S \ \text{ is a homeomorphism and } \ S \circ U = U \circ S\}.$$

The methods used in the proof of the next proposition, as well as in the rest of the paper, are adapted from the proof of Theorem 1.4 and other parts of [2]. The main difficulty is that, in contrary to the measure theoretical case, we do not have at our disposal continuous selectors for a continuous map, in general. So for instance not every group extension is a cocycle extension and we are not able to use transform functions (which are constructed with use of measurable selectors) to define some maps. Moreover, on various groups we have different topological structures (in this paper, on the centralizer we consider the uniform topology, while in the measure theoretical case the centralizer is furnished with the weak topology).

Proposition 2. *Assume that*

$$t : (Z \times G'/K', U_{G',K',\alpha'}) \longrightarrow (Z \times G/K, U_{G,K,\alpha})$$

is a homomorphism between two minimal isometric extensions of $\mathcal{Z}$ for which there exists $s \in \mathrm{Aut}(\mathcal{Z})$ such that $s \circ \pi' = \pi \circ t$. Let $K < G$ be irreducible (i.e. K does not contain nontrivial normal subgroups of G). Then

(i) *there exist a continuous group epimorphism $l : G' \longrightarrow G$ and a continuous function $f : Z \longrightarrow G$ such that*

$$t(z, g'K') = (s(z), f(z)l(g')K), \tag{2}$$

$l(K') \subset K$ and

$$l(\alpha'(z)) = f(Uz)^{-1}\alpha(sz)f(z); \tag{3}$$

(ii) *$l^{-1}(K) = K'$ if and only if t is an isomorphism;*

(iii) *if $l|_{K'}$ is one-to-one then $K' < G'$ is irreducible;*

(iv) *if $K' < G'$ is irreducible and t is an isomorphism then l is one-to-one and $l(K') = K$.*

Proof: (i) First observe that $(z, g', sz, g) \overset{J}{\longmapsto} (z, g', g)$ establishes an isomorphism between $((\varphi' \times \varphi)^{-1}\, \mathrm{Graph}(s), U_{\alpha'} \times U_\alpha)$ and $(Z \times G' \times G, U_{\alpha' \times \alpha \circ s})$, where $(\alpha' \times \alpha \circ s)(z) = (\alpha'(z), \alpha \circ s(z))$. Let us take a $U_{\alpha' \times \alpha\circ s}$-minimal subset $M \subset J((\psi' \times \psi)^{-1}\, \mathrm{Graph}(t)) \subset J((\varphi' \times \varphi)^{-1}\, \mathrm{Graph}(s))$.

Let $\rho : Z \times G'/K' \longrightarrow G/K$ be a continuous map determined by

$$t(z, g'K') = (s(z), \rho(z, g'K')). \tag{4}$$

Since t is a homomorphism of flows, ρ satisfies

$$\rho(Uz, \alpha'(z)g'K') = \alpha \circ s(z)\rho(z, g'K'). \tag{5}$$

From (5) it follows that a continuous map

$$Z \times G' \times G \ni (z, g', g) \xrightarrow{F} g^{-1}\rho(z, g'K') \in G/K$$

is $U_{\alpha' \times \alpha \circ s}$-invariant. Therefore $F|_M$ is constant, hence there is $k \in G$ such that

$$\rho(z, g'K') = gkK \tag{6}$$

for every $(z, g', g) \in M$. Without loss of generality we may assume that $k = 1_G$, for if $k \neq 1_G$, replace α, M and t by $\alpha_1(z) = k\alpha(z)k^{-1}$, $M_1 = \{(z, g', kgk^{-1}) : (z, g', g) \in M\}$ and $t_1(z, g'K') = (s(z), kgk^{-1}K)$ if $t(z, g'K') = (z, gkK)$, respectively. Then all corresponding objects are isomorphic and $\rho_1 : Z \times G'/K' \longrightarrow G/K$ (defined as above for t_1) satisfies $\rho_1(z, g'K') = gK$ for all $(z, g', g) \in M_1$.

Put $H = \{(g', g) |\ M(g', g) = M\}$. Then H is a closed subgroup of $G' \times G$. $\square$

Theorem 3 ([3, Theorem 3.1])**.** *The flow $(M, U_{\alpha' \times \alpha \circ s})$ is an H-extension of $\mathcal{Z}$.*

Now let $N_{G'} = \{g' \in G' :\ (g', 1_G) \in H\}$ and $N_G = \{g \in G :\ (1_{G'}, g) \in H\}$. Both $N_{G'}$ and N_G are closed normal subgroups of H (by the assumption of minimality, H has full projections onto both coordinates).

Lemma 4 ([3, Lemma 3.4])**.** *There exists a continuous group homomorphism $\xi : G'/N_{G'} \longrightarrow G/N_G$ such that*

$$H = \bigcup_{g' \in G'} \tau'^{-1}(g'N_{G'}) \times \tau^{-1}(\xi(g'N_{G'})),$$

where $\tau' : G' \longrightarrow G'/N_{G'}$ and $\tau : G \longrightarrow G/N_G$ are natural homomorphisms.

Proof: Let us take $k \in N_G$. For each $(z, g', g) \in M$, $(z, g', gk) \in M$ since $(1_{G'}, k) \in H$. By the choice of M, we have $t(z, g'K') = (s(z), gK) = (s(z), gkK)$, hence $k \in K$. Thus $N_G < K$, hence by irreducibility of $K < G$, $N_G = \{1_G\}$. It follows that if $l = \xi \circ \tau'$ then $H = \{(g', l(g')) :\ g' \in G'\}$. Now put $f : Z \longrightarrow G$,

$$f(z) = gl(g')^{-1}, \tag{7}$$

if $(z, g', g) \in M$. Observe that if $(z, g', g), (z, g'_1, g_1) \in M$ then $({g'}^{-1} g'_1,\ g^{-1} g_1) \in H$ and since l is a group homomorphism, f is well-defined. Obviously f is continuous. Therefore (4), (6) (with $k = 1_G$) and (7) give (2). Also, by (7) and the invariance of M, we have (3).

Now take $k' \in K'$. Find $z \in Z$ and $k \in K$ such that $(z, k', k) \in M$. Then

$$(s(z), f(z)K) = t(z, K') = t(z, k'K') = (s(z), f(z)l(k')K)$$

and thus $l(k') \in K$. Therefore $l(K') \subset K$.

(ii) If t is an isomorphism and $g' \in l^{-1}(K)$ then we have

$$t(z, g'K') = (s(z), f(z)l(g')K) = (s(z), f(z)K) = t(z, K'),$$

hence $g' \in K'$.

Now if $l^{-1}(K) = K'$ and $t(z, g'K') = t(z_1, g'_1 K')$ then $z = z_1$ and $l({g'}^{-1} g'_1) \in K$, hence ${g'}^{-1} g'_1 \in K'$.

Now (iii) and (iv) are obvious. □

Given a flow (Z, U), by $C(Z, U)$ ($C(U)$ for short, if the phase space Z is established) let us denote the *centralizer* of (Z, U), i.e. the set of all continuous transformations from Z onto Z which commute with U. If we endow the subset $\mathrm{Aut}(\mathcal{Z}) = \mathrm{Aut}(U) \subset C(Z, U)$ of all invertible elements with the topology of uniform convergence (we denote the closure operation in this topology by "cl"), it becomes a Polish group.

As an immediate consequence of the above proposition we have the following.

Proposition 5. *Let $(Z \times G/K, U_{G,K,\alpha})$ be a minimal isometric extension of $\mathcal{Z}$. Then every lift $\tilde{s} \in C(U_{G,K,\alpha})$ of $s \in \mathrm{Aut}(\mathcal{Z})$ is of the form*

$$\tilde{s}(z, gK) = s_{f,l}(z, gK) = (s(z), f(z)l(g)K),$$

where $f : Z \longrightarrow G$ is a continuous function and $l : G \longrightarrow G$ is a continuous group epimorphism such that $l(K) \subset K$ and

$$l(\alpha(z)) = f(Uz)^{-1} \alpha(sz) f(z). \tag{8}$$

Moreover $s_{f,l}$ is invertible if and only if so is l. In such a case $l(K) = K$.

Denote $\mathrm{Ad}_k : G \longrightarrow G$, $\mathrm{Ad}_k(g) = k^{-1} g k$ and $N_G(K) = \{g \in G :\ g^{-1} K g = K\}$.

Corollary 6. (i) *Every $s \in \mathrm{Aut}(Z)$ which can be lifted to $C(U_{G,K,\alpha})$ can also be lifted to $C(U_\alpha)$.*

(ii) *If $\mathcal{Z}$ is a minimal rotation then every element of $C(U_{G,K,\alpha})$ is a lift of some $s \in \mathrm{Aut}(\mathcal{Z})$.*

(iii) *If $\mathcal{Z}$ is a minimal rotation and all elements of $C(U_\alpha)$ are invertible then so are all elements of $C(U_{G,K,\alpha})$.*

(iv) *If $\widetilde{\mathrm{Id}} \in C(U_{G,K,\alpha})$ is a lift of Id_Z then*

$$\widetilde{\mathrm{Id}}(z,gK) = (z,gkK) = (\mathrm{Id}_Z)_{k,\mathrm{Ad}_k}(z,gK),$$

where $k \in N_G(K)$.

(v) *All lifts of the identity are invertible.*

(vi) *If s, $s^{-1} \in \mathrm{Aut}(\mathcal{Z})$ and s, s^{-1} can both be lifted to $C(U_{G,K,\alpha})$ then all lifts of s are invertible.*

PROOF: One can easily check that (i), (ii) and (iii) follow. To show (iv) put $\widetilde{\mathrm{Id}} = (\mathrm{Id}_Z)_{f,l}$ and observe that, by (8) (with $s = \mathrm{Id}_Z$), a continuous map $Z \times G \ni (z,g) \longmapsto g^{-1}f(z)l(g) \in G$ is U_α-invariant, hence constant. Thus there exists $k \in G$ such that for every $(z,g) \in Z \times G$

$$f(z)l(g) = gk. \tag{9}$$

Putting $g = 1_G$ in (9), we get $f(z) = k$ for every $z \in Z$ and then $l(g) = k^{-1}gk$ for each $g \in G$, so that $l = \mathrm{Ad}_k$. Since Ad_k is invertible, we immediately have that $k \in N_G(K)$.

Now (v) and (vi) follow from (iv) and Proposition 5. □

Remark 4. Proposition 5 generalizes some results of Section 3 of [3] and Section 2 of [4] where the case of group extensions (instead of isometric extensions) is considered.

Now put

$$L_K(U,\alpha) = \{s \in \mathrm{Aut}(\mathcal{Z}) :\ s,\ s^{-1}\ \text{ can be lifted to }\ C(U_{G,K,\alpha})\},$$
$$\tilde{C}(U_{G,K,\alpha}) = \{\tilde{s} \in C(U_{G,K,\alpha}) :\ \tilde{s}\ \text{ is a lift of some }\ s \in L_K(U,\alpha)\}.$$

From Corollary 6(vi) it follows that the above sets are groups. Notice also that $\tilde{C}(U_{G,K,\alpha})$ is a closed subgroup of $\mathrm{Aut}(U_{G,K,\alpha})$ (from Proposition 5 it follows that $\tilde{s} \in \tilde{C}(U_{G,K,\alpha})$ iff the factor $\mathcal{Z}$ is invariant under $\tilde{s}$).

Let us define $\sigma : N_G(K)/K \longrightarrow \tilde{C}(U_{G,K,\alpha})$ by $\sigma(kK) = \mathrm{Id}_{k,\mathrm{Ad}_k}$. We easily see that σ is an injective group homomorphism.

Using Proposition 5 define a map $\pi_K : \tilde{C}(U_{G,K,\alpha}) \longrightarrow \mathrm{Aut}(\mathcal{Z})$ by $\pi_K(s_{f,l}) = s$, which is a continuous group homomorphism. Moreover, by Corollary 6(iv), $\mathrm{Im}(\sigma) = \mathrm{Ker}(\pi_K)$, hence the image of σ is a closed subgroup. If we endow $N_G(K)/K$ with the quotient topology, σ becomes a continuous injective homomorphism between Polish groups. Therefore σ is open.

On $L_K(U,\alpha) = \tilde{C}(U_{G,K,\alpha})/\,\mathrm{Ker}(\pi_K)$ consider the quotient topology and call it the L_K-topology. By [1], $L_K(U,\alpha)$ furnished with the L_K-topology becomes a Polish group. Notice also that the L_K-topology is stronger than the uniform topology on $\mathrm{Aut}(\mathcal{Z})$ restricted to $L_K(U,\alpha)$.

We have constructed the following short exact sequence of Polish topological groups.

$$1 \longrightarrow N_G(K)/K \xrightarrow{\sigma} \tilde{C}(U_{G,K,\alpha}) \xrightarrow{\pi_K} L_K(U,\alpha) \longrightarrow 1. \tag{10}$$

Proposition 7. *If* $\mathrm{Aut}(\mathcal{Z}) = \mathrm{cl}\{U^n:\ n\in\mathbb{Z}\} = L_K(U,\alpha)$, *then*

$$\tilde{C}(U_{G,K,\alpha}) = \mathrm{cl}\{\sigma(kK)\circ(U_{G,K,\alpha})^n:\ n\in\mathbb{Z},\ k\in N_G(K)\}.$$

Proof: Since $\mathrm{Aut}(\mathcal{Z}) = L_K(U,\alpha)$ the L_K-topology and uniform topology are equal. To end the proof notice that, by Corollary 6(iv),

$$\pi_K^{-1}(\{U^n:\ n\in\mathbb{Z}\}) = \{\sigma(kK)\circ(U_{G,K,\alpha})^n:\ n\in\mathbb{Z},\ k\in N_G(K)\}.$$

Since the uniform topology on $\mathrm{Aut}(\mathcal{Z})$ is equal to the quotient topology (with respect to $\mathrm{Ker}(\pi_k)$), the pre-image of the dense in $\mathrm{Aut}(\mathcal{Z})$ set $\{U^n:\ n\in\mathbb{Z}\}$ is dense in $\tilde{C}(U_{G,K,\alpha})$. □

Corollary 8. *Let $\mathcal{Z}$ be a minimal rotation. If $K \neq \{1_G\}$ and $K < G$ is irreducible, then $L_K(U,\alpha) \neq C(U)$.*

Proof: Recall that an arbitrary minimal flow $\mathcal{Z}$ is a rotation iff $\mathrm{Aut}(\mathcal{Z})$ is compact, notice that $N_G(K)/K$ is compact and use (10) to finish the proof. □

References

[1] Brown L.G., *Topologically complete groups*, Proc. Amer. Math. Soc. **35** (1972), 593–600.

[2] Danilenko A., Lemańczyk M., *Isometric extensions, 2-cocycles and ergodicity of skew products*, Studia Math. **137** (1999), 123–142.

[3] Glasner E., Mentzen M.K., Siemaszko A., *A natural family of factors for minimal flows*, Contemp. Math. **215** (1998), 19–42.

[4] Siemaszko A., *Ergodic and Topological Properties of Distal Dynamical Systems*, PhD Thesis, Toruń, 1996.

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